Research paper

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Problems of structural optimization for post-buckling behaviour

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Abstract A proposal of a new approach to the optimal design of structures under stability constraints is presented. It is shown that the standard problem of maximization of the instability load may be modified so as to obtain a specified post-critical behaviour of the designed structure. The modified optimal structure represents stable post-buckling behaviour either locally, that is, in the vicinity of the critical point, or for a specified range of generalized displacements. First, some rigid–elastic finite-degree-of-freedom models are optimized, giving an insight into the modified design problems. Then a classification of the new optimization problems is presented. Various forms of instability are taken into account and a broad selection of objective as well as constraint functions is proposed. Based on the presented classification and following the proposed optimization concept, detailed formulations of nonlinear problems of design for post-buckling behaviour are given.

Key words instability, post-buckling behaviour, optimal design

1
Introduction

The maximization of the instability load for a prescribed volume of a designed element is a standard problem of optimization under stability constraints. The analysis of nonlinear post-buckling behaviour and the influence of imperfections are, in general, not included in such a standard formulation and therefore important information about the behaviour of a designed element after buckling is not provided. Very often the standard optimal structure represents unstable post-buckling behaviour and is very sensitive to imperfections. This is a drawback of the design and it indicates that the combination of geometrically nonlinear analysis with the design procedure is necessary, especially from a practical point-of-view. Because of its complexity, this area of research has not been broadly investigated so far. Only recently have papers been published dealing with the optimization of geometrically nonlinear structures exposed to a loss of stability (Godoy 1996; Mróz and Piekarski 1996, 1998; Perry and Gürdal 1996; Pietrzak 1996; Cardoso *et al.* 1997; Sousa *et al.* 1999; Sorokin and Terentiev 2001). It has been shown that if geometrical nonlinearity is allowed for and nonlinear instability analysis is performed, more accurate information about the behaviour of the optimized structure can be provided. It is possible to evaluate the quality of the design and, if necessary, to reject solutions that are not applicable. Furthermore, it is possible to implement nonlinear constraints into the formulation of the optimization problem and hence to modify the design. Post-buckling constraints of a special form that depends on the type of instability are added to the mathematical programming problem, which allows the nonlinear equilibrium path of the optimized structure to be altered and a stable post-buckling path to be created. This concept was proposed by Bochenek (1993), and then applied to solving many nonlinear optimization problems (Bochenek 1996, 1997a,b, 1999a,b; Bochenek and Krużelecki 2001; Bochenek and Bielski 2001).

2
A concept of modified optimization

The aim of this section is to present the idea of a new approach to optimization against instability. Several simple rigid–elastic finite-degree-of-freedom models that consist

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of rigid rods connected by elastic joints and equipped with extensional and rotational springs are chosen for this purpose. For each of them, an instability analysis based on energy considerations is performed, leading to an analytical expression for the load as a function of the generalized displacement that describes the nonlinear equilibrium path.

The first model shown in Fig. 1 consists of two rigid rods connected by an elastic joint. A rotational spring of stiffness C and an extensional spring K, both with linearly elastic characteristics, are added to the system. The model is loaded by a conservative force P that retains its direction after buckling. If the angle φ is chosen as the generalized displacement that controls the nonlinear post-critical deformation, the total potential energy of the system can be written as

$$\Pi = \frac{1}{2}C\varphi^2 + \frac{1}{2}K(L\sin\varphi)^2 - \\ P(L + D - L\cos\varphi - D\cos\psi)\,. \tag{1}$$

From the stationarity condition $\mathrm{d}\Pi/\mathrm{d}\varphi = 0$ follows an expression for the load p vs displacement φ that describes the nonlinear equilibrium path:

$$p(\varphi) = \frac{\sqrt{\varkappa^2 - \sin^2\varphi}\left(\varphi + \frac{1}{2}\gamma\sin 2\varphi\right)}{\sqrt{\varkappa^2 - \sin^2\varphi}\sin\varphi + \frac{1}{2}\sin 2\varphi}\,, \tag{2}$$

in which a geometrical relation and dimensionless quantities are introduced as

$$\sin\psi = \frac{1}{\varkappa}\sin\varphi\,, \quad p = \frac{PL}{C}\,, \quad \gamma = \frac{KL^2}{C}\,, \quad \varkappa = \frac{D}{L}\,. \tag{3}$$

The critical (bifurcation) load can be found directly from (2) as

$$p_{\mathrm{cr}} = (1+\gamma)\frac{\varkappa}{1+\varkappa}\,. \tag{4}$$

If post-critical paths found for various values of the stiffness γ are analyzed, one can see from Fig. 2 that the post-buckling behaviour of the system can be either stable or unstable depending on the value of γ. This means that by selecting appropriate values of γ, the creation of a specified behaviour of the structure is possible. Hence, for γ as the design variable, the following optimization problem can be formulated. For a given value of C, find γ so as to maximize the critical load and simultaneously assure stable post-buckling behaviour of the optimized structure:

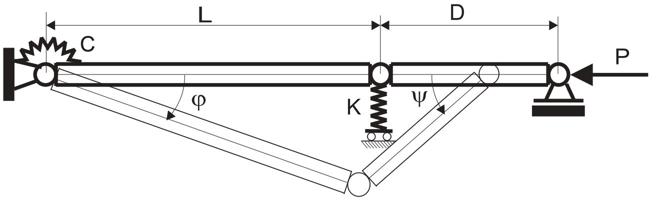

Fig. 1 Rigid–elastic one-degree-of-freedom system, symmetric bifurcation

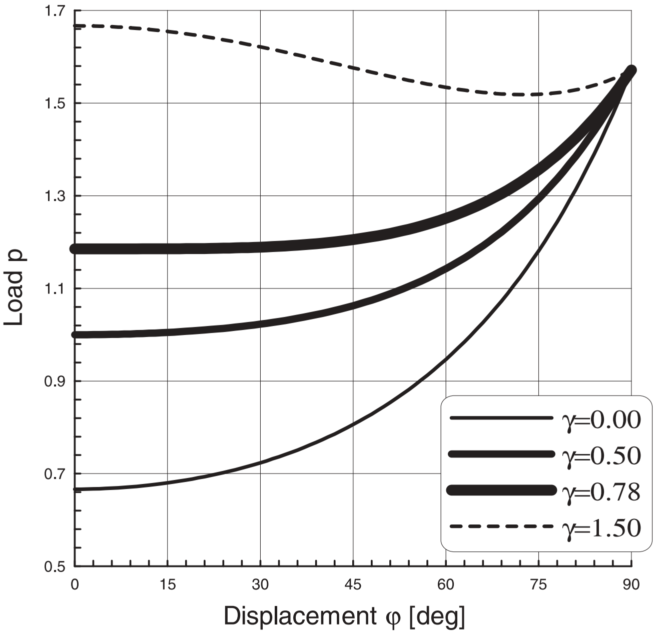

Fig. 2 Post-critical paths for selected values of γ

$$\text{maximize} \quad p_{\mathrm{cr}}(\gamma) = (1+\gamma)\frac{\varkappa}{1+\varkappa}\,,$$

$$\text{subject to} \quad \frac{\partial^2 p}{\partial\varphi^2}\,(0;\gamma) \geq 0\,. \tag{5}$$

Solving the problem formulated above, in which the post-buckling constraint is set locally for $\varphi = 0$, one obtains

$$\gamma_{\mathrm{opt}} = \frac{\varkappa^3 + 4\varkappa^2 - 3}{3(\varkappa^3 + 1)}\,, \tag{6}$$

which, for $\varkappa = 2.0$, gives $\gamma = \gamma_{\mathrm{opt}} = 7/9$ and $p_{\mathrm{cr}} = 32/27$. The post-buckling path for the optimal solution is represented in Fig. 2 by a thick solid line.

