Chapter 10

Statistical Background and Concepts

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10.1 Introduction

Statistics do a fine job of enumerating what has already occurred. Industry's most urgent needs are to estimate what will happen in the future. Will the product be profitable? How often will defects occur? The job of statistics is to help estimate the future based on the past.

When designing any part or system, it is necessary to estimate and account for the variation that is likely to occur in the parts, materials, and product features. Statistics can help estimate or model the most likely outcome, and how much variation there is likely to be in that outcome. From these models, estimates of manufacturability and product performance can be made long before production. Knowledge of the probabilities of defects prior to production is important to the financial success of the product. Changes to the design or manufacturing processes that are completed prior to production are far less costly than changes made during production or changes made after the product is fielded. Statistics can help estimate these probabilities.

10.2 Shape, Locations, and Spread

Historical data or data from a designed experiment when displayed in a histogram will:

- Have a shape
- Have a location relative to some important values such as the average or a specification limit
- Have a spread of values across a range.

For example, Fig. 10-1 contains full indicator movement (FIM) runout values of 1,000 steel shafts, measured in thousandths of an inch (mils). Ideally, these 1,000 shafts would all be the same, but the histogram begins to reveal some information about these shafts and the processes that made them. The thousand data points are displayed in a histogram in Fig. 10-1. A histogram displays the frequency (how often) a range of values is present. The histogram has a shape, its location is concentrated between the values 0.000 and 0.005, and is spread out between the values 0 and 0.030. The range that occurs most often is 0.000 to 0.002, but there are many shafts that are larger than this. Statistics can help quantify the histogram. With knowledge of the type of distribution (shape), the mean of the sample (location), and the standard deviation of the sample (spread), one can estimate the chance that a shaft will exceed a certain value like a specification. We will come back to this example later.

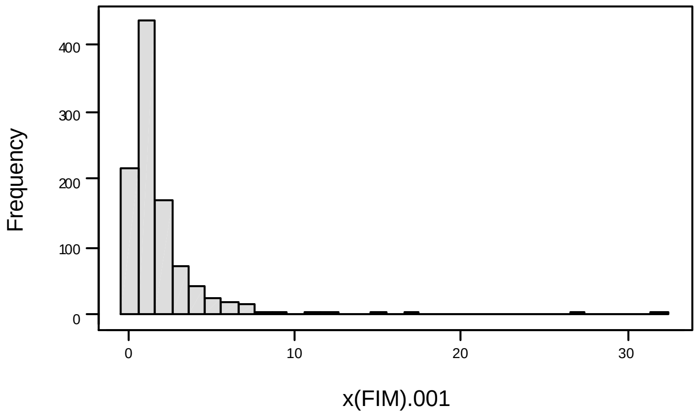


Figure 10-1 Histogram of runout (FIM) data

10.3 Some Important Distributions

Data that is measured on a continuous scale like inches, ohms, pounds, volts, etc. is referred to as variables data. Data that is classified by pass or fail, heads or tails, is called attributes data. Variables data may be more expensive to gather than attributes data, but is much more powerful in its ability to make estimates about the future.

10.3.1 The Normal Distribution

The normal distribution is a mathematical model. All mathematical models are wrong, in that there is always some error. Some models are useful. This is one of them.

Karl Frederick Gauss described this distribution in the eighteenth century. Gauss found that repeated measurements of the same astronomical quantity produced a pattern like the curve in Fig. 10-2. This pattern has since been found to occur almost everywhere in life. Heights, weights, IQs, shoe sizes, various standardized test scores, economic indicators, and a host of measurements in service and manufacturing are all examples of where the normal distribution applies. (Reference 4) A normal distribution:

- Has one central value (the average).
- Is symmetrical about the average.
- Tails off asymptotically in each direction.

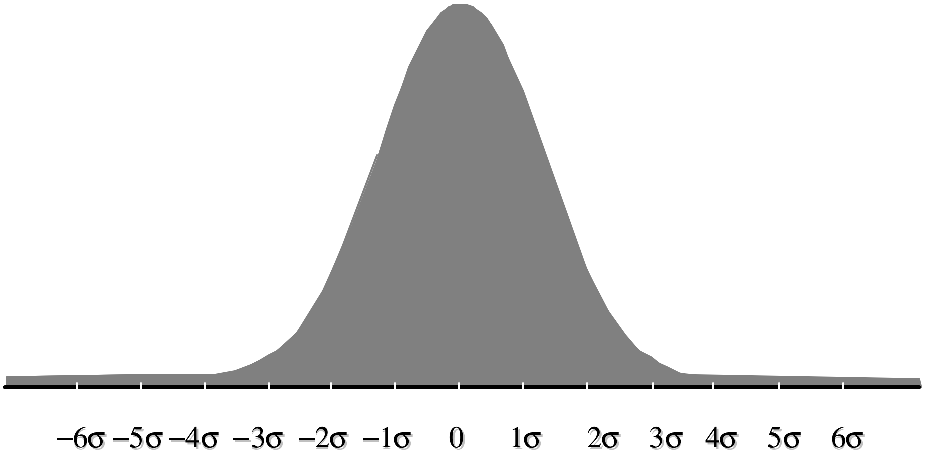


Figure 10-2 The normal distribution

The normal distribution is defined by: $f(x) = \frac{1}{s\sqrt{2p}} e^{-(1/2)[(x-m)/s]^2}$

The mean *(m)* is: $m = \frac{\sum_{i=1}^{n} x_i}{n}$

The standard deviation *(s)* is: $s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - m)^2}{n}}$

where

N is the size of the population

x_i is value of the *i*th component in the population

It is important to note that the definitions for the mean (μ) and the standard deviation (σ) are not dependent on the distribution *f(x)*. We will see other functions later, but the definitions for the mean and the standard deviation are the same.

Data that appear to be normally distributed occur often in science and engineering. In my many years of practice and study, I have never seen a perfectly normal distribution. To illustrate, the following histograms (Figs. 10-3 to 10-6) were generated by picking random numbers from a true normal distribution with a mean of 10 and a standard deviation of 1.

Five samples from a true normal distribution yield a histogram with very little information (Fig. 10-3). The curve is a normal distribution with an average and a standard deviation calculated from the five samples. It is used to compare the data with a normal curve produced from that data.

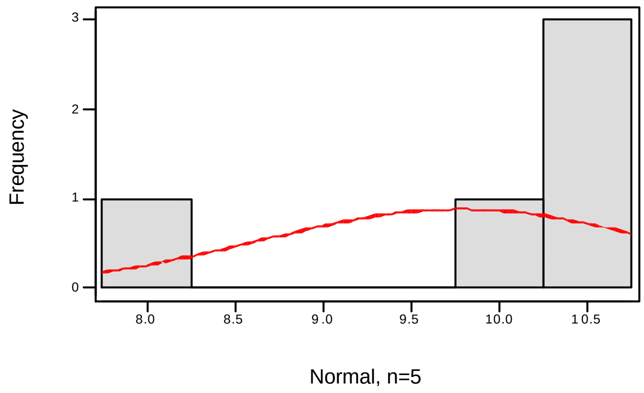


Figure 10-3 Histogram of normal, n=5, with normal curve

When 50 samples are taken from a normal distribution we see the following histogram and a normal curve generated from the 50 samples (Fig. 10-4). Here we begin to see a central tendency between 10.0 and 10.5 and a gradual decline in frequency as we move away from the center.

