

CHAPTER 8

ELASTIC THEORY OF HYDRAULIC TRANSIENTS (WATER HAMMER)*

In situations where the velocity can change suddenly and the pipeline is relatively long, the elastic properties of the pipe and liquid become significant factors. In Chapter 7 we saw how a pipeline responded to the sudden closure of a valve. This valve motion caused an increase in pressure head ΔH to occur, which propagated at a speed a . In this chapter governing relations for ΔH and a will be developed, thereby broadening the range of applications from that of the simple example in Chapter 7.

8.1 THE EQUATION FOR PRESSURE HEAD CHANGE ΔH

The linear momentum equation will be used to develop an equation for ΔH . We know that a change in velocity ΔV will cause a pressure head change ΔH to propagate at some speed a . In Fig. 8.1a ΔV is actually negative, leading to an increase in head ΔH . We begin with a section of pipe of incremental length δL , where δL is arbitrarily small but not differentially small as would be dL . The pressure wave and the pipe bulge (which is caused by the pressure head change ΔH) propagate at speed a . The wave speed is the speed relative to an observer at rest with respect to the pipe rather than the speed relative to the water velocity. For relatively rigid pipes either choice for a reference would give essentially the same results. Because this is an unsteady flow situation, the linear momentum

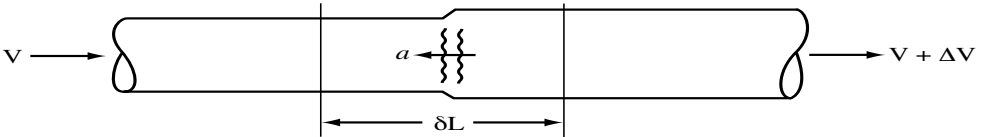


Figure 8.1a The unsteady flow control volume for a momentum analysis.

equation for steady flow does not apply. However, here it is possible to use a translating coordinate system so that the unsteady flow appears to be steady, as shown in Fig. 8.1b:

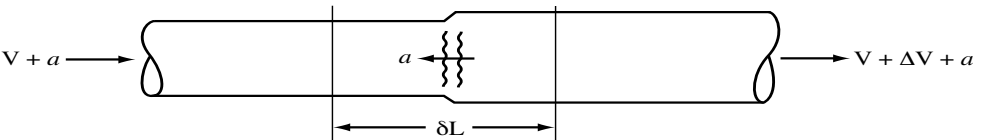


Figure 8.1b The steady flow control volume for a momentum analysis.

* Pages 305-315 are adapted from *Elementary Fluid Mechanics*, by R. L. Street, G. Z. Watters, and J. K. Vennard, Ed. 7, Copyright 1996 by John Wiley & Sons, Inc. Reprinted by permission.

If we move our reference system to the left at a speed a , we have for all appearances a steady flow. From basic fluid mechanics we may then apply the steady one-dimensional linear momentum equation

$$\Sigma \mathbf{F}_{ext} = (\Sigma Q\rho\mathbf{V})_{out} - (\Sigma Q\rho\mathbf{V})_{in} \quad (8.1)$$

where Q is the discharge, ρ is the fluid density, and $\Sigma \mathbf{F}_{ext}$ is the sum of the external forces acting. The momentum correction factor for nonuniform velocity profiles is assumed to be 1.00 in this case.

Considering only the component of this vector equation parallel to the pipe and noting that the momentum flux enters and leaves the pipe section of length δL at only one cross section each, we can write

$$(\Sigma F_{ext})_x = Q\rho(V_{out} - V_{in}) \quad (8.2)$$

To apply the momentum equation, we must specify a control volume and account for all forces acting on the fluid in the control volume at a particular instant and at that same instant evaluate the momentum fluxes into and out of the control volume. We choose the control volume to coincide with the inside of the pipe walls over the length δL and include the flow cross section at each end of this pipe section. This control volume, the fluid in it, and the external forces acting are shown in Fig. 8.2:

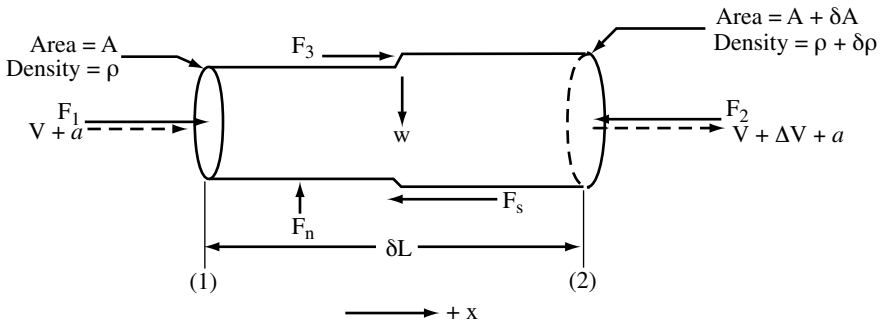


Figure 8.2 The steady-flow control volume with all forces shown.

The side shear force F_s caused by friction will be neglected because its size is proportional to δL . Also, because we consider only relatively strong pipe materials (steel, concrete, etc.), the pipe bulge is very small and so F_3 is also negligible.

Application of Eq. 8.2 gives

$$F_1 - F_2 = Q\rho[(V + \Delta V + a) - (V + a)] = Q\rho[\Delta V] \quad (8.3)$$

in which $Q\rho = (V + a)A\rho$.