As can be seen in the above example, by implementing an appropriate local post-buckling constraint into the formulation of the optimization problem, the desired modification of the symmetric post-critical path was achieved. It is worth stressing that such a local constraint may not be sufficient in all cases. To make matters even more complicated, the behaviour of the structure at the critical point does not have to be symmetric. What can be done if this happens? We shall discuss this issue by analyzing another model, shown in Fig. 3. Once again the model consists of two bars, but this time only the one of length L is rigid. The length of the second one can vary and its extensibility is modelled by a spring K. The total potential energy for the model is given by

$$\Pi = \frac{1}{2}C\varphi^2 + \frac{1}{2}K(L_k - L_0)^2 - PL(1-\cos\varphi)\,, \tag{7}$$

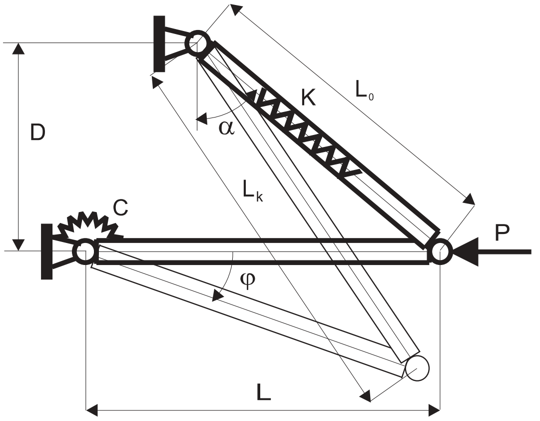

Fig. 3 Rigid–elastic one-degree-of-freedom model, asymmetric bifurcation

which leads to

$$p(\varphi)=\frac{\varphi}{\sin\varphi}+\gamma[1-\varkappa\tan\alpha+\varkappa\cot\varphi]- \frac{\gamma\varkappa}{\cos\alpha}\frac{(1-\varkappa\tan\alpha)+\varkappa\cot\varphi}{\sqrt{(\varkappa\tan\alpha+\cos\varphi-1)^2+(\varkappa+\sin\varphi)^2}} \tag{8}$$

and

$$p_{\mathrm{cr}}=1+\gamma\cos^2\alpha\,. \tag{9}$$

The definitions of the dimensionless quantities are the same as in the previous example, and once again, the maximal critical load subject to stable post-buckling behaviour is sought. However, the formulation of the optimization problem is different. Since the behaviour of the structure at the critical point is asymmetric, post-buckling constraints that are independent of each other for positive and negative values of the generalized displacement must be imposed. In addition, setting constraints that ensure symmetric behaviour in the vicinity of the critical point is necessary. For given values C and α, values of γ and $\varkappa$ are sought for which the critical load is maximal with respect to the constraints forcing the optimized structure to behave in a stable way in a specified interval of the angular displacement φ:

$$\text{maximize}\quad p_{\mathrm{cr}}(\gamma,\varkappa)=1+\gamma\cos^2\alpha\,,$$

$$\text{subject to}\quad \frac{\partial p}{\partial\varphi}(0;\gamma,\varkappa)=\varkappa-\sin\alpha\cos\alpha=0\,, \tag{10}$$

$$\frac{\partial p}{\partial\varphi}(\varphi;\gamma,\varkappa)\,\mathrm{sign}\,\varphi\geq 0\quad\text{for }\varphi\in[\varphi_1,\varphi_2]\,. \tag{11}$$

In Fig. 4, selected solutions (for $\alpha=60°$) that fulfill equality constraint (10) are shown. The optimal solution $\varkappa_{\mathrm{opt}}=\sqrt{3}/4$, $\gamma_{\mathrm{opt}}=0.74$, $p_{\mathrm{cr}}=1.185$, found for $\varphi_1=-90°$, $\varphi_2=90°$, is represented by a thick solid line.

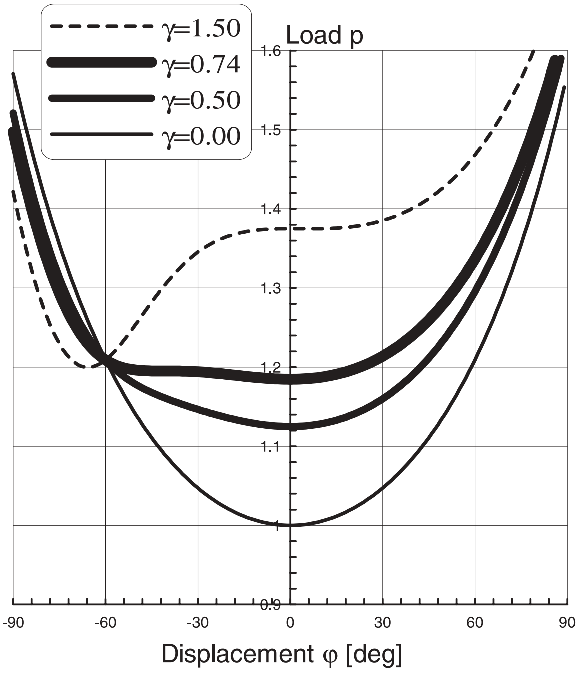

Fig. 4 Post-critical paths for selected values of γ

In the examples discussed above, instability was caused by applied external forces. It is known that in some cases, loss of stability can be the result of an increase in temperature. A simple model that is exposed to thermal buckling is shown in Fig. 5. The extensibility of a bar axis is represented by a spring of stiffness K and a coefficient of thermal expansion α. The total potential energy of deformation (without thermal energy) can be written as

$$\Pi=\frac{1}{2}C\varphi^2+\frac{1}{2}K\left[\alpha L_0T-L+L_0\right]^2\,. \tag{12}$$

A nonlinear equilibrium path $t(\varphi)$ that follows from the stationarity condition is given by

$$t(\varphi)=\frac{1}{\gamma}\frac{\varphi}{\sin\varphi}\cos^2\varphi+\frac{1-\cos\varphi}{\cos\varphi}\,. \tag{13}$$

The quantities in (13) are defined as

$$t=\alpha T\,,\qquad \gamma=\frac{KL_0^2}{C}\,. \tag{14}$$

The optimal value of γ is sought so as to maximize the critical temperature t (strain caused by temperature in-

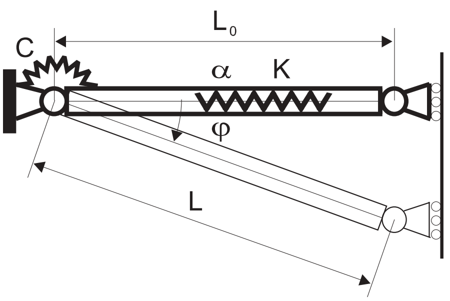

Fig. 5 Rigid–elastic one-degree-of-freedom model, thermal buckling

crement T) subject to stable post-critical behaviour of the system:

$$\text{maximize} \quad t_{\mathrm{cr}} = \frac{1}{\gamma}\,,$$

$$\text{subject to} \quad \frac{\partial^2 t}{\partial \varphi^2}(0;\gamma) \geq 0\,. \tag{15}$$

Solving (15) one obtains $\gamma_{\mathrm{opt}} = 5/3$, $t_{\mathrm{cr}} = 3/5$, and the post-buckling path for the optimal solution is given by a thick solid line in Fig. 6.