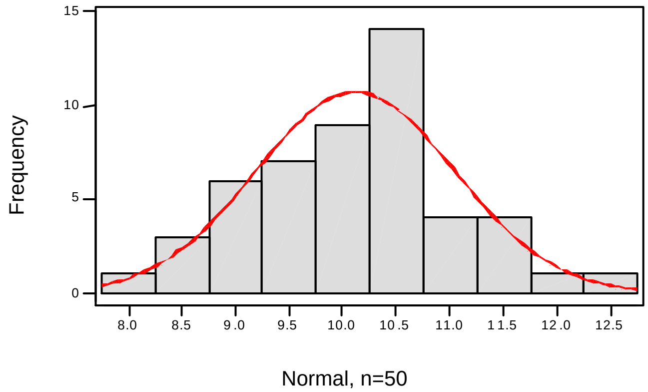


Figure 10-4 Histogram of normal, n=50, with normal curve

The histogram for 500 samples (Fig. 10-5) was taken from a truly normal distribution. Even with 500 samples the histogram does not quite fit the normal model. In this example, the mode (highest peak) is around 9.75.

The histogram for 5000 samples (Fig. 10-6) taken from a normal distribution is still not a perfect fit. Be aware of this behavior when you examine data and distributions. There are statistical tests for judging whether or not a distribution could be from a normal distribution. In these examples, all of the histograms passed the Anderson-Darling test for normality. (Reference 1)

How do I calculate the percent of the population that will be beyond a certain value?

The mathematical answer is to integrate the function *f(x)*. The practical answer is to use a *Z* table found in statistics books (see Appendix at the end of this chapter), or a statistical software package like *Minitab 12*. (Reference 6) Statisticians long ago prepared a table called a *Z* table to make this easier.

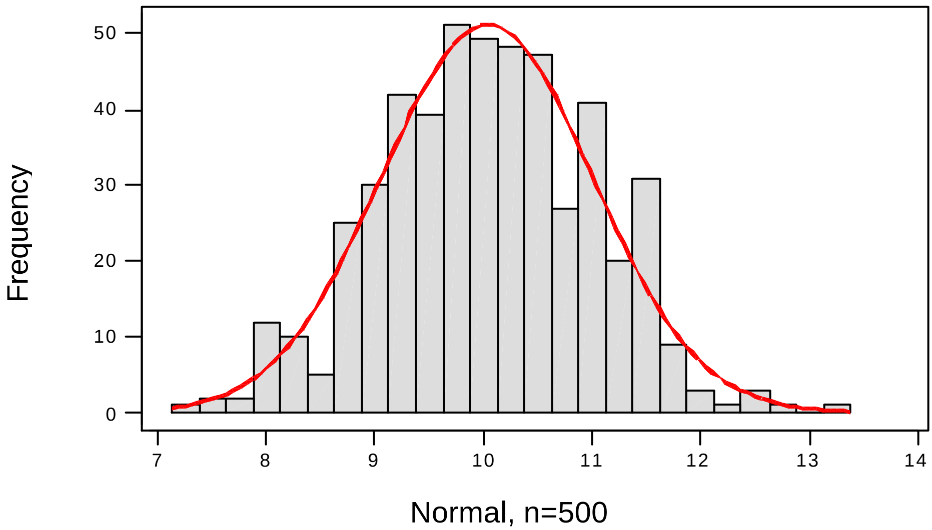


Figure 10-5 Histogram of normal, n=500, with normal curve

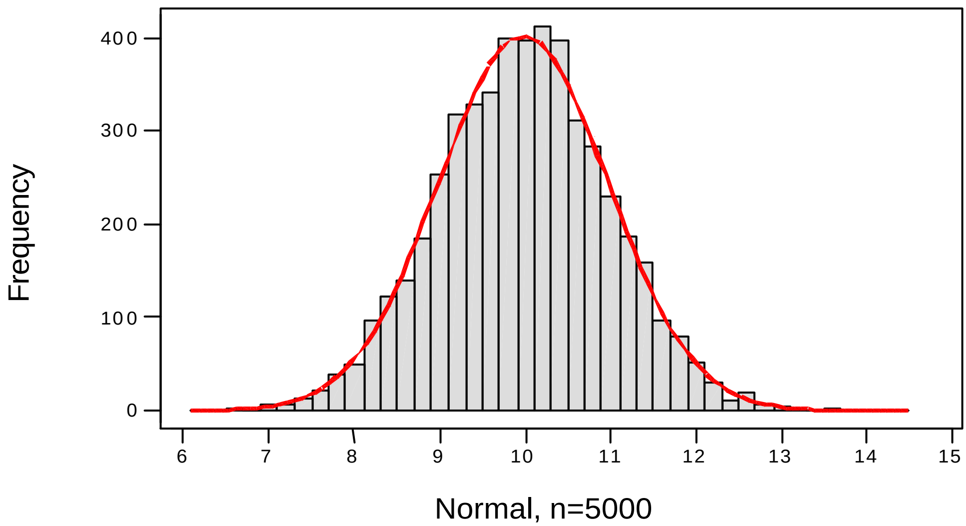


Figure 10-6 Histogram of normal, n=5000, with normal curve

There are different types of *Z* tables. The Appendix shows a *Z* table for the unilateral tail area under a normal curve beyond a given *Z* value. To use the table, we need a *Z* value. *Z* is a statistic that is defined as:

$Z = (x\text{-}m)/s$, where:

x is a value we are interested in, a specification limit, for example

m is the mean (average)

s is the standard deviation

Continuing with Fig. 10-7 as an example, suppose we are interested in knowing the probability of *x* being greater than 2.5σ. (Remember that σ is a value that has a unit of measure like inches.) Using the *Z* table in the Appendix for *Z* = 2.5, we find the value 0.00621, which is the probability that *x* will be greater than 2.5σ.