If the pressure at (1) were p_o , then the pressure at (2) would be $p_o + \Delta p$, and

$$p_o A - (p_o + \Delta p)(A + \delta A) = (V + a)A\rho(\Delta V) \quad (8.4)$$

Expanding this equation and recognizing that $\Delta p = \gamma\Delta H$ and δA is very small compared to ΔH , A , and γ , we can neglect the small terms with the result

$$-\Delta H\gamma A = (V + a)A\rho(\Delta V) \quad (8.5)$$

This equation can also be written as

$$\Delta H = -\frac{\rho}{\gamma} \Delta V (V + a) \quad (8.6)$$

or

$$\Delta H = -\frac{a \Delta V}{g} \left(1 + \frac{V}{a}\right) \quad (8.7)$$

In most rigid pipe situations (even PVC with a wave speed of only 1200 ft/s), the value of V/a is less than 0.01. Accordingly, Eq. 8.7 is generally (and always in this work) applied as

$$\Delta H = -\frac{a}{g} \Delta V \quad (8.8)$$

From Eq. 8.8 we see that a decrease in velocity ΔV causes an increase in head ΔH . Further, ΔH depends on the wave speed a and cannot be determined until a value of a is established.

8.2 WAVE SPEED FOR THIN-WALLED PIPES

To develop an equation for the wave speed, we will consider conservation of mass in the pipe section δL long that was used to develop Eq. 8.4. We examine the mass flow into and out of the pipe section over the time period required for the wave to pass through that portion of the pipe. The net inflow of mass will be equated to the increase in mass storage in δL to produce an equation for wave speed a . Again we assume that a decrease in velocity occurs, hence mass accumulates.

To begin we note the situation when the wave has first reached the control volume and then at the time the wave has just passed through the section at a time δt later.

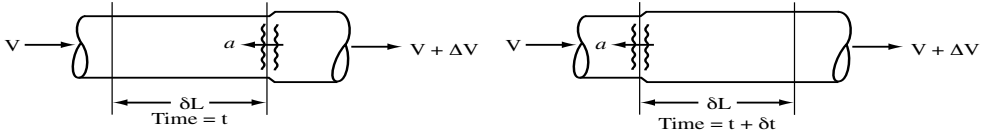


Figure 8.3 Propagation of the pressure wave at two instants.

It is clear that δL and δt are related via the wave speed as $\delta L = a \delta t$.

8.2.1. NET MASS INFLOW

During the time interval required for the wave to pass through the control volume, mass has accumulated in the section in the amount

$$\delta M = V A \rho \delta t - (V + \Delta V)(\rho + \delta \rho)(A + \delta A) \delta t \quad (8.9)$$

Expanding the parentheses and neglecting small terms gives

$$\delta M = -A \rho \Delta V \delta t \quad (8.10)$$

or, in terms of wave speed and δL ,

$$\delta M = -A\rho\Delta V \frac{\delta L}{a} \quad (8.11)$$

This extra liquid is stored in the control volume partly by being compressed slightly to a larger density and partly by occupying additional space provided by stretching the pipe cross section a small amount.

We now proceed to quantify the volume changes for the liquid and the pipe.

8.2.2. CHANGE IN LIQUID VOLUME DUE TO COMPRESSIBILITY

Because the pressure has increased during the passage of a positive pressure wave caused by a decrease in velocity, the volume of liquid in the section is compressed to a slightly higher density. The equation relating the increase in pressure and decrease in volume is the equation defining the bulk modulus of elasticity for a liquid, as can be found in any elementary fluid mechanics text:

$$K = - \frac{dp}{d\mathcal{V}/\mathcal{V}} \quad (8.12)$$

Here K is the bulk modulus of elasticity of the liquid and p and \mathcal{V} are the pressure and volume of the liquid, respectively. Since K is relatively constant over a wide pressure range (assuming no entrained gases in the liquid), we can let $dp = \Delta p$ and write Eq. 8.6 as

$$\delta\mathcal{V} = - \Delta p \frac{\mathcal{V}}{K} \quad (8.13)$$

where $\delta\mathcal{V}$ is the change in liquid volume in the control volume resulting from the pressure change Δp .

8.2.3. CHANGE IN PIPE VOLUME DUE TO ELASTICITY

When the increased pressure stretches the pipe, more space is available to store the accumulated net inflow of liquid. The pipe may stretch both circumferentially and longitudinally, so we must consider both contributions to the change in pipe volume.

Developments that are basic to the mechanics of solid materials show the relation between the pipe wall strains in the two perpendicular directions. If a material is strained in one direction by an amount ε_1 , then a strain ε_2 will occur in the perpendicular direction (provided the material is free to strain without developing a stress in that direction) according to $\varepsilon_2 = \mu\varepsilon_1$, where μ is Poisson's ratio. If there is a restriction to free strain in either direction caused either by restraint or applied stress, the relation is more complicated. A text on the mechanics of materials will provide the following equations for two-dimensional stress which can be applied to thin-walled pipes:

$$\sigma_1 = \frac{\varepsilon_1 + \mu\varepsilon_2}{1 - \mu^2} E \quad \text{or} \quad \varepsilon_1 = \frac{\sigma_1 - \mu\sigma_2}{E} \quad (8.14a)$$

$$\sigma_2 = \frac{\varepsilon_2 + \mu\varepsilon_1}{1 - \mu^2} E \quad \text{or} \quad \varepsilon_2 = \frac{\sigma_2 - \mu\sigma_1}{E} \quad (8.14b)$$

Here σ_1 and ε_1 are the stress and strain, respectively, in the direction along the pipe axis, σ_2 and ε_2 are the values in the circumferential direction, and E is the modulus of elasticity of the pipe wall material. Of course, if the wall material is not homogeneous and isotropic, then a more complex analysis is required.