The results obtained so far show that by changing quantities that describe the stiffness of the structure or its geometry, the post-buckling behaviour can be modified and the desired stable behaviour after buckling can be obtained. It is known that the design variables in the modified design problems can also be chosen from quantities describing additional support or additional loading. The next example shows that even a parameter controlling the behaviour of the loading after buckling can be a design variable. The analyzed structure is shown in Fig. 7. The quantity η describes the direction of loading in post-critical regime. Although the problem is non-conservative, the static criterion of stability is sufficient as long as the analysis is limited to negative values of η. From the equation of equilibrium one can obtain

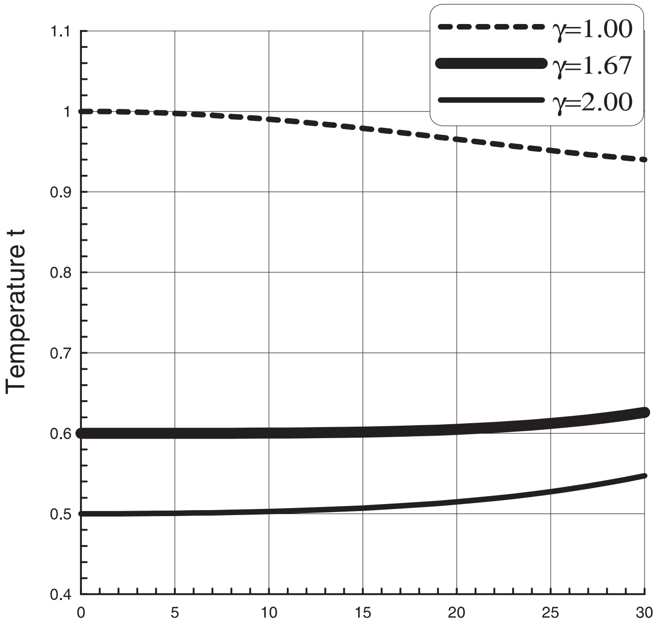

Fig. 6 Post-critical paths for selected values of γ

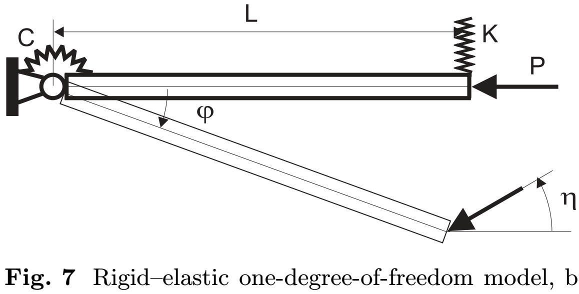

Fig. 7 Rigid–elastic one-degree-of-freedom model, buckling under subtangential force

$$C\varphi + KL^2 \sin\varphi\cos\varphi = PL\cos\eta\varphi\sin\varphi -$$

$$PL\sin\eta\varphi\cos\varphi\,, \tag{16}$$

which leads to (the former definitions of dimensionless quantities hold)

$$p(\varphi) = \frac{\varphi + \gamma\sin\varphi\cos\varphi}{\sin(1-\eta)\varphi}\,. \tag{17}$$

For a given γ the following modified design problem,

$$\text{maximize} \quad p_{\mathrm{cr}}(\eta) = \frac{1+\gamma}{1-\eta}\,,$$

$$\text{subject to} \quad \frac{\partial^2 p}{\partial \varphi^2}(0;\eta) \geq 0\,, \tag{18}$$

leads to the optimal value of the design variable η,

$$\eta_{\mathrm{opt}} = 1 - 2\sqrt{\frac{\gamma}{1+\gamma}}\,. \tag{19}$$

Selected post-buckling paths for $\gamma = 1$ are shown in Fig. 8, in which the thick solid line represents the optimal solution ($\eta_{\mathrm{opt}} = 1 - \sqrt{2}$, $p_{\mathrm{cr}} = \sqrt{2}$).

Summarizing the discussion of this section, one can state that modification of the standard optimization problem is possible and the proposed approach allows the specified behaviour of the optimized structure after buckling to be obtained. The modified optimal structure exhibits stable post-critical behaviour either locally,

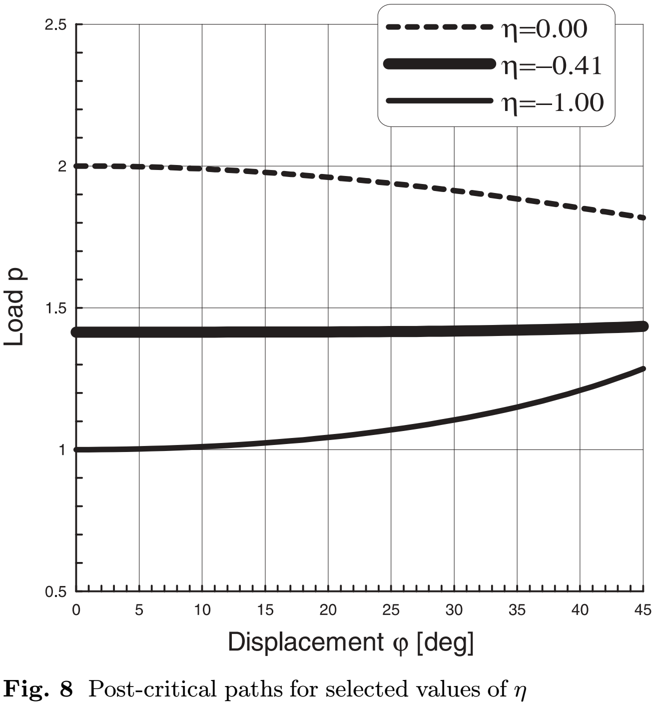

Fig. 8 Post-critical paths for selected values of η

that is, in the vicinity of the critical point or in specified range of a generalized displacement. Moreover various cases of loadings or design variables show that the implementation of nonlinear post-buckling analysis in the formulation of optimization problems opens many possibilities for new design problems. The proposed new concept of optimization under stability constraints is called the **modified optimization**.

3
General classification

In the modified design problems, the most important decision to be made is the choice of post-buckling **constraints**. One can impose these constraints either locally (i.e. in the vicinity of the critical point) or for the specified range of a generalized displacement. The latter approach is called "extended local" here. If constraints are set for any specific value of a generalized displacement, it is called a "global" approach. The concept of post-buckling constraints is presented in Fig. 9.

The **design variables** in the modified design problem can be chosen from quantities that describe the stiffness of a structure, the shape of its cross-section or the shape of its axis, additional active or passive (additional support) loads, and even the behaviour of the load after buckling.

The **objective** in the modified design problem is usually the same as in the standard problem of optimization against instability, i.e., bifurcation or snap-through load. Since nonlinear analysis is allowed for, the objective can also be chosen as the maximal load on the nonlinear post-buckling path or the minimal load if the maximal load is absent. When design variables do not affect the buckling load but can change the post-critical behaviour, the objective can be chosen as a specified function.

Selecting the objective now and implementing the post-buckling constraints, many new modified design tasks may be proposed. These modified problems for structures exposed to elastic instability can be classified according to the form of instability. Selected objective functions for standard and modified problems of structural optimization against instability are presented in Figs. 10 and 11. The following notation was applied to describe particular optimization tasks:

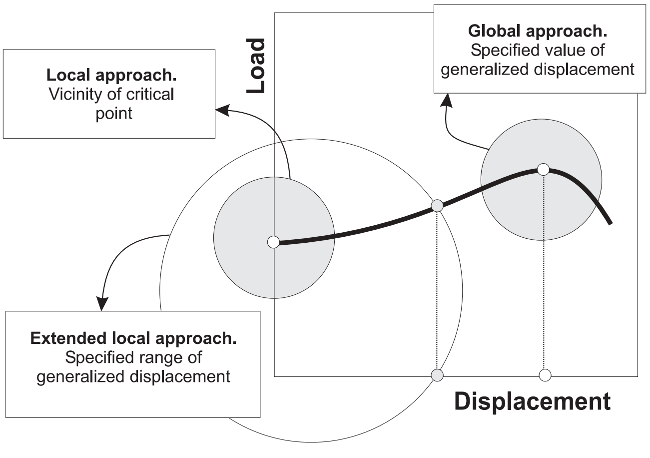

Fig. 9 Local, extended local, and global post-buckling constraints

- Upper-case letters – type of instability loading: B-Bifurcation, M-Multimodal bifurcation, S-Snap-through loading, L-Lower critical load, U-Upper critical loading (leading to exhaustion of carrying capacity), F-Flutter load, O-denotes the absence of a relevant formulation;
- Lower-case letters – type of formulation: s-standard formulation, m-modified formulation;
- Superscripts – e-elasticity (modified problems can be formulated for inelastic instability and then p-plasticity, c-creep are used), (1)-single criterion optimization, (2)-multi-criteria optimization;
- Subscripts – 2-second order bifurcation, o-objective function different from critical load, d-displacement for snap-through load as the objective;
- Lower-case letters in parentheses – type of approach for post-buckling constraints: (l)-local approach, (f)-extended local approach (for finite interval), (g)-global approach.