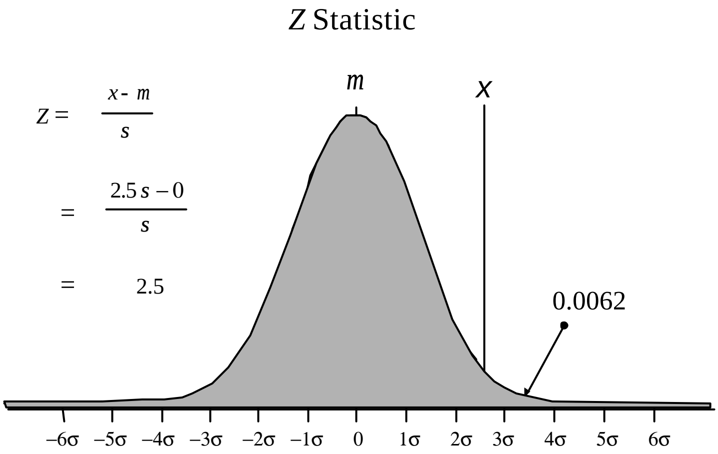


Figure 10-7 *Z* Statistic

What if the histogram does not look like a normal distribution?

There are many continuous distributions that occur in science and engineering that are not normal. Some of the most common continuous distributions are:

1. Beta
2. Cauchy
3. Exponential
4. Gamma
5. Laplace
6. Logistic
7. Lognormal
8. Weibull

We will look at the lognormal briefly here for illustration, although I think it is best to refer to texts on statistics and reliability for more detail. (References 3 and 4)

10.3.2 Lognormal Distribution

Recall the above example of the FIM of the shafts. (Fig. 10-1) Certainly this is not normally distributed. Fig. 10-8 is a test for normality. The plot points do not follow the expected line for a normal distribution and the p value is 0.000. The chance that this data came from a normal distribution is almost zero.

This has the shape of a lognormal distribution, which occurs often in mechanical and electrical measurements. The measurements tend to stack up near zero because that is the natural limit. For example, shafts cannot be better than zero FIM and electrical resistance cannot be less than zero.

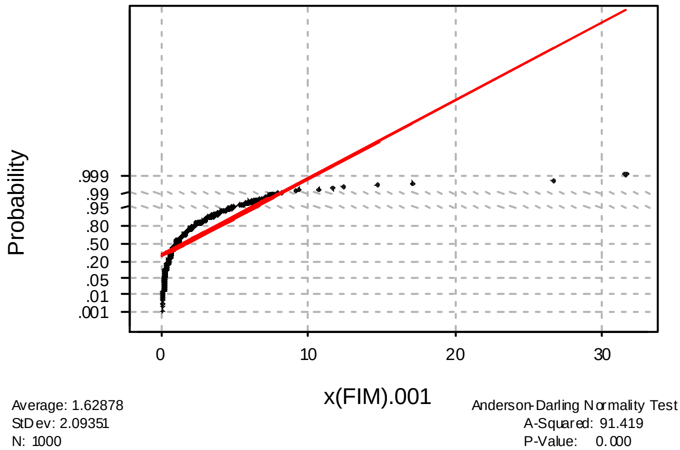


Figure 10-8 Normality test FIM

There are two ways to handle the lognormal distribution. One is to transform the value of the x's by using the relationship:

$$y = ln(x),$$

And plot a new histogram (Fig. 10-9).

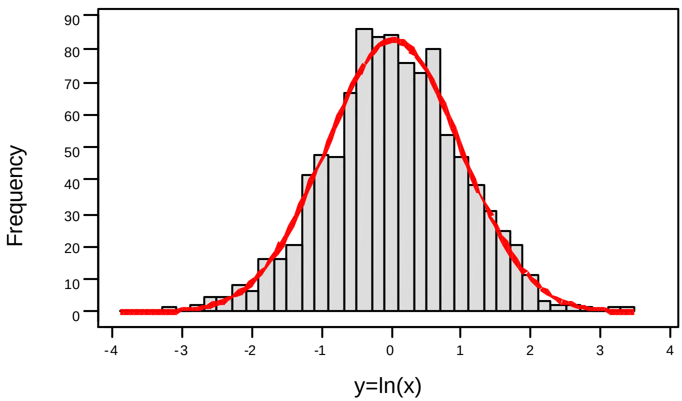


Figure 10-9 Histogram of transformed FIM measurements

This new histogram looks like a good approximation to a normal curve. It passes the Anderson-Darling test for normality (Fig. 10-10), and we can now apply the usual statistics to this transformed set of data.

The second way to work with lognormal distributions is to perform the calculations directly on the lognormal data using a statistical software package like *Minitab 12*. This software can calculate and plot all the relevant statistics from most distributions.

In either case, we can determine the probability of exceeding a value like a specification limit.

The probabilities are *additive* for each dimension or feature of a part or system. This additive property allows a design team to estimate the probability of a defect at any level in the system.

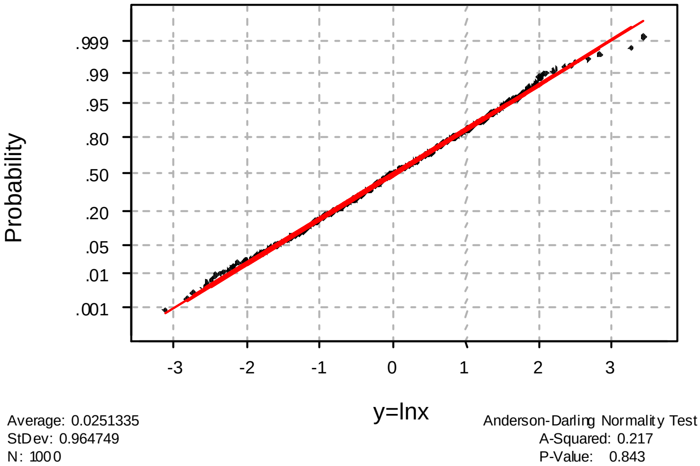


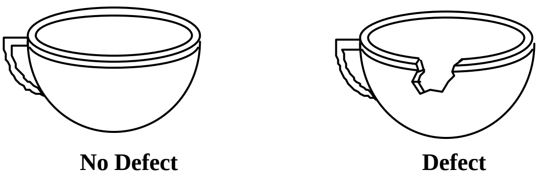
Figure 10-10 Normality tests for transformed data

10.3.3 Poisson Distribution

Discrete data that is classified by pass or fail, heads or tails, is called attributes data. Attributes data can be distributed according to:

- A uniform distribution of probability
- The hypergeometric distribution
- The binomial distribution or
- The Poisson distribution

Figure 10-11 shows an example of attributes data.



$$\text{DPU} = \frac{\text{\# defects found}}{\text{\# units inspected}} = \frac{1}{200} = .005$$

Figure 10-11 Attributes data

The Poisson can be applied to many randomly occurring phenomena over time or space. Consider the following scenarios:

- The number of disk drive failures per month for a particular type of disk drive
- The number of dental cavities per 12-year-old child
- The number of particles per square centimeter on a silicon wafer
- The number of calls arriving at an emergency dispatch station per hour
- The number of defects occurring in a day’s production of radar units
- The number of chocolate chips per cookie

The Poisson can model each of these scenarios. The Poisson random variable is characterized by the form "the number of occurrences per unit interval," where an occurrence could be a defect, a mechanical or electrical failure, an arrival, a departure, or a chocolate chip. The unit could be a unit of time, or a unit of space, or a physical unit like a radar or a cookie, or a person.