For water hammer pressure waves there is usually a stress and strain already resident in the pipe caused by the steady state flow. Hence we write the preceding equations in incremental form

$$\Delta\sigma_1 = \frac{\Delta\varepsilon_1 + \mu\Delta\varepsilon_2}{1 - \mu^2} E \quad \text{or} \quad \Delta\varepsilon_1 = \frac{\Delta\sigma_1 - \mu\Delta\sigma_2}{E} \quad (8.15a)$$

$$\Delta\sigma_2 = \frac{\Delta\varepsilon_2 + \mu\Delta\varepsilon_1}{1 - \mu^2} E \quad \text{or} \quad \Delta\varepsilon_2 = \frac{\Delta\sigma_2 - \mu\Delta\sigma_1}{E} \quad (8.15b)$$

The change in volume caused by circumferential stretching is

$$\delta V_c = \pi D \frac{\delta D}{2} \delta L \quad (8.16)$$

where $\pi\delta D = \pi D\Delta\varepsilon_2$. Combining the two equations gives

$$\delta V_c = \frac{1}{2} \pi D^2 \delta L \Delta\varepsilon_2 \quad (8.17)$$

The change in volume caused by longitudinal stretching is

$$\delta V_l = \frac{\pi}{4} D^2 \delta L \Delta\varepsilon_1 \quad (8.18)$$

Combining Eqs. 8.17 and 8.18 gives the total volume change due to pipe stretching as

$$\delta V = \frac{\pi}{4} D^2 \delta L (\Delta\varepsilon_1 + 2\Delta\varepsilon_2) \quad (8.19)$$

We now begin the process of replacing the expressions for strain with those for the stress and pressure which cause the strain. The change in circumferential stress in the pipe wall under static conditions is

$$\Delta\sigma_2 = \frac{\Delta p D}{2e} \quad (8.20)$$

where e is the pipe wall thickness. However, the transient conditions of water hammer would in general cause the pipe to respond dynamically in a manner which can only be analyzed accurately by carefully considering the mass of the pipe and fitting materials as well as pipe restraints. That is, any valves, fittings, and other attachments in addition to the weight of the pipe must be displaced by pressure changes. These displacements are in turn affected by the type and elastic behavior of the pipe restraints. This type of analysis would be entirely too complex to accomplish in general, so we assume that the static conditions adequately approximate the dynamic behavior. Experimental results over the years have generally validated this approach. Substituting the above equation into the first of Eqs. 8.15b gives

$$\frac{\Delta p D}{2e} = \frac{\Delta\varepsilon_2 + \mu\Delta\varepsilon_1}{1 - \mu^2} E \quad (8.21)$$

While the relation between circumferential stress and pressure is valid for all types of restraint, the relation between longitudinal stress and strain varies with restraint type. For

example, if the pipe were anchored at one end and free to stretch longitudinally (much like a long slender pressure vessel), the longitudinal stress would be

$$\Delta\sigma_1 = \frac{\Delta p D}{4e} \quad (8.22)$$

under static conditions. On the other hand, if the pipe were rigidly anchored to prevent any axial strain, then $\Delta\sigma_1 = \mu\Delta\sigma_2$ because $\Delta\varepsilon_1 = 0$. However, if the pipe contained functioning expansion joints throughout its length, then $\Delta\sigma_1 = 0$ and $\Delta\varepsilon_1$ is of no interest. Following the nomenclature of Wylie and Streeter (1993), we identify the above cases as the following:

- Case (a) pipe anchorage only at the upstream end;
- Case (b) full pipe restraint from axial movement.
- Case (c) longitudinal expansion joints along the pipeline.

In a practical sense the actual pipe restraint situation probably will not conform precisely to any of these cases but lies somewhere in this range of possibilities.

Here we will now choose one restraint case to develop relatively fully as an example of how the analysis of each case should proceed. Because buried pipelines are relatively common and might be expected to be fully restrained axially by soil friction and anchor blocks, we will examine Case (b) restraint next as we move ahead to compute a wave speed.

Wave Speed Solution for Case (b) Restraint

For this restraint choice $\Delta\varepsilon_1 = 0$ and Eqs. 8.15a become

$$\Delta\sigma_1 = \frac{\mu\Delta\varepsilon_2}{1-\mu^2} E \quad \text{or} \quad \Delta\sigma_1 = \mu\Delta\sigma_2 \quad (8.23)$$

and Eq. 8.21 becomes

$$\frac{\Delta p D}{2e} = \frac{\Delta\varepsilon_2}{1-\mu^2} E \quad (8.24)$$

Substituting this equation into Eq. 8.19 in place of $\Delta\varepsilon_2$ gives the total volume change as

$$\delta\mathcal{V} = \frac{\pi}{4} D^2 \delta L \left(\frac{1-\mu^2}{E} \right) \left(\frac{\Delta p D}{e} \right) \quad (8.25)$$

Now recall that Eq. 8.11, based on conservation of mass, computes the incremental mass which has accumulated in the pipe control volume in δt seconds. If we subtract the mass in the pipe before the passage of the wave from the amount existing after the passage of the wave, we obtain

$$\delta M = (\rho + \delta\rho)(A\delta L + \delta\mathcal{V}) - \rho A\delta L \quad (8.26)$$

Equating this expression for δM with that of Eq. 8.11, expanding, and dropping small terms gives

$$\delta\rho A\delta L + \rho\delta\mathcal{V} = A\rho\Delta V \frac{\delta L}{a} \quad (8.27)$$

To arrange this equation into a more usable form, note for a given mass of material that an increase in pressure causes a decrease in volume and an increase in density. That is, $\rho V = \text{constant}$ so $V\delta\rho + \rho\delta V = 0$ and

$$\delta\rho = -\frac{\delta V}{V}\rho \quad (8.28)$$

Substituting this result into Eq. 8.13 gives

$$\delta\rho = \rho\left(\frac{\Delta p}{K}\right) \quad (8.29)$$

Replacing Δp with $\gamma\Delta H$ in the preceding equation, substituting it and Eq. 8.25 into Eq. 8.27 leads to

$$\gamma\Delta H\left[\frac{1}{K} + \left(\frac{1-\mu^2}{E}\right)\frac{D}{e}\right] = \frac{\Delta V}{a} \quad (8.30)$$

Combining this equation with Eq. 8.8 results in

$$a^2\rho\left[\frac{1}{K} + \frac{D}{e}\left(\frac{1-\mu^2}{E}\right)\right] = 1 \quad (8.31)$$

or, in the conventional form for wave speed,

$$a = \frac{\sqrt{K/\rho}}{\sqrt{1 + \frac{K}{E}\frac{D}{e}(1-\mu^2)}} \quad \text{Case (b)} \quad (8.32)$$

It is now possible to compute the wave speed and pressure increase directly in simple situations where Eq. 8.8 applies.