4
Detailed formulations

Based on the presented classification and following the proposed optimization concept, detailed formulations of selected nonlinear problems of design for post-buckling behaviour are given. The particular tasks are defined within the groups of problems specified in Sect. 3. Mathematical formulae for those tasks are presented, as well as a graphical illustration of each subproblem. The figures show the results of application of the modified formulation compared with the results of the standard optimization.

4.1
Structural optimization against instability leading to maximization of single buckling load

Maximization of the bifurcation load subject to a constant total volume for the optimized structure is a standard problem of optimization under stability constraints:

$$\text{maximize} \quad p_{\mathrm{cr}}(a_i)\,,$$

$$\text{subject to} \quad V(a_i) = V_0\,. \tag{20}$$

In (20), a_i stands for the design variables and V is the volume of the structure. The standard problem is now modified by implementing suitable post-buckling constraints either in local or in extended local form. Both symmetric and asymmetric bifurcation are taken into account.

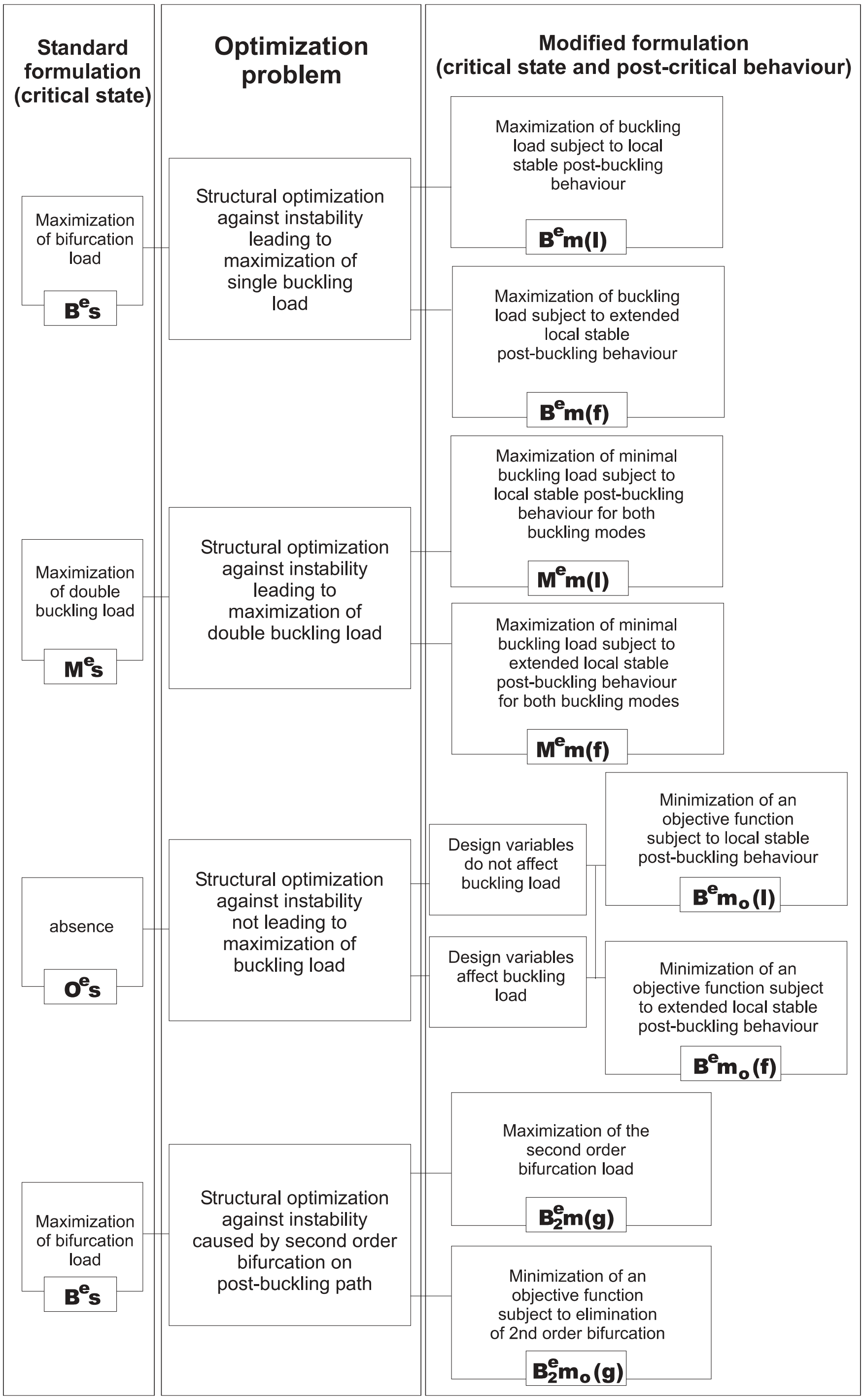

Fig. 10 Selected objective functions for standard and modified problems of structural optimization against instability

4.1.1
Maximization of buckling load subject to local stable post-buckling behaviour – problem $B^e m(l)$

Symmetric bifurcation (Fig. 12):

$$\begin{aligned} &\text{maximize} && p_{\text{cr}}(a_i), \\ &\text{subject to} && V(a_i) = V_0\,, \\ &&& \frac{\partial^2 p}{\partial \delta^2}(0; a_i) \geq 0\,. \end{aligned} \tag{21}$$

The quantity δ in (21) stands for a generalized displacement that controls post-buckling deformation.
Asymmetric bifurcation (Fig. 13):

$$\begin{aligned} &\text{maximize} && p_{\text{cr}}(a_i)\,, \\ &\text{subject to} && V(a_i) = V_0\,, \\ &&& \frac{\partial p}{\partial \delta}(0; a_i) = 0\,, \\ &&& \frac{\partial^2 p}{\partial \delta^2}(0; a_i) \geq 0\,. \end{aligned} \tag{22}$$