The probability distribution function for the Poisson is: $P(X = x) = (l^{x} e^{-l}) / x!$

where

P is the probability that a single unit has x occurrences

λ is a positive constant representing "the average number of occurrences per unit interval"

x is a nonnegative integer and is the specified number of occurrences per unit interval

e is the number whose natural logarithm is 1, and is equal to approximately 2.71828.

For example, suppose we had the following information about a product:

- 1,000 units were inspected and 519 defects were observed.

We want to:

- calculate the number of defects per unit (DPU), and
- estimate the number of units that have exactly three defects (X=3).

The overall rate (λ) that defects occur is: 519/1000 = 0.519 defects per unit (DPU). For X = 3 defects (exactly 3 defects on a unit), the probability is:

$$P(X = 3) = [(l^{3})(e^{-l})] / 3!$$

$$l = 519 / 1000 = 0.519$$

$$P(X = 3) = 0.01387$$

The probability that a unit has exactly 3 defects is 0.01387. So, for 1,000 units we would expect 14 units to have exactly 3 defects each. Table 10-1 enumerates the distribution of the 519 defects.

Table 10-1 Distribution of defects

X (number of defects)	*P(X)*	Number of Units	Defects
0	0.5951	595	0
1	0.3088	309	309
2	0.0802	80	160
3	0.0139	14	42
4	0.0018	2	8
5	0.0002	0	0
6	0.0000	0	0
7	0.0000	0	0
Total	1.0000	1,000	519

The distribution appears graphically in Fig. 10-12.

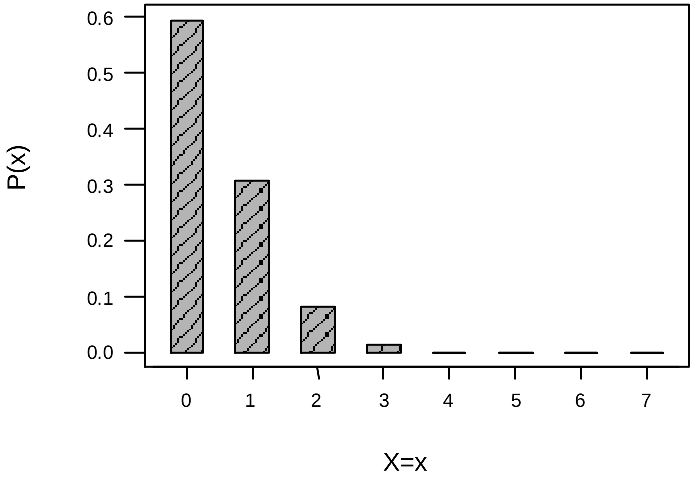


Figure 10-12 Plot of Poisson probabilities

How do I estimate yield from DPU?

To produce a unit of product with zero defects, we need to know the probability of zero defects. Recalling the Poisson equation above,

$$P(X = x) = (\lambda^x e^{-\lambda}) / x!$$

Substituting DPU for λ, and solving for x = 0, we have $P(0) = e^{-DPU}$

To yield good product, there must be no defects. Therefore, the first time yield is : $FTY = e^{-DPU}$. First time yield is a function of how many defects there are. Zero *DPU* means that *FTY*=100%. This agrees with our intuition that if there are no defects, the yield must be 100%.

How do I estimate parts per million (PPM) from yield?

PPM is a measure of the estimated number of defects that are expected from a process if a million units were made. Parts per million defective is: *PPM* = (1-*FTY*)(1,000,000).

10.4 Measures of Quality and Capability

10.4.1 Process Capability Index

Historically, process capability has been defined by industry as + or - 3σ (Fig. 10-13). For any one feature or process output, plus or minus 3 sigma gives good results 99.73% of the time with a normal

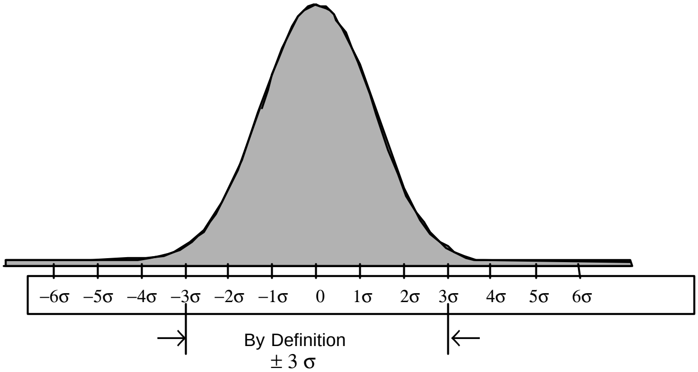


Figure 10-13 Process capability

distribution. This is certainly adequate, especially when dealing with a few features. From this concept came the Process Capability Index (Cp), defined in Fig. 10-14.

$$Cp = \frac{\text{Spec Width}}{\text{Mfg Capability}} = \frac{\text{USL - LSL}}{\pm 3\sigma}$$

"Concurrent Engineering Index"

Design / Manufacturing

Figure 10-14 Capability index

The automotive industry, with leadership from Ford Motor Company, set the design standard of Cp=1.33 in the early 1980s, which corresponds to a process capability of ±4 sigma (Fig. 10-15). This standard has been upgraded since that time, but it is important to note that the product designers had a standard to meet, and that *implied knowing the capability of the process*.

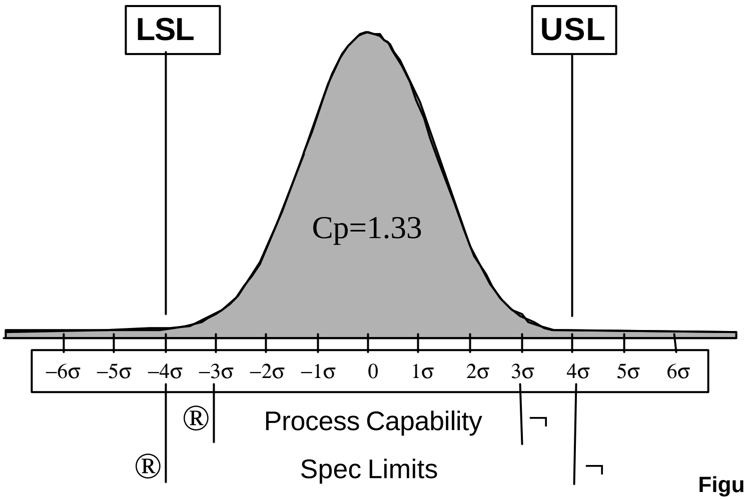


Figure 10-15 Capability index at ± 4 sigma

The Cp index can be thought of as the *concurrent engineering index*. The design engineers have responsibility for the specifications (the numerator), and the process engineers have responsibility for the capability (the denominator). Today's integrated product teams should know the Cp index for each critical-to-quality characteristic.