Wylie and Streeter (1993) show that the equation for wave speed can be conveniently expressed in the general form

$$a = \frac{\sqrt{K/\rho}}{\sqrt{1 + \frac{K}{E}\frac{D}{e}(C)}} \quad (8.33)$$

where

$$\text{for Case (a) restraint} \quad C = 5/4 - \mu \quad (8.34a)$$

$$\text{for Case (b) restraint} \quad C = 1 - \mu^2 \quad (8.34b)$$

and

$$\text{for Case (c) restraint} \quad C = 1.0 \quad (8.34c)$$

Keep in mind that this set of equations for wave speed applies only to thin-walled pipes. If the pipe walls are sufficiently thick, the above equations must be modified (see Section 8.3 for a discussion of what constitutes a thin-walled pipe).

The following table of values for E and μ should be useful in calculating wave speeds in pipes made of common materials. The bulk modulus K for water is approximately 300,000 lb/in², although some references cite K 's as high as 320,000 lb/in².

Table 8.1 Moduli of Elasticity and Poisson Ratios for Common Pipe Materials

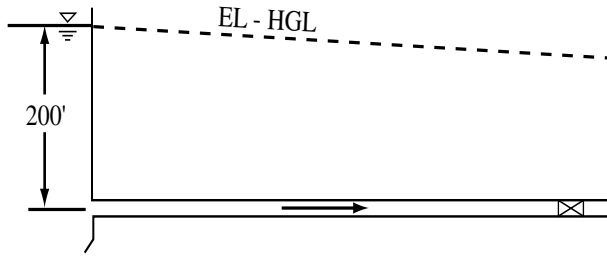
Material	E (lb/in ²)	Poisson ratio μ
Steel	30×10^6	0.30
Ductile cast iron	24×10^6	0.28
Copper	16×10^6	0.36
Brass	15×10^6	0.34
Aluminum	10.5×10^6	0.33
PVC	4×10^5	0.45
Fiberglass-reinforced plastic	$E_2 = 4 \times 10^6$	$\mu_2 = 0.27-0.30$
	$E_1 = 1.3 \times 10^6$	$\mu_1 = 0.20-0.24$
Asbestos Cement	3.4×10^6	0.30
Concrete	$57,000 \sqrt{f'_c}$ $f'_c = 28\text{-day strength}$	dynamic value 0.24

However, even a small amount of free air or gas suspended in the liquid can drastically reduce the value of K and the resultant wave speed (see Section 8.4). Unfortunately, evaluating the amount of air, its distribution, its pervasiveness, and its effect on wave speed is most difficult. Consequently, in design situations it is common practice to take a conservative approach and assume there is no air present. This *generally* leads to a prediction of higher water hammer pressures. Any presence of entrained air or gas in the system would be a fortuitous circumstance, at least in the sense of reducing the wave speed.

One should note that a pipe can become infinitely strong ($E \rightarrow \infty$) without the wave speed also becoming infinite. From Eq. 8.33 it is clear that this situation causes the denominator of the equation to go to 1.0, resulting in a wave speed $a = \sqrt{K/\rho}$ which for water is about 4720 ft/s. This number has no practical significance in design because it is far too high to serve as even an approximate wave speed value for preliminary design. With even a limited amount of experience, a designer can make a far better estimate of the wave speed in the pipe under consideration.

Example Problem 8.1

To get a "feel" for the pressure head changes and the resultant elastic deformations of pipe and liquid caused by a typical water hammer situation and to demonstrate the effects of different pipe restraints, the following problem is analyzed.



Water flows in this 24-inch steel pipeline at a velocity of 6 ft/s. The pipeline has a wall thickness of 0.25 inches. First we will calculate the wave speed for the three types of restraint by using Eqs. 8.33 and 8.34:

$$\text{Case (a)} \quad a = \frac{\sqrt{K/\rho}}{\sqrt{1 + \frac{K D}{E e} (C)}} = \frac{4720}{\sqrt{1 + \frac{3 \times 10^5}{3 \times 10^7} \frac{24}{0.25} (5/4 - 0.30)}} = 3410 \text{ ft/s}$$

$$\text{Case (b)} \quad a = \frac{4720}{\sqrt{1 + \frac{3 \times 10^5}{3 \times 10^7} \frac{24}{0.25} (1 - 0.30^2)}} = 3450 \text{ ft/s}$$

$$\text{Case (c)} \quad a = \frac{4720}{\sqrt{1 + \frac{3 \times 10^5}{3 \times 10^7} \frac{24}{0.25} (1.00)}} = 3370 \text{ ft/s}$$

As one can see clearly here, the differences are for all practical purposes insignificant for this pipe.

Now we will compute with Eq. 8.8 the pressure head changes resulting from sudden valve closure for all three cases of restraint:

$$\text{Case (a)} \quad \Delta H = -\frac{a}{g} \Delta V = -\frac{3410}{32.2} (-6) = 635 \text{ ft}$$

$$\text{Case (b)} \quad \Delta H = -\frac{3450}{32.2} (-6) = 643 \text{ ft}$$

$$\text{Case (c)} \quad \Delta H = -\frac{3370}{32.2} (-6) = 628 \text{ ft}$$

Because the head increase depends directly on the wave speed, the negligible difference in wave speed translates into a head difference of no more than 2%, which is usually a negligible difference.