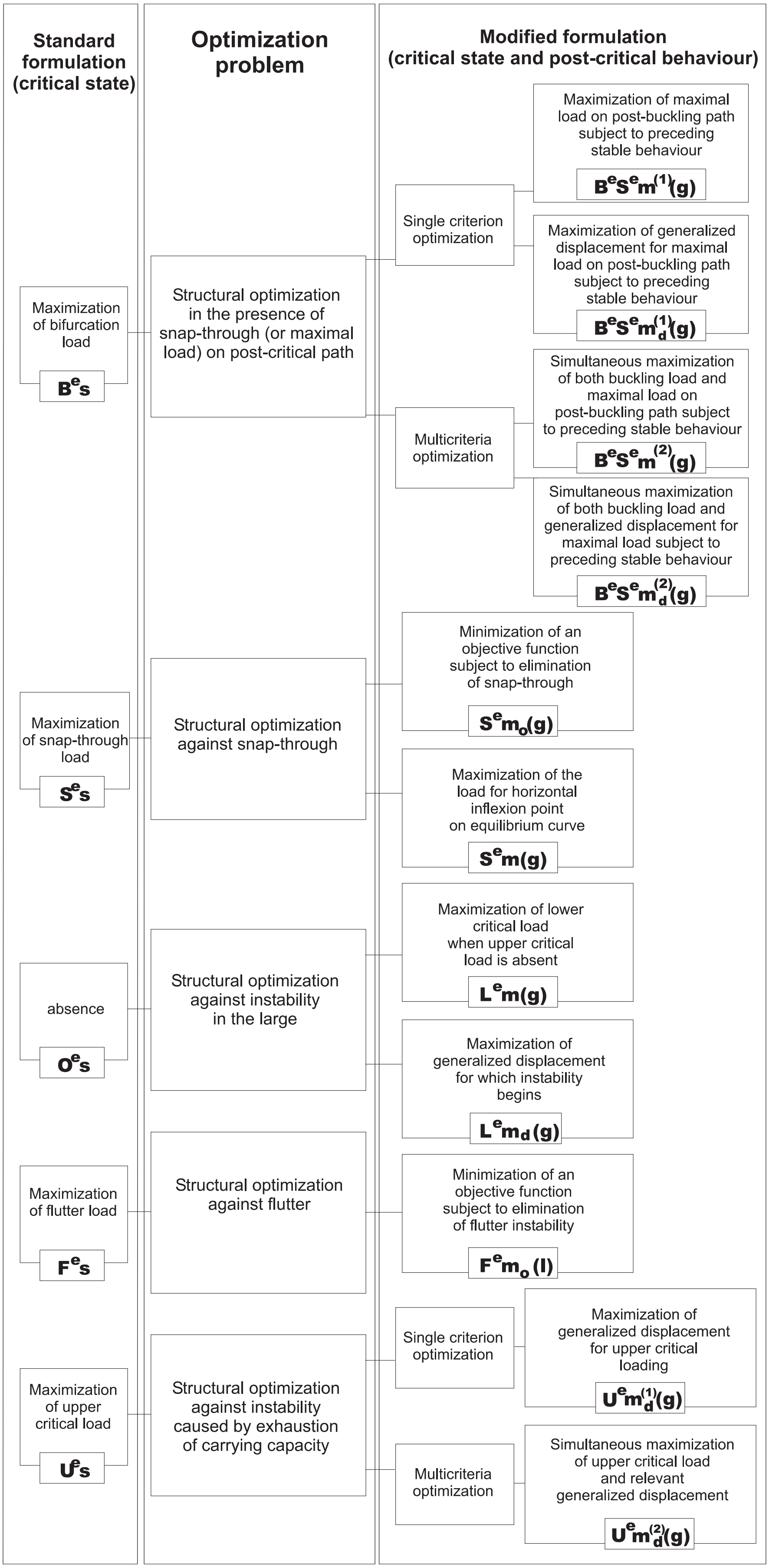

Fig. 11 Selected objective functions for standard and modified problems of structural optimization against instability

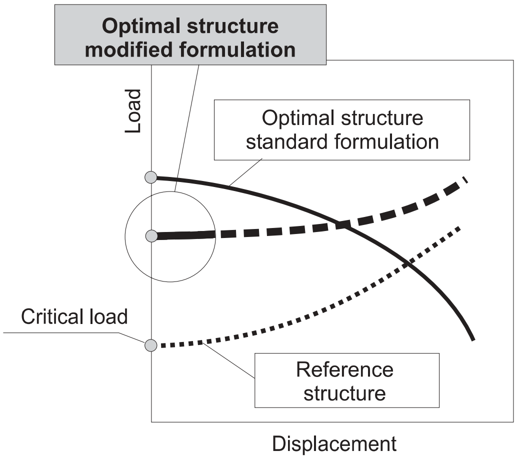

Fig. 12 Maximization of buckling load subject to local stable post-buckling behaviour, symmetric bifurcation

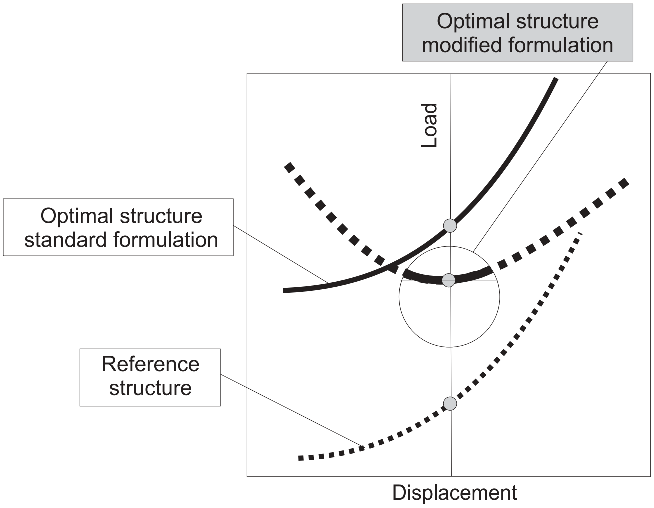

Fig. 13 Maximization of buckling load subject to local stable post-buckling behaviour, asymmetric bifurcation

4.1.2
Maximization of buckling load subject to extended local stable post-buckling behaviour – problem $B^e m(f)$

Formulating constraints imposed on the post-buckling behaviour in the extended local approach, the post-critical path is discretized, which leads to a set of constraints for specified values of the generalized displacement δ_j.

Symmetric bifurcation (Fig. 14):

$$\begin{aligned}
&\text{maximize} && p_{\mathrm{cr}}(a_i)\,,\\
&\text{subject to} && V(a_i) = V_0\,,\\
& && p(\delta_j; a_i) - p(\delta_{j+1}; a_i) \le 0\,,\\
& && j = 1, 2 \ldots m\,.
\end{aligned} \tag{23}$$

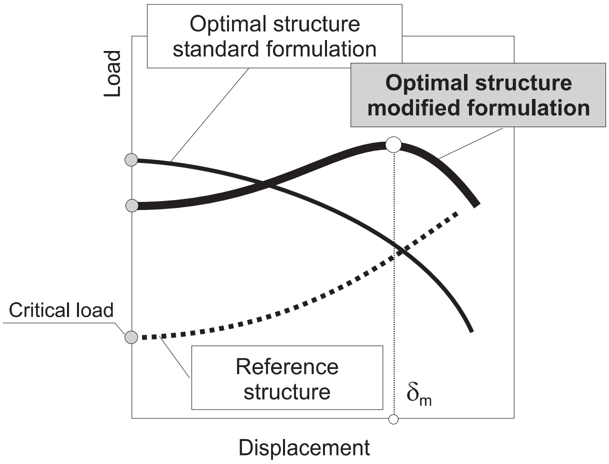

Fig. 14 Maximization of buckling load subject to extended local stable post-buckling behaviour, symmetric bifurcation

Asymmetric bifurcation (Fig. 15):

$$\begin{aligned}
&\text{maximize} && p_{\mathrm{cr}}(a_i)\,,\\
&\text{subject to} && V(a_i) = V_0\,,\\
& && \frac{\partial p}{\partial \delta}(0; a_i) = 0\,,\\
& && p(\delta_j; a_i) - p(\delta_{j+1}; a_i) \le 0\,,\\
& && \quad \delta_j \ge 0, \quad j = 1, 2 \ldots m\,,\\
& && p(\delta_{k+1}; a_i) - p(\delta_k; a_i) \le 0\,,\\
& && \quad \delta_k \le 0, \quad k = 1, 2 \ldots l\,.
\end{aligned} \tag{24}$$