10.4.2 Process Capability Index Relative to Process Centering (Cpk)

The Cp index has a shortcoming. It does not account for shifts and drifts that occur during the long-term course of manufacturing. Another index is needed to account for shifts in the centering. See Fig. 10-16.

With Six Sigma, the process mean can shift 1.5 standard deviations (see Chapter 1) even when the process is monitored using modern statistical process control (SPC). Certainly, once the shift is detected, corrective action is taken, but the ability to detect a shift in the process on the next sample is small. (It can be shown that for the common x-bar and range chart method with sample size of 5, the probability of detecting a 1.5 sigma shift on the next sample is about 0.50.)

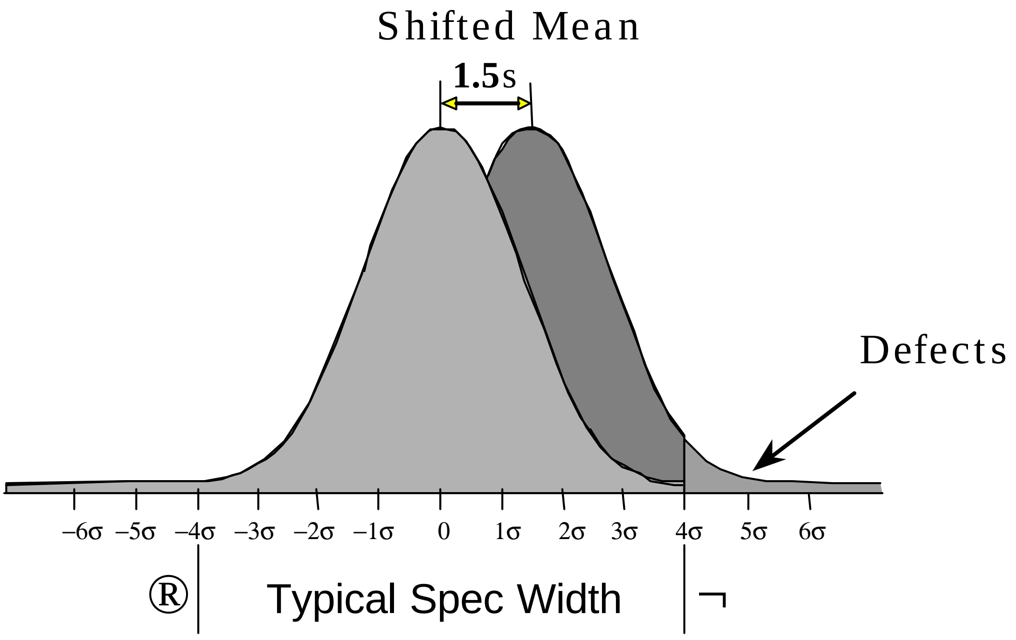


Figure 10-16 The reality

Another index is needed to indicate process centering. Cpk is the process capability index adjusted for centering. It is defined as:

Cpk = Cp(1-k)

where k is the ratio of the amount the center has moved off target divided by the amount from the center to the nearest specification limit. See Fig. 10-17.

If the design target is ±6 sigma, then Cp = 2, and Cpk = 1.5. If every critical-to-quality (CTQ) characteristic is at ±6 sigma, then the probability of all the CTQs being good simultaneously is very high. There would be only 3.4 defects for every 1 million CTQs. See Figs. 10-17 and 10-18.

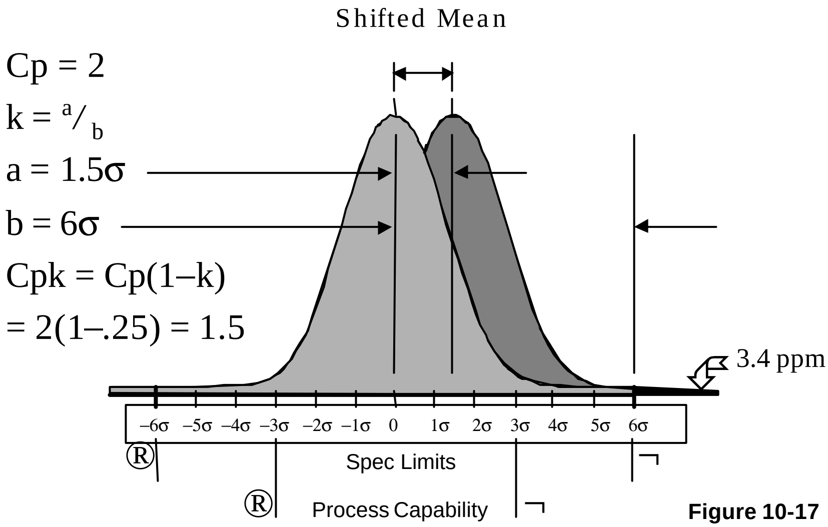


Figure 10-17 Cp and Cpk at Six Sigma

Distribution Shifted 1.5σ

CTQs	± 3s	± 4s	± 5s	± 6s
1	**93.32%**	**99.379%**	**99.9767%**	**99.99966%**
10	**50.08**	**93.96**	**99.768**	**99.9966**
30	**12.57**	**82.95**	**99.30**	**99.99**
50	**---**	**73.24**	**98.84**	**99.98**
100	**---**	**53.64**	**97.70**	**99.966**
150	**---**	**39.28**	**96.57**	**99.948**
200	**---**	**28.77**	**95.45**	**99.931**
300	**---**	**15.43**	**93.26**	**99.897**
400	**---**	**8.28**	**91.11**	**99.862**
500	**---**	**4.44**	**89.02**	**99.828**
800	**---**	**00.69**	**83.02**	**99.724**
1200	**---**	**00.06**	**75.63**	**99.587**

Figure 10-18 Yields through multiple CTQs

10.5 Summary

"We should design products in light of that variation which we know is inevitable rather than in the darkness of chance." –Mikel J. Harry

Estimating the variation that will occur in the parts, materials, processes, and product features is the responsibility of the design team. Estimates of product performance and manufacturability can be made long before production. Statistics can help estimate the most likely outcome, and how much variation there is likely to be in that outcome. Changes made early in the design process are easier and less costly than changes made after production has started. Six Sigma design is the application of statistical techniques to analyze and optimize the inherent system design margins. The objective is a design that can be built error free.

10.6 References

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6. Minitab, Inc. 1997. Minitab Release 12 for Windows. State College, PA: Minitab, Inc.