We now calculate the change in pipe wall stress caused by these head increases for all three types of restraint:

$$\text{Case (a)} \quad \Delta \sigma_2 = \frac{\Delta p D}{2e} = \frac{\left(635 \times \frac{62.4}{144}\right) \times 24}{2 \times 0.25} = 13,210 \text{ lb/in}^2$$

$$\Delta\sigma_1 = \frac{\Delta p D}{4e} = \frac{1}{2} \Delta\sigma_2 = 6600 \text{ lb/in}^2$$

$$\text{Case (b)} \quad \Delta\sigma_2 = \frac{\Delta p D}{2e} = \frac{\left(643 \times \frac{62.4}{144}\right) \times 24}{2 \times 0.25} = 13,370 \text{ lb/in}^2$$

$$\Delta\sigma_1 = \mu \Delta\sigma_2 = 0.30 \times 13,370 = 4010 \text{ lb/in}^2$$

$$\text{Case (c)} \quad \Delta\sigma_2 = \frac{\Delta p D}{2e} = \frac{\left(628 \times \frac{62.4}{144}\right) \times 24}{2 \times 0.25} = 13,060 \text{ lb/in}^2$$

$$\Delta\sigma_1 = 0$$

Next we calculate the percentage increase in the pipe diameter caused by the pressure head increase for all three cases of restraint. Using Eq. 8.15b,

$$\text{Percent change in } D = 100 \frac{\delta D}{D} = 100 \times \Delta\varepsilon_2 = \frac{100}{E} (\Delta\sigma_2 - \mu \Delta\sigma_1)$$

$$\text{Case (a)} \quad \text{Percent change} = \frac{100}{30 \times 10^6} (13,210 - 0.30 \times 6600) = 0.037\%$$

$$\text{Case (b)} \quad \text{Percent change} = \frac{100}{30 \times 10^6} (13,370 - 0.30 \times 4010) = 0.041\%$$

$$\text{Case (c)} \quad \text{Percent change} = \frac{100}{30 \times 10^6} (13,060 - 0.30 \times 0) = 0.044\%$$

These results, showing the relatively small elastic deformation of the pipe, substantiate many of the previous assumptions that were used in neglecting small terms in equations.

Finally, we will look at the water entering the pipe section during the passage of the pressure wave and determine what percentage is accommodated by pipe stretching and what portion is relegated to water compression. The fraction of liquid directed to liquid compression is given by the first term in Eq. 8.31.

$$\text{Percent change in water volume} = 100 \rho a^2 \left(\frac{1}{K} \right) = 100 \frac{\rho a^2}{K}$$

The remainder is due to pipe stretching.

$$\text{Case (a)} \quad \text{Percent change in water volume} = 100 \frac{1.94 \times 3410^2}{300,000 \times 144} = 52\%$$

$$\text{Percent accommodated by pipe stretching} = 48\%$$

$$\text{Case (b)} \quad \text{Percent change in water volume} = 100 \frac{1.94 \times 3450^2}{300,000 \times 144} = 53\%$$

$$\text{Percent accommodated by pipe stretching} = 47\%$$

Wave Speed, ft/s

Restraint	Thin-walled	Thick-walled	Error %
Case (a)	4020	4000	0.5
Case (b)	4050	4020	0.7
Case (c)	4000	3970	0.7

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8.3.2. CIRCULAR TUNNELS

The wave speed equations for circular tunnels can be derived from the thick-walled equations by letting the wall thickness go to infinity. When C for case (a) is inserted into the wave speed equation 8.33 and the D/e ratio is allowed to go to zero, the result is

$$a = \frac{\sqrt{K / \rho}}{\sqrt{1 + \frac{2K}{E}(1 + \mu)}} \quad (8.36)$$

For tunnels which are concrete-lined or steel-lined with concrete backing, the analysis is a good deal more complex. Refer to Halliwell (1963) for the rather lengthy equations required for computing wave speed in this situation.

8.3.3. REINFORCED CONCRETE PIPE

For reinforced concrete pipe the transformed-section method can be used to convert the pipe into an equivalent homogeneous pipe. Then the computation of the wave speed can be accomplished by using the homogeneous pipe equations. To be assured of an accurate wave speed, it is necessary to know exactly how the pipe was fabricated. Only that concrete which can sustain load under pressure should be used in the computation. This type of concrete pipe is generally prestressed to assure this capability.

The transformed-section method replaces the concrete cross-sectional area with an equivalent cross-sectional area of steel using the formula

$$A_{st} = \frac{E_{con}}{E_{st}} A_{con} \quad (8.37)$$

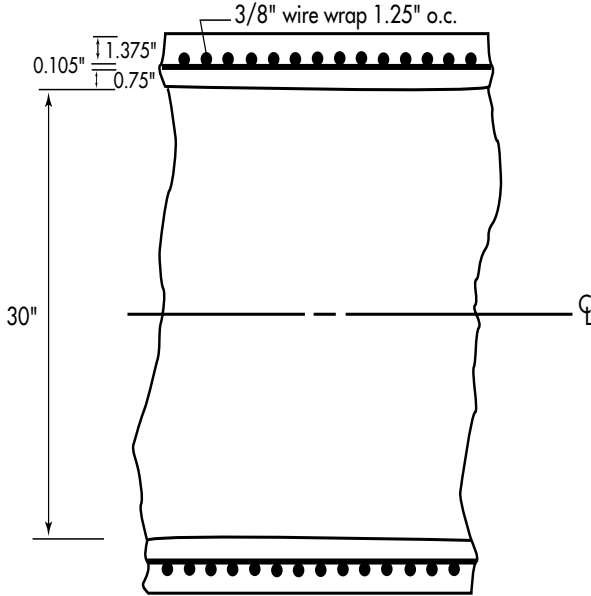
neglecting any variation in stress over the thickness of the concrete. We always convert the concrete to steel; doing the reverse would create an exceedingly thick-walled pipe.

In working with reinforced concrete pipe that is not prestressed, the concrete is assumed to carry no load. The reinforcing steel is regarded as a thin-walled steel pipe having the same cross-sectional area as the circumferential reinforcement has, and the wave speed is computed with the thin-walled equations.

If the reinforced concrete pipe is pretensioned or post-tensioned, the area of concrete placed in compression by the pre- or post-tensioning process must be included in the transformed section. This prestressing makes the pipe much stronger, but it also results in higher wave speeds which generally lead to higher water hammer pressures. The following illustrative problem demonstrates the transformed-section technique.