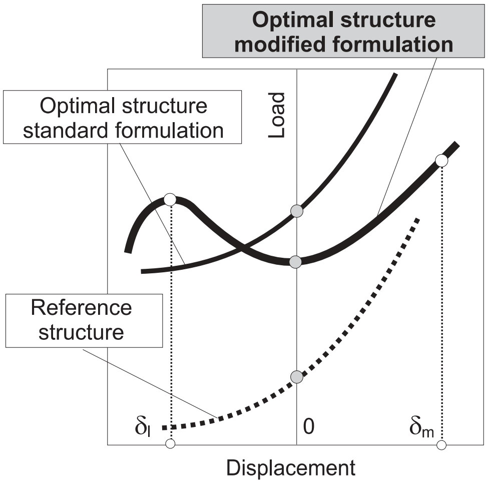

Fig. 15 Maximization of buckling load subject to extended local stable post-buckling behaviour, asymmetric bifurcation

4.2
Structural optimization against instability leading to maximization of double buckling load

The standard optimization problem is the maximization of the minimal buckling load subject to a constraint imposed on the total volume of the optimized structure.

4.2.1

Maximization of minimal buckling load subject to local stable post-buckling behaviour for both buckling modes – problem M^{e}m(l)

In the local approach, the minimal critical load is maximized with respect to constraints, ensuring stable behaviour of both buckling modes in the vicinity of the critical points (Fig. 16):

$$
\begin{aligned}
&\text{maximize minimal} && p_{\text{cr}}(a_i)\,,\\
&\text{subject to} && V(a_i) = V_0\,,\\
&&& \frac{\partial^2 p^{(1)}}{\partial \delta^2}(0; a_i) \geq 0\,,\\
&&& \frac{\partial^2 p^{(2)}}{\partial \delta^2}(0; a_i) \geq 0\,.
\end{aligned}
\tag{25}
$$

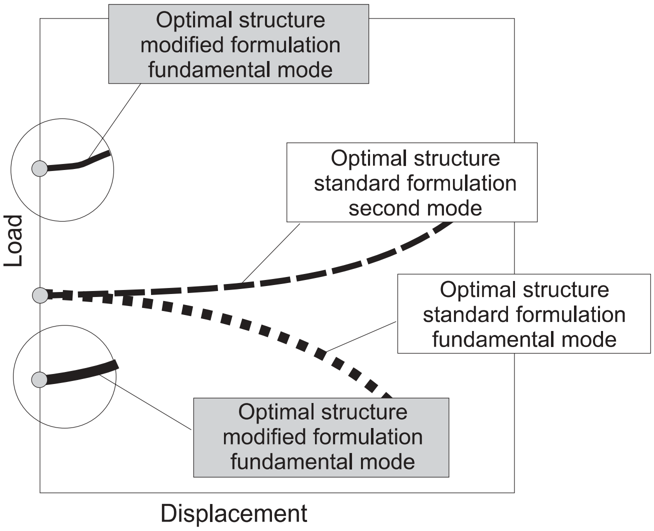

Fig. 16 Maximization of minimal buckling load subject to local stable post-buckling behaviour for both buckling modes

4.2.2

Maximization of minimal buckling load subject to extended local stable post-buckling behaviour for both buckling modes – problem M^{e}m(f)

As for the local approach, stable post-buckling behaviour for both buckling modes is required, this time in a specified range of the generalized displacement. The implementation of extended local constraints leads to the separation of critical loads (Fig. 17):

$$
\begin{aligned}
&\text{maximize minimal} && p_{\text{cr}}(a_i)\,,\\
&\text{subject to} && V(a_i) = V_0\,,\\
&&& p^{(1)}(\delta_j; a_i) - p^{(1)}(\delta_{j+1}; a_i) \leq 0\,,\\
&&& \quad j = 1, 2 \ldots m\,,\\
&&& p^{(2)}(\delta_k; a_i) - p^{(2)}(\delta_{k+1}; a_i) \leq 0\,,\\
&&& \quad k = 1, 2 \ldots l\,.
\end{aligned}
\tag{26}
$$

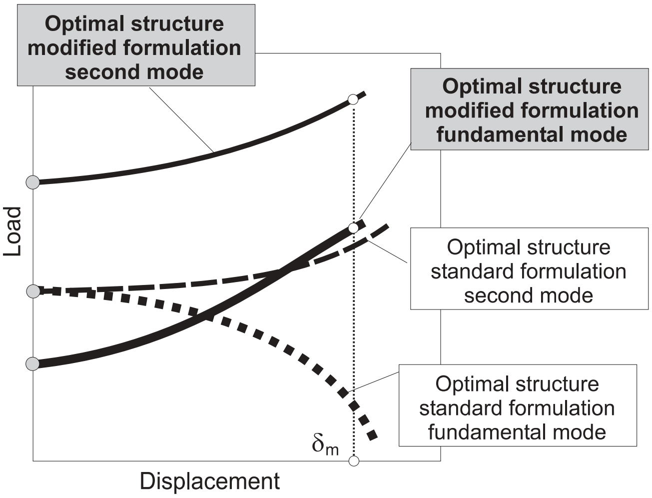

Fig. 17 Maximization of minimal buckling load subject to extended local stable post-buckling behaviour for both buckling modes

4.2.3

Maximization of minimal buckling load subject to stable post-buckling behaviour for fundamental buckling mode provided that post-critical path for the other mode goes above the fundamental one – problem M^{e}m(f)

In many cases, the requirement of stable post-critical behaviour of both modes is not necessary. It is sufficient if only the fundamental path is stable and the second one goes above it. This leads to the following alternative formulation (Fig. 18):

$$
\begin{aligned}
&\text{maximize} && p^{(1)}(a_i)\,,\\
&\text{subject to} && V(a_i) = V_0\,,\\
&&& p^{(1)}(\delta_j; a_i) - p^{(1)}(\delta_{j+1}; a_i) \leq 0\,,\\
&&& \quad j = 1, 2 \ldots m\,,\\
&&& p^{(1)}(\delta_k; a_i) - p^{(2)}(\delta_k; a_i) \leq 0\,,\\
&&& \quad k = 1, 2 \ldots l\,.
\end{aligned}
\tag{27}
$$

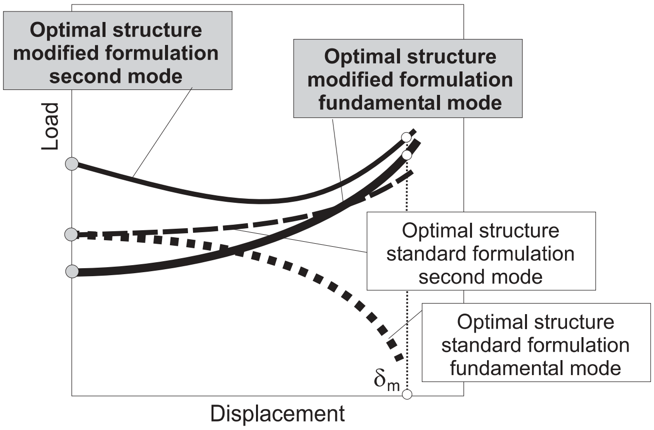

Fig. 18 Maximization of minimal buckling load subject to stable post-buckling behaviour for fundamental buckling mode, provided that post-critical path for the other mode goes above the fundamental one

4.3
Structural optimization against instability not leading to maximization of buckling load

If the design variables do not affect the critical load, the standard formulation of the optimization problem is not possible. When design variables influence the post-buckling behaviour, the modified problems can be posed (Figs. 19 and 20).