Table of Unilateral Tail Under the Normal Curve Beyond Selected *Z* Values

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0	5.0000E-01	4.9601E-01	4.9202E-01	4.8803E-01	4.8405E-01	4.8006E-01	4.7608E-01	4.7210E-01	4.6812E-01	4.6414E-01
0.1	4.6017E-01	4.5620E-01	4.5224E-01	4.4828E-01	4.4433E-01	4.4038E-01	4.3644E-01	4.3251E-01	4.2858E-01	4.2465E-01
0.2	4.2074E-01	4.1683E-01	4.1294E-01	4.0905E-01	4.0517E-01	4.0129E-01	3.9743E-01	3.9358E-01	3.8974E-01	3.8591E-01
0.3	3.8209E-01	3.7828E-01	3.7448E-01	3.7070E-01	3.6693E-01	3.6317E-01	3.5942E-01	3.5569E-01	3.5197E-01	3.4827E-01
0.4	3.4458E-01	3.4090E-01	3.3724E-01	3.3360E-01	3.2997E-01	3.2636E-01	3.2276E-01	3.1918E-01	3.1561E-01	3.1207E-01
0.5	3.0854E-01	3.0503E-01	3.0153E-01	2.9806E-01	2.9460E-01	2.9116E-01	2.8774E-01	2.8434E-01	2.8096E-01	2.7760E-01
0.6	2.7425E-01	2.7093E-01	2.6763E-01	2.6435E-01	2.6109E-01	2.5785E-01	2.5463E-01	2.5143E-01	2.4825E-01	2.4510E-01
0.7	2.4196E-01	2.3885E-01	2.3576E-01	2.3270E-01	2.2965E-01	2.2663E-01	2.2363E-01	2.2065E-01	2.1770E-01	2.1476E-01
0.8	2.1186E-01	2.0897E-01	2.0611E-01	2.0327E-01	2.0045E-01	1.9766E-01	1.9489E-01	1.9215E-01	1.8943E-01	1.8673E-01
0.9	1.8406E-01	1.8141E-01	1.7879E-01	1.7619E-01	1.7361E-01	1.7106E-01	1.6853E-01	1.6602E-01	1.6354E-01	1.6109E-01
1	1.5866E-01	1.5625E-01	1.5386E-01	1.5151E-01	1.4917E-01	1.4686E-01	1.4457E-01	1.4231E-01	1.4007E-01	1.3786E-01
1.1	1.3567E-01	1.3350E-01	1.3136E-01	1.2924E-01	1.2714E-01	1.2507E-01	1.2302E-01	1.2100E-01	1.1900E-01	1.1702E-01
1.2	1.1507E-01	1.1314E-01	1.1123E-01	1.0935E-01	1.0749E-01	1.0565E-01	1.0383E-01	1.0204E-01	1.0027E-01	9.8525E-02
1.3	9.6800E-02	9.5098E-02	9.3417E-02	9.1759E-02	9.0123E-02	8.8508E-02	8.6915E-02	8.5343E-02	8.3793E-02	8.2264E-02
1.4	8.0757E-02	7.9270E-02	7.7804E-02	7.6358E-02	7.4934E-02	7.3529E-02	7.2145E-02	7.0781E-02	6.9437E-02	6.8112E-02
1.5	6.6807E-02	6.5522E-02	6.4255E-02	6.3008E-02	6.1780E-02	6.0571E-02	5.9380E-02	5.8207E-02	5.7053E-02	5.5917E-02
1.6	5.4799E-02	5.3699E-02	5.2616E-02	5.1551E-02	5.0503E-02	4.9471E-02	4.8457E-02	4.7460E-02	4.6479E-02	4.5514E-02
1.7	4.4565E-02	4.3633E-02	4.2716E-02	4.1815E-02	4.0930E-02	4.0059E-02	3.9204E-02	3.8364E-02	3.7538E-02	3.6727E-02
1.8	3.5930E-02	3.5148E-02	3.4380E-02	3.3625E-02	3.2884E-02	3.2157E-02	3.1443E-02	3.0742E-02	3.0054E-02	2.9379E-02
1.9	2.8717E-02	2.8067E-02	2.7429E-02	2.6804E-02	2.6190E-02	2.5588E-02	2.4998E-02	2.4419E-02	2.3852E-02	2.3296E-02
2	2.2750E-02	2.2216E-02	2.1692E-02	2.1178E-02	2.0675E-02	2.0182E-02	1.9699E-02	1.9226E-02	1.8763E-02	1.8309E-02
2.1	1.7865E-02	1.7429E-02	1.7003E-02	1.6586E-02	1.6177E-02	1.5778E-02	1.5386E-02	1.5004E-02	1.4629E-02	1.4262E-02
2.2	1.3904E-02	1.3553E-02	1.3209E-02	1.2874E-02	1.2546E-02	1.2225E-02	1.1911E-02	1.1604E-02	1.1304E-02	1.1011E-02
2.3	1.0724E-02	1.0444E-02	1.0170E-02	9.9031E-03	9.6419E-03	9.3867E-03	9.1375E-03	8.8940E-03	8.6563E-03	8.4242E-03
2.4	8.1975E-03	7.9762E-03	7.7602E-03	7.5494E-03	7.3436E-03	7.1428E-03	6.9468E-03	6.7556E-03	6.5691E-03	6.3871E-03
2.5	6.2096E-03	6.0365E-03	5.8677E-03	5.7030E-03	5.5425E-03	5.3861E-03	5.2335E-03	5.0848E-03	4.9399E-03	4.7987E-03