Example Problem 8.3

A 30-in. inside diameter reinforced concrete pipe is pretensioned using 3/8-in diameter wrapping wire placed 1.25 in on center. The pipe was manufactured by first fabricating a thin steel cylinder 0.105 in thick and then centrifugally placing a dense cement mortar lining 0.75 in thick inside the steel cylinder. After curing the liner, the pretensioning is accomplished by stressing the wire as it is wrapped around the steel cylinder. This process places the cement liner in compression. The ends of the wrapping wire are welded to the steel cylinder to maintain the pretensioning. A one-inch-thick concrete cover is placed over the wrapping wire as a protective coating. This cover carries no load. A longitudinal pipe section is shown below. Assume the 28-day strength of the concrete is 6000 lb/in².



Using the formula in Table 8.1 for concrete, E is

$$E_{con} = 57,000\sqrt{f'_c} = 57,000\sqrt{6000} = 4.4 \times 10^6 \text{ lb/in}^2$$

The area of steel wire per inch of pipe is $\frac{\pi}{4} \times 0.375^2 / 1.25 = 0.0884 \text{ in}^2/\text{in}$.

The equivalent area of steel that is required to replace the cement lining which was prestressed during the wrapping process is

$$A_{st} = \frac{E_{con}}{E_{st}} A_{con} = \frac{4.4 \times 10^6}{30 \times 10^6} \times 0.75 = 0.110 \text{ in}^2/\text{in}$$

Now the thickness of the equivalent steel pipe is

$$e_{eq} = 0.0884 + 0.110 + 0.105 = 0.303 \text{ in}$$

We compute the diameter of the equivalent pipe by locating the centroid of the section as follows:

This same equation and a detailed description of the difficulties encountered in the solution of water hammer problems which have entrained air is given by Tullis et al. (1976).

It is clear from Eq. 8.40 that the wave speed in the pipeline depends on the pressure in the pipeline because the values of α and K_{air} depend on pressure. As a consequence, the wave speed varies with the passage of a pressure wave. This factor greatly complicates an analysis and makes the accurate prediction of water hammer pressures most difficult.

An example is presented below to demonstrate the dramatic effect that small fractions of entrained air can have on the wave speed. The first step which must be taken is to establish a method of determining K_{air} . As elementary fluid mechanics texts show, K_{air} depends on the thermodynamic process followed by the air as it compresses or expands. Wylie and Streeter (1993) suggest using an isothermal process with $K_{air} = p$. The other extreme is to use an isentropic process where $K_{air} = kp = 1.4p$. If some provision for heat transfer is made, then a polytropic process with $K_{air} = np = 1.2p$ (as used later with surge tanks) may be appropriate. The effects of these various alternatives are shown below.

Example Problem 8.4

Consider the pipeline of Example Problem 8.1 and consider air entrainment percentages of 0.10, 0.50, 1.0, and 2.0%. We assume the atmospheric pressure to be 14.7 lb/in² and use Case (b) restraint. For a polytropic process and an entrained air percentage of 0.10%, the wave speed from Eq. 8.40 is

$$a = \frac{\sqrt{\frac{300,000 \times 144}{1.94(1 - 0.001)}}}{\sqrt{1 + \frac{3 \times 10^5}{3 \times 10^7} \frac{24}{0.25} (1 - 0.30^2) + 0.001 \times \frac{3 \times 10^5}{1.2 \times \left(\frac{200 \times 62.4}{144} + 14.7 \right)}}} = 2270 \text{ ft/s}$$

The following table summarizes the wave speeds for the three separate thermodynamic processes:

% Air Entrainment	Isothermal Process	Polytropic Process	Isentropic Process
0.1	2150 ft/s	2270 ft/s	2360 ft/s
0.5	1160 ft/s	1260 ft/s	1340 ft/s
1.0	845 ft/s	920 ft/s	988 ft/s
2.0	610 ft/s	666 ft/s	777 ft/s

Regardless of the process, the differences in results among the assumed processes are not great in view of the other uncertainties.

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8.5 DIFFERENTIAL EQUATIONS OF UNSTEADY FLOW

Up to this point we have seen for a given impulsive change in velocity ΔV at a pipeline section that we can compute the pressure head change ΔH which will result. This ability will now be extended so the velocity and pressure head at any pipe section at any time can be determined as the result of boundary and initial conditions imposed at any section of the system. To accomplish this, we will use Euler's equation from Chapter 7 and develop another equation based on conservation of mass.

8.5.1. CONSERVATION OF MASS

We apply conservation of mass to a control volume that coincides with the interior of the pipe and is of length ds :

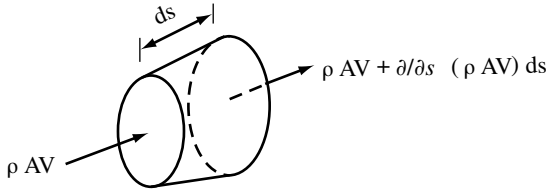


Figure 8.4 Control volume coinciding with the interior surface of the pipe.

The result of this application is

$$\rho AV - \left[\rho AV + \frac{\partial}{\partial s}(\rho AV) ds \right] = \frac{\partial}{\partial t}(\rho A ds) \quad (8.41)$$

or

$$- \frac{\partial}{\partial s}(\rho AV) ds = \frac{\partial}{\partial t}(\rho A ds) \quad (8.42)$$

At this point we employ a rather unconventional form of the control volume concept in that we require the sides of the control volume to be attached to the pipe wall. Thus the control volume will elongate as the pipe stretches longitudinally. The only exception is in Case (c) where we keep the control volume at a constant length even though the pipe elongates (the total length of the pipeline remains constant even as the pipe slips in its expansion joints). This technique is used because the pipe stretching affects the volume of storage, and the relation between pipe elasticity and the available volume for the liquid is identical to that in Section 8.2.