4.3.1
Minimization of an objective function subject to local stable post-buckling behaviour when design variables do not affect buckling load – problem $\mathbf{B^e m_o(l)}$

$$
\begin{aligned}
\text{minimize}\quad & F(a_i)\,,\\
\text{subject to}\quad & V(a_i) = V_0\,,\\
& \frac{\partial p}{\partial \delta}(0;a_i) = 0\,,\\
& \frac{\partial^2 p}{\partial \delta^2}(0;a_i) \geq 0\,.
\end{aligned}
\tag{28}
$$

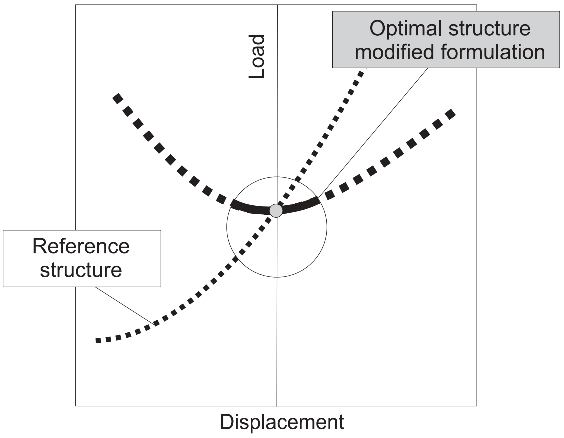

Fig. 19 Minimization of an objective function subject to local stable post-buckling behaviour when design variables do not affect buckling load

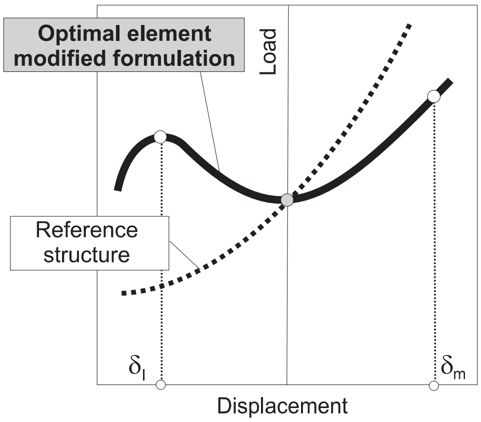

Fig. 20 Minimization of an objective function subject to extended local stable post-buckling behaviour when design variables do not affect buckling load

4.3.2
Minimization of an objective function subject to extended local stable post-buckling behaviour when design variables do not affect buckling load – problem $\mathbf{B^e m_o(f)}$

$$
\begin{aligned}
\text{minimize}\quad & F(a_i)\,,\\
\text{subject to}\quad & V(a_i) = V_0\,,\\
& \frac{\partial p}{\partial \delta}(0,a_i) = 0\,,\\
& p(\delta_j;a_i) - p(\delta_{j+1};a_i) \leq 0\,,\\
& \quad \delta_j \geq 0, \quad j = 1,2\ldots m\,,\\
& p(\delta_{k+1};a_i) - p(\delta_k;a_i) \leq 0\,,\\
& \quad \delta_k \leq 0, \quad k = 1,2\ldots l\,.
\end{aligned}
\tag{29}
$$

In (28) and (29), F stands for a specified objective function.

4.4
Structural optimization in the presence of snap-through (or maximal load) on post-critical path

After buckling, a maximal load on the post-critical path may appear. This can happen for a reference structure, but such behaviour can also be observed for the standard optimal one. The following three modified problems are proposed (Figs. 21, 22, and 23).

4.4.1
Maximization of maximal load on post-buckling path subject to preceding stable behaviour – problem $\mathbf{B^e S^e m^{(1)}(g)}$

$$
\begin{aligned}
\text{maximize}\quad & p^{\max}(a_i)\,,\\
\text{subject to}\quad & V(a_i) = V_0\,,\\
& p(\delta_j;a_i) - p(\delta_{j+1};a_i) \leq 0\,,\\
& \quad j = 1,2\ldots m\,.
\end{aligned}
\tag{30}
$$

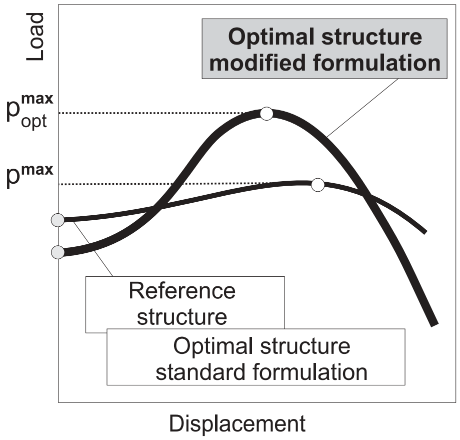

Fig. 21 Maximization of maximal load on post-buckling path subject to preceding stable behaviour

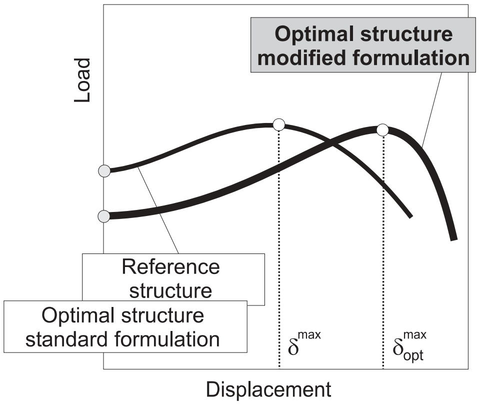

Fig. 22 Maximization of generalized displacement for maximal load on post-buckling path subject to preceding stable behaviour

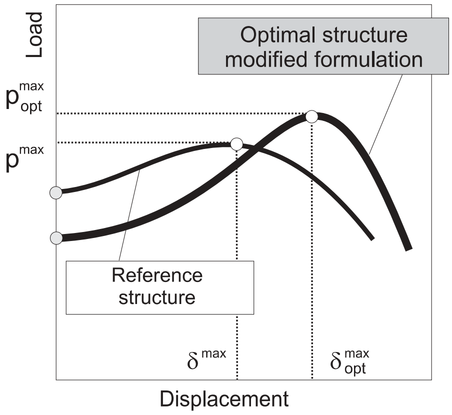

Fig. 23 Maximization of both buckling load and maximal load on post-buckling path subject to preceding stable behaviour

4.4.2

Maximization of generalized displacement for maximal load on post-buckling path subject to preceding stable behaviour – problem $\mathbf{B^eS^em_d^{(1)}(g)}$

$$\begin{aligned} &\text{maximize} && \delta^{\max}(a_i)\,, \\ &\text{subject to} && V(a_i) = V_0\,, \\ & && p(\delta_j; a_i) - p(\delta_{j+1}; a_i) \le 0\,, \\ & && \quad j = 1, 2 \ldots m\,. \end{aligned} \tag{31}$$

4.4.3

Maximization of both buckling load and maximal load on post-buckling path subject to preceding stable behaviour – problem $\mathbf{B^eS^em^{(2)}(g)}$

$$\begin{aligned} &\text{maximize} && \eta p_{\mathrm{cr}}(a_i) + (1-\eta) p^{\max}(a_i)\,, \\ &\text{subject to} && V(a_i) = V_0\,, \\ & && p(\delta_j; a_i) - p(\delta_{j+1}; a_i) \le 0\,, \\ & && \quad j = 1, 2 \ldots m\,. \end{aligned} \tag{32}$$

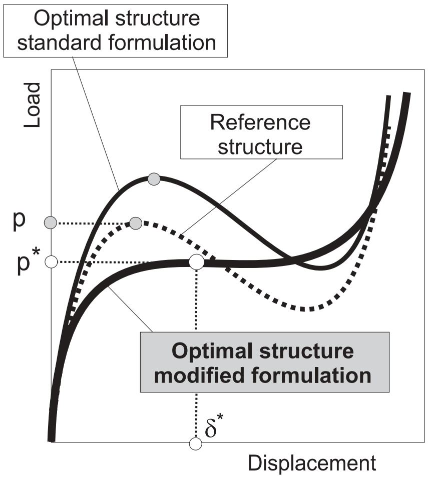

Fig. 24 Minimization of an objective function subject to elimination of snap-through

4.5

Structural optimization against snap-through

Maximization of the snap-through load is the standard optimization problem in this case. Following the approach of this paper, the proposed modifications lead to the elimination of instability (Fig. 24).