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.6	4.6611E-03	4.5270E-03	4.3964E-03	4.2691E-03	4.1452E-03	4.0245E-03	3.9069E-03	3.7924E-03	3.6810E-03	3.5725E-03
2.7	3.4668E-03	3.3640E-03	3.2640E-03	3.1666E-03	3.0718E-03	2.9796E-03	2.8899E-03	2.8027E-03	2.7178E-03	2.6353E-03
2.8	2.5550E-03	2.4769E-03	2.4011E-03	2.3273E-03	2.2556E-03	2.1858E-03	2.1181E-03	2.0522E-03	1.9883E-03	1.9261E-03
2.9	1.8657E-03	1.8070E-03	1.7500E-03	1.6947E-03	1.6410E-03	1.5888E-03	1.5381E-03	1.4889E-03	1.4411E-03	1.3948E-03
3	1.3498E-03	1.3062E-03	1.2638E-03	1.2227E-03	1.1828E-03	1.1441E-03	1.1066E-03	1.0702E-03	1.0349E-03	1.0007E-03
3.1	9.6755E-04	9.3539E-04	9.0421E-04	8.7400E-04	8.4471E-04	8.1632E-04	7.8882E-04	7.6217E-04	7.3636E-04	7.1135E-04
3.2	6.8713E-04	6.6367E-04	6.4095E-04	6.1896E-04	5.9766E-04	5.7704E-04	5.5708E-04	5.3776E-04	5.1906E-04	5.0097E-04
3.3	4.8346E-04	4.6652E-04	4.5013E-04	4.3427E-04	4.1894E-04	4.0411E-04	3.8977E-04	3.7590E-04	3.6249E-04	3.4953E-04
3.4	3.3700E-04	3.2489E-04	3.1318E-04	3.0187E-04	2.9094E-04	2.8038E-04	2.7017E-04	2.6032E-04	2.5080E-04	2.4160E-04
3.5	2.3272E-04	2.2415E-04	2.1587E-04	2.0788E-04	2.0017E-04	1.9272E-04	1.8554E-04	1.7860E-04	1.7191E-04	1.6545E-04
3.6	1.5922E-04	1.5322E-04	1.4742E-04	1.4183E-04	1.3644E-04	1.3124E-04	1.2623E-04	1.2140E-04	1.1674E-04	1.1225E-04
3.7	1.0793E-04	1.0376E-04	9.9739E-05	9.5868E-05	9.2138E-05	8.8546E-05	8.5086E-05	8.1753E-05	7.8543E-05	7.5453E-05
3.8	7.2477E-05	6.9613E-05	6.6855E-05	6.4201E-05	6.1646E-05	5.9187E-05	5.6822E-05	5.4545E-05	5.2355E-05	5.0249E-05
3.9	4.8222E-05	4.6273E-05	4.4399E-05	4.2597E-05	4.0864E-05	3.9198E-05	3.7596E-05	3.6057E-05	3.4577E-05	3.3155E-05
4	3.1789E-05	3.0476E-05	2.9215E-05	2.8003E-05	2.6839E-05	2.5721E-05	2.4648E-05	2.3617E-05	2.2627E-05	2.1676E-05
4.1	2.0764E-05	1.9888E-05	1.9047E-05	1.8241E-05	1.7466E-05	1.6723E-05	1.6011E-05	1.5327E-05	1.4671E-05	1.4042E-05
4.2	1.3439E-05	1.2860E-05	1.2305E-05	1.1773E-05	1.1263E-05	1.0774E-05	1.0306E-05	9.8568E-06	9.4264E-06	9.0140E-06
4.3	8.6189E-06	8.2403E-06	7.8777E-06	7.5303E-06	7.1976E-06	6.8790E-06	6.5739E-06	6.2817E-06	6.0020E-06	5.7343E-06
4.4	5.4780E-06	5.2327E-06	4.9979E-06	4.7732E-06	4.5582E-06	4.3525E-06	4.1558E-06	3.9675E-06	3.7875E-06	3.6153E-06
4.5	3.4506E-06	3.2932E-06	3.1426E-06	2.9987E-06	2.8611E-06	2.7295E-06	2.6038E-06	2.4837E-06	2.3689E-06	2.2592E-06
4.6	2.1544E-06	2.0543E-06	1.9586E-06	1.8673E-06	1.7800E-06	1.6967E-06	1.6171E-06	1.5412E-06	1.4686E-06	1.3994E-06
4.7	1.3333E-06	1.2702E-06	1.2101E-06	1.1526E-06	1.0978E-06	1.0455E-06	9.9562E-07	9.4803E-07	9.0263E-07	8.5934E-07
4.8	8.1805E-07	7.7868E-07	7.4115E-07	7.0536E-07	6.7124E-07	6.3872E-07	6.0772E-07	5.7818E-07	5.5003E-07	5.2320E-07
4.9	4.9764E-07	4.7329E-07	4.5009E-07	4.2800E-07	4.0695E-07	3.8691E-07	3.6782E-07	3.4965E-07	3.3234E-07	3.1587E-07
5	3.0019E-07	2.8526E-07	2.7105E-07	2.5753E-07	2.4466E-07	2.3242E-07	2.2077E-07	2.0969E-07	1.9915E-07	1.8912E-07

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
5.1	1.7958E-07	1.7051E-07	1.6189E-07	1.5369E-07	1.4589E-07	1.3848E-07	1.3143E-07	1.2473E-07	1.1837E-07	1.1231E-07
5.2	1.0656E-07	1.0110E-07	9.5910E-08	9.0978E-08	8.6293E-08	8.1843E-08	7.7616E-08	7.3602E-08	6.9790E-08	6.6170E-08
5.3	6.2733E-08	5.9469E-08	5.6371E-08	5.3431E-08	5.0640E-08	4.7991E-08	4.5477E-08	4.3091E-08	4.0827E-08	3.8680E-08
5.4	3.6642E-08	3.4709E-08	3.2876E-08	3.1137E-08	2.9488E-08	2.7924E-08	2.6441E-08	2.5035E-08	2.3702E-08	2.2438E-08
5.5	2.1240E-08	2.0104E-08	1.9028E-08	1.8008E-08	1.7042E-08	1.6126E-08	1.5258E-08	1.4436E-08	1.3657E-08	1.2919E-08
5.6	1.2221E-08	1.1559E-08	1.0932E-08	1.0338E-08	9.7764E-09	9.2443E-09	8.7405E-09	8.2636E-09	7.8121E-09	7.3848E-09
5.7	6.9804E-09	6.5976E-09	6.2354E-09	5.8927E-09	5.5684E-09	5.2616E-09	4.9714E-09	4.6968E-09	4.4371E-09	4.1915E-09
5.8	3.9592E-09	3.7395E-09	3.5318E-09	3.3353E-09	3.1496E-09	2.9740E-09	2.8081E-09	2.6512E-09	2.5029E-09	2.3627E-09
5.9	2.2303E-09	2.1051E-09	1.9868E-09	1.8751E-09	1.7695E-09	1.6698E-09	1.5755E-09	1.4865E-09	1.4024E-09	1.3230E-09
6	1.2481E-09	1.1773E-09	1.1104E-09	1.0473E-09	9.8765E-10	9.3138E-10	8.7825E-10	8.2811E-10	7.8078E-10	7.3611E-10
6.1	6.9395E-10	6.5417E-10	6.1663E-10	5.8121E-10	5.4779E-10	5.1626E-10	4.8651E-10	4.5845E-10	4.3199E-10	4.0702E-10
6.2	3.8348E-10	3.6128E-10	3.4034E-10	3.2060E-10	3.0198E-10	2.8443E-10	2.6788E-10	2.5228E-10	2.3758E-10	2.2372E-10
6.3	2.1065E-10	1.9834E-10	1.8674E-10	1.7580E-10	1.6550E-10	1.5579E-10	1.4665E-10	1.3803E-10	1.2991E-10	1.2226E-10
6.4	1.1506E-10	1.0827E-10	1.0188E-10	9.5864E-11	9.0196E-11	8.4858E-11	7.9833E-11	7.5100E-11	7.0645E-11	6.6450E-11
6.5	6.2502E-11	5.8784E-11	5.5285E-11	5.1992E-11	4.8892E-11	4.5975E-11	4.3229E-11	4.0646E-11	3.8214E-11	3.5927E-11
6.6	3.3775E-11	3.1750E-11	2.9845E-11	2.8053E-11	2.6367E-11	2.4781E-11	2.3290E-11	2.1887E-11	2.0568E-11	1.9327E-11
6.7	1.8160E-11	1.7063E-11	1.6032E-11	1.5062E-11	1.4150E-11	1.3293E-11	1.2487E-11	1.1729E-11	1.1017E-11	1.0348E-11
6.8	9.7185E-12	9.1272E-12	8.5715E-12	8.0493E-12	7.5585E-12	7.0974E-12	6.6641E-12	6.2570E-12	5.8745E-12	5.5151E-12
6.9	5.1775E-12	4.8604E-12	4.5625E-12	4.2827E-12	4.0198E-12	3.7730E-12	3.5411E-12	3.3234E-12	3.1189E-12	2.9269E-12
7	2.7466E-12	2.5773E-12	2.4183E-12	2.2691E-12	2.1290E-12	1.9974E-12	1.8740E-12	1.7580E-12	1.6492E-12	1.5471E-12
7.1	1.4512E-12	1.3612E-12	1.2768E-12	1.1975E-12	1.1232E-12	1.0534E-12	9.8787E-13	9.2642E-13	8.6875E-13	8.1465E-13
7.2	7.6389E-13	7.1627E-13	6.7159E-13	6.2968E-13	5.9036E-13	5.5348E-13	5.1888E-13	4.8643E-13	4.5600E-13	4.2745E-13
7.3	4.0068E-13	3.7558E-13	3.5203E-13	3.2995E-13	3.0925E-13	2.8983E-13	2.7163E-13	2.5456E-13	2.3855E-13	2.2355E-13
7.4	2.0948E-13	1.9629E-13	1.8393E-13	1.7234E-13	1.6148E-13	1.5129E-13	1.4175E-13	1.3280E-13	1.2441E-13	1.1655E-13
7.5	1.0919E-13	1.0228E-13	9.5813E-14	8.9749E-14	8.4068E-14	7.8743E-14	7.3754E-14	6.9080E-14	6.4700E-14	6.0596E-14