Expanding the parentheses of Eq. 8.42 yields

$$- \left(\rho A \frac{\partial V}{\partial s} ds + \rho V \frac{\partial A}{\partial s} ds + AV \frac{\partial \rho}{\partial s} ds \right) = \rho A \frac{\partial}{\partial t}(ds) + \rho ds \frac{\partial A}{\partial t} + A ds \frac{\partial \rho}{\partial t} \quad (8.43)$$

Regrouping and dividing by the control volume mass $\rho A ds$,

$$\frac{1}{\rho} \left(\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial s} \right) + \frac{1}{A} \left(\frac{\partial A}{\partial t} + V \frac{\partial A}{\partial s} \right) + \frac{1}{ds} \frac{\partial}{\partial t}(ds) + \frac{\partial V}{\partial s} = 0 \quad (8.44)$$

Recognizing that $\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial s} = \frac{d\rho}{dt}$ and $\frac{\partial A}{\partial t} + V \frac{\partial A}{\partial s} = \frac{dA}{dt}$, Eq. 8.44 becomes

$$\frac{1}{\rho} \frac{d\rho}{dt} + \frac{1}{A} \frac{dA}{dt} + \frac{\partial V}{\partial s} + \frac{1}{ds} \frac{d}{dt}(ds) = 0 \quad (8.45)$$

From Section 8.2, $K = - \frac{dp}{dV/V} = \frac{dp}{d\rho/\rho}$ so that

$$\frac{1}{\rho} \frac{d\rho}{dt} = \frac{1}{K} \frac{dp}{dt} \quad (8.46)$$

To develop a useful expression for dA/dt in terms of p , the elastic pipe deformations must be considered. For the change in cross-sectional area, Eq. 8.17 shows that

$$dA = \frac{dV_c}{dL} = \frac{1}{2} \pi D^2 d\epsilon_2 = \frac{1}{2} \pi \frac{D^2}{E} (d\sigma_2 - \mu d\sigma_1) \quad (8.47)$$

$$\frac{1}{A} dA = \frac{2}{E} (d\sigma_2 - \mu d\sigma_1) \quad (8.48)$$

In evaluating these stresses we will again examine Case (b) restraint; hence

$$d\sigma_2 = \frac{D}{2e} dp \quad \text{and} \quad d\sigma_1 = \mu d\sigma_2 \quad (8.49)$$

so

$$d\sigma_2 - \mu d\sigma_1 = (1 - \mu^2) d\sigma_2 = (1 - \mu^2) \frac{D}{2e} \frac{dp}{dt} \quad (8.50)$$

Finally,

$$\frac{1}{A} \frac{dA}{dt} = (1 - \mu^2) \frac{D}{eE} \frac{dp}{dt} \quad (8.51)$$

Considering longitudinal expansion,

$$d(ds) = d\epsilon_1 ds \quad (8.52)$$

which is zero for Case (b). Thus

$$\frac{1}{ds} \frac{d}{dt} (ds) = 0 \quad (8.53)$$

Combining all these results in Eq. 8.45 gives

$$\frac{1}{K} \frac{dp}{dt} + (1 - \mu^2) \frac{D}{eE} \frac{dp}{dt} + \frac{\partial V}{\partial s} = 0 \quad (8.54)$$

$$\frac{dp}{dt} \left[\frac{1}{K} + (1 - \mu^2) \frac{D}{eE} \right] + \frac{\partial V}{\partial s} = 0 \quad (8.55)$$

From Eq. 8.31 it is clear that the term in the brackets is $\frac{1}{a^2 \rho}$. This statement is also correct for Case (a) and Case (c) pipe restraint. Making this substitution for the terms in brackets leads to

$$\frac{1}{\rho} \frac{dp}{dt} + a^2 \frac{\partial V}{\partial s} = 0 \quad (8.56)$$

When we combine this result with the Euler equation of motion, Eq. 7.19, we have two independent partial differential equations for $p(s,t)$ and $V(s,t)$:

$$\frac{dV}{dt} + \frac{1}{\rho} \frac{\partial p}{\partial s} + g \frac{dz}{ds} + \frac{f}{2D} V|V| = 0 \quad (8.57)$$

$$a^2 \frac{\partial V}{\partial s} + \frac{1}{\rho} \frac{dp}{dt} = 0 \quad (8.58)$$

8.5.2. INTERPRETATION OF THE DIFFERENTIAL EQUATIONS

Before moving to the solution of these equations in Chapter 9, we can learn about the nature of these solutions by looking at a linearized subset of the full equations. If we first express the pressure p in terms of the piezometric head H via the relation $p = \rho g(H - z)$, then Eqs. 8.57 and 8.58 become

$$\frac{dV}{dt} + g \frac{\partial H}{\partial s} + \frac{f}{2D} V|V| = 0 \quad (8.59)$$

and

$$a^2 \frac{\partial V}{\partial s} + g \frac{dH}{dt} = 0 \quad (8.60)$$

in which the variation of the density ρ is presently assumed to be negligible. The derivative d/dt actually represents both the temporal and convective partial derivative terms. For example,

$$\frac{dV}{dt} = \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial s} \quad (8.61)$$

and similarly for dH/dt . Thus Eqs. 8.59 and 8.60 actually contain two nonlinear convective terms in addition to the nonlinear friction term.

Let us assume for the moment that the linear terms in Eqs. 8.59 and 8.60 are larger than the nonlinear terms and discard the nonlinear terms; we can evaluate later the consequences of this simplification. Then these equations become

$$\frac{\partial V}{\partial t} + g \frac{\partial H}{\partial s} = 0 \quad (8.62)$$

and

$$a^2 \frac{\partial V}{\partial s} + g \frac{\partial H}{\partial t} = 0 \quad (8.63)$$

Since the equations are now linear, cross-differentiation and some algebra will allow us to eliminate either one of the two dependent variables V and H in favor of the other one. Thus, if we take the partial derivative of Eq. 8.62 with respect to s and the partial derivative of Eq. 8.63 with respect to t , the algebra leads to

$$\frac{\partial^2 H}{\partial t^2} = a^2 \frac{\partial^2 H}{\partial s^2} \quad (8.64)$$

By interchanging the differentiation roles of s and t , we can demonstrate that V is also governed by this equation.

Equation 8.64 is a basic equation of mathematical physics called the wave equation. The parameter a in Eq. 8.64 is known as the wave propagation speed. Hence we expect the solutions of Eq. 8.64 for either H or V to display the behavior of waves. By means of a change in the independent variables, we can deduce the general solution of Eq. 8.64 for H ; the procedure is identical for V . We begin with $H = H(s, t)$ and $V = V(s, t)$, and we choose as new independent variables $u = t + s/a$ and $v = t - s/a$. Application of the chain rule of differentiation to compute the new partial derivatives of H then leads to

$$4 \frac{\partial^2 H}{\partial u \partial v} = 0 \quad (8.65)$$

as the new form of the governing equation. The general solution immediately follows as

$$H - H_0 = F_1\left(t + \frac{s}{a}\right) + F_2\left(t - \frac{s}{a}\right) \quad (8.66)$$

in which F_1 and F_2 are each an entirely general function of their one argument, and H_0 is an additive constant of integration which fixes the reference level for the head H .

We turn now to the interpretation of the result expressed in Eq. 8.66. Functions F_1 and F_2 are each wave forms; either or both may exist in a particular problem, depending on the particular initial and boundary conditions. We focus first on F_1 : at any instant t_1 the wave form H described by the function F_1 can be any function of the distance variable s ,

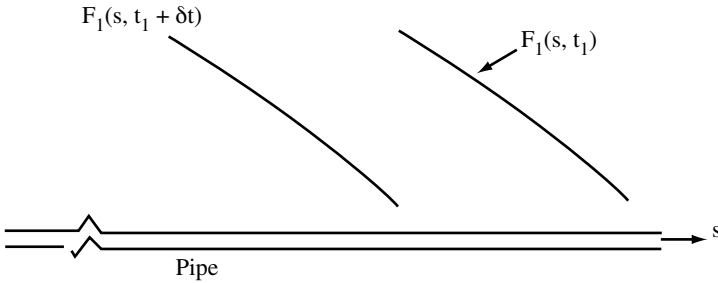


Figure 8.5 Motion of the wave form F_1 .

as we see on the right side of Fig. 8.5. So long as the argument $t + s/a$ is unchanged, the

wave form is unchanged. But as the time t advances, the argument $t + s/a$ can only remain constant if s/a decreases by δt as t increases by δt ; thus the constant wave form moves in the negative s -direction (to the left), as Fig. 8.5 also shows. We conclude that F_1 describes a left-moving wave. By similar reasoning we find that F_2 describes a right-moving wave form. The general solution to Eq. 8.66 is then a superposition of any number, one or many, of left- and/or right-moving waves.

Is the neglect of the nonlinear convective acceleration terms justified? These are the terms

$$V \frac{\partial V}{\partial s} \quad \text{and} \quad V \frac{\partial H}{\partial s} \quad (8.67)$$

If we apply the scaling $s \sim at$ to the terms in Eq. 8.67, we quickly find

$$\begin{aligned} V \frac{\partial V}{\partial s} &\sim \frac{V}{a} \frac{\partial V}{\partial t} \\ V \frac{\partial H}{\partial s} &\sim \frac{V}{a} \frac{\partial H}{\partial t} \end{aligned} \quad (8.68)$$

Since it is almost always true that $V/a \ll 1$, the convective terms, initially dropped, are nearly always much smaller than the linear terms that were retained. In later solutions the neglect of these terms will often be an acceptable approximation, although in some ways the quality of the resulting solution is less precise. Only in rare problems where V/a is not much smaller than unity is it essential to retain these terms.

8.6 PROBLEMS*

8.1 A high-pressure water system is being designed for use in a lumber mill to remove bark from logs. The main portion of the pipe system is 6-inch steel pipe with walls 0.219 inches thick. The inside diameter of the pipe is 6.187 inches, and the working water pressure in the pipe is 1094 lb/in^2 . The allowable stress in the steel pipe walls is $15,000 \text{ lb/in}^2$. Under steady flow conditions the water pressure in the pipe is about 750 lb/in^2 with a flow velocity of 10 ft/s . Because of the nature of the process, there is a need for a very rapid valve closure.

- Compute the wave speed for all three types of restraint using the thin-walled pipe formulas.
- Make a recommendation as to whether the water hammer pressures developed under sudden valve closure could overstress the pipe.

8.2 A 12-inch PVC line is laid above ground using bell-and-spigot joints. It is restrained laterally to prevent buckling but is free to strain longitudinally. However, concrete anchors at each bend prevent the pipe from blowing apart. The inside diameter of the pipe is 12.09 inches and the wall thickness is 0.311 inches.

- Choose the proper restraint case, and compute the wave speed using the thin-walled pipe formula.
- Also calculate the percent increase in pipe volume that is caused when a water velocity of 10 ft/s is brought suddenly to rest.
- What change in diameter does this represent?

8.3 A long water line is to be constructed of T-30 Transite pipe (asbestos cement). The pipe is rated for a maximum pressure of 300 lb/in^2 . The inside diameter of the pipe is 18 inches, and the outside diameter is 19.70 inches. Lengths of pipe are joined with couplings and ring gaskets.

- Compute the wave speed for all three types of restraint, assuming the thin-walled pipe formulas apply.
- Which wave speed would you recommend using? Why?

8.4 When water hammer occurs in the pipe of Problem 8.3, the water is compressed and the pipe is stretched. What percentage of the volume change can be attributed to water compression and what percentage to pipe expansion? Assume Case (b) restraint applies.

8.5 Calculate the wave speed in the following situations for Case (b) restraint using the thin-walled pipe formulas:

- Steel pipe, 36-inch inside diameter, 0.375-inch wall thickness
- Ductile