4.5.1

Minimization of an objective function subject to elimination of snap-through – problem $\mathbf{S^em_o(g)}$

$$\begin{aligned} &\text{minimize} && F(a_i)\,, \\ &\text{subject to} && V(a_i) = V_0\,, \\ & && \frac{\partial p}{\partial \delta}(\delta^*; a_i) = \frac{\partial^2 p}{\partial \delta^2}(\delta^*; a_i) = 0\,. \end{aligned} \tag{33}$$

4.5.2

Maximization of the load for horizontal inflexion point on equilibrium curve – problem $\mathbf{S^em(g)}$

$$\begin{aligned} &\text{maximize} && p^*(\delta^*; a_i)\,, \\ &\text{subject to} && V(a_i) = V_0\,, \\ & && \frac{\partial p}{\partial \delta}(\delta^*; a_i) = \frac{\partial^2 p}{\partial \delta^2}(\delta^*; a_i) = 0\,. \end{aligned} \tag{34}$$

4.6

Structural optimization against instability in the large[1]

In some cases, the critical state does not exist and instability occurs only at finite displacements. The stan-

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[1] "instability in the large" refers to the case when the system is stable for small disturbances but loses stability for large ones.

dard optimization problem cannot be formulated, but design against instability in the large can be performed (Figs. 25 and 26).

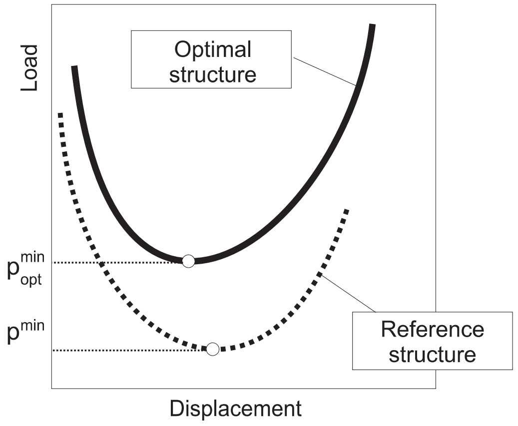

Fig. 25 Maximization of lower critical loading when upper critical load is absent

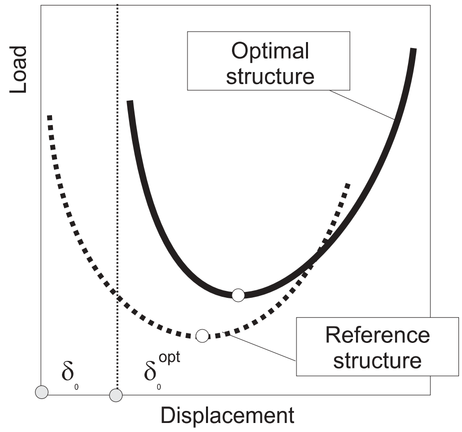

Fig. 26 Maximization of generalized displacement for which instability begins

4.6.1
Maximization of lower critical load when upper critical load is absent – problem $\mathbf{L^e m(g)}$

$$\begin{aligned} &\text{maximize} \quad p^{\min}(\delta; a_i)\,, \\ &\text{subject to} \quad V(a_i) = V_0\,. \end{aligned} \tag{35}$$

4.6.2
Maximization of generalized displacement for which instability begins – problem $\mathbf{L^e m_d(g)}$

$$\begin{aligned} &\text{maximize} \quad \delta_0(\delta; a_i)\,, \\ &\text{subject to} \quad V(a_i) = V_0\,. \end{aligned} \tag{36}$$

5
The modified design of elastic structures

Many of the above classified problems have already been illustrated by the optimization of structural elements like columns, arches, frames, and shells. The first example presented by Bochenek (1993) was concerned with the maximization of the buckling load with respect to the extended local stable post-buckling behaviour of the Stern column (problem $\mathbf{B^e m(f)}$). Maximization of the buckling load subject to the extended local stable post-buckling behaviour for an arch with an extensible axis is performed in Bochenek (1996) (problem $\mathbf{B^e m(f)}$). Minimization of an objective function subject to the extended local stable post-buckling behaviour for the Koiter frame with an additional support is presented in Bochenek (1997a) (problem $\mathbf{B^e m_o(f)}$). The modified optimization in the case in which standard optimization leads to the maximization of the double buckling load (problem $\mathbf{M^e m(f)}$) is discussed in Bochenek (1999a,b). The first paper deals with columns exposed to thermal buckling, whereas the second one is devoted to the optimization of columns in an elastic medium. Bochenek and Bielski (2001) showed the modified design of the Stern column in the presence of the maximal load on the post-buckling path (problem $\mathbf{B^e S^e m^{(1)}(g)}$), as well as the maximization of the buckling load subject to the extended local post-critical behaviour for a toroidal shell under an external uniform pressure and bending (problem $\mathbf{B^e m(f)}$). The problem of the elimination of snap-through instability for an axisymmetric shell (problem $\mathbf{S^e m_o(g)}$) and optimization against instability in the large for a compressed beam resting on a rigid foundation (problem $\mathbf{L^e m(g)}$) is presented in Bochenek (1997b).

In addition, nonlinear analysis and modified optimization have been performed for some discrete models. The use of such elastic–rigid systems is very convenient. It simplifies calculations significantly, but still allows characteristic features of the proposed concept of optimization to be shown for a specified post-buckling behaviour. The models of the Stern column and an arch with an extensible axis are optimized in Bochenek (1996) (problem $\mathbf{B^e m(f)}$) and the models of the Koiter frame with an additional support and a cylindrical shell under radial pressure with an additional tensile loading (problem $\mathbf{B^e m_o(f)}$) are discussed by Bochenek and Krużelecki (2001).

6
Closure

It has been shown that the standard problem of maximization of the instability load may be modified so as to obtain a specified post-critical behaviour of the designed structure. The post-buckling constraints of a special form that depends on the type of instability added to the mathematical programming problem allow the nonlinear equilibrium path of the optimized structure to be altered and a stable post-buckling path to be created. The detailed classification of modified problems for structures exposed to elastic instability has been presented. The next step is to combine geometrical nonlinearity with physical nonlinearity and to investigate the possibility of formulating modified optimization problems for elastic–plastic structures. The proposals for the modified design problems are as follows:

- Structural optimization against instability leading to the maximization of a single elastic–plastic buckling load (maximization of the elastic–plastic buckling load subject to either the local or the extended local stable post-buckling behaviour);
- Structural optimization in the presence of elastic–plastic snap-through on the post-critical path emerging from the elastic critical point (maximization of either the maximal load on the post-buckling path or the generalized displacement for the maximal load subject to preceding stable behaviour, maximization of both the buckling load and the maximal load on the post-buckling path subject to preceding stable behaviour);
- Structural optimization in the presence of plastic snap-through on the post-critical path emerging from the plastic critical point;
- Structural optimization against plastic snap-through (minimization of an objective function subject to the elimination of snap-through, maximization of the load for the horizontal inflexion point on the equilibrium curve).

Some results for a discrete elastic–plastic–rigid system that illustrate the modified optimization in the elastic–plastic range have already been obtained by Bochenek and Bielski (2001).

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