	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
7.6	5.6750E-14	5.3148E-14	4.9773E-14	4.6611E-14	4.3648E-14	4.0873E-14	3.8274E-14	3.5839E-14	3.3558E-14	3.1421E-14
7.7	2.9420E-14	2.7546E-14	2.5790E-14	2.4146E-14	2.2606E-14	2.1164E-14	1.9813E-14	1.8548E-14	1.7364E-14	1.6255E-14
7.8	1.5216E-14	1.4243E-14	1.3333E-14	1.2480E-14	1.1682E-14	1.0934E-14	1.0234E-14	9.5786E-15	8.9651E-15	8.3906E-15
7.9	7.8529E-15	7.3494E-15	6.8781E-15	6.4370E-15	6.0239E-15	5.6373E-15	5.2754E-15	4.9367E-15	4.6196E-15	4.3228E-15
8	4.0450E-15	3.7850E-15	3.5417E-15	3.3139E-15	3.1008E-15	2.9013E-15	2.7145E-15	2.5398E-15	2.3763E-15	2.2233E-15
8.1	2.0801E-15	1.9460E-15	1.8206E-15	1.7033E-15	1.5935E-15	1.4907E-15	1.3946E-15	1.3046E-15	1.2205E-15	1.1417E-15
8.2	1.0680E-15	9.9906E-16	9.3455E-16	8.7420E-16	8.1773E-16	7.6491E-16	7.1548E-16	6.6924E-16	6.2599E-16	5.8552E-16
8.3	5.4766E-16	5.1224E-16	4.7911E-16	4.4812E-16	4.1913E-16	3.9201E-16	3.6664E-16	3.4291E-16	3.2071E-16	2.9994E-16
8.4	2.8052E-16	2.6236E-16	2.4536E-16	2.2947E-16	2.1460E-16	2.0070E-16	1.8769E-16	1.7553E-16	1.6415E-16	1.5351E-16
8.5	1.4356E-16	1.3425E-16	1.2554E-16	1.1740E-16	1.0979E-16	1.0267E-16	9.6007E-17	8.9779E-17	8.3954E-17	7.8507E-17
8.6	7.3412E-17	6.8648E-17	6.4193E-17	6.0026E-17	5.6130E-17	5.2486E-17	4.9079E-17	4.5892E-17	4.2913E-17	4.0126E-17
8.7	3.7521E-17	3.5084E-17	3.2806E-17	3.0675E-17	2.8683E-17	2.6820E-17	2.5078E-17	2.3449E-17	2.1926E-17	2.0501E-17
8.8	1.9169E-17	1.7924E-17	1.6760E-17	1.5671E-17	1.4653E-17	1.3701E-17	1.2810E-17	1.1978E-17	1.1200E-17	1.0472E-17
8.9	9.7916E-18	9.1553E-18	8.5604E-18	8.0042E-18	7.4841E-18	6.9978E-18	6.5431E-18	6.1180E-18	5.7204E-18	5.3487E-18
9	5.0012E-18	4.6762E-18	4.3724E-18	4.0883E-18	3.8227E-18	3.5744E-18	3.3421E-18	3.1250E-18	2.9220E-18	2.7322E-18
9.1	2.5547E-18	2.3888E-18	2.2336E-18	2.0885E-18	1.9529E-18	1.8260E-18	1.7074E-18	1.5966E-18	1.4929E-18	1.3959E-18
9.2	1.3053E-18	1.2206E-18	1.1413E-18	1.0672E-18	9.9795E-19	9.3317E-19	8.7260E-19	8.1597E-19	7.6301E-19	7.1350E-19
9.3	6.6720E-19	6.2391E-19	5.8343E-19	5.4559E-19	5.1020E-19	4.7710E-19	4.4616E-19	4.1723E-19	3.9017E-19	3.6487E-19
9.4	3.4122E-19	3.1910E-19	2.9841E-19	2.7907E-19	2.6099E-19	2.4407E-19	2.2826E-19	2.1347E-19	1.9964E-19	1.8671E-19
9.5	1.7462E-19	1.6331E-19	1.5274E-19	1.4285E-19	1.3360E-19	1.2495E-19	1.1687E-19	1.0930E-19	1.0223E-19	9.5617E-20
9.6	8.9432E-20	8.3648E-20	7.8238E-20	7.3179E-20	6.8448E-20	6.4023E-20	5.9885E-20	5.6015E-20	5.2395E-20	4.9010E-20
9.7	4.5844E-20	4.2883E-20	4.0114E-20	3.7524E-20	3.5101E-20	3.2836E-20	3.0716E-20	2.8734E-20	2.6880E-20	2.5146E-20
9.8	2.3525E-20	2.2008E-20	2.0589E-20	1.9261E-20	1.8020E-20	1.6859E-20	1.5772E-20	1.4756E-20	1.3806E-20	1.2917E-20
9.9	1.2085E-20	1.1307E-20	1.0579E-20	9.8985E-21	9.2616E-21	8.6658E-21	8.1084E-21	7.5870E-21	7.0992E-21	6.6429E-21

This table was generated using Microsoft ® Excel (Reference 4) and the *Z* equation from Reference 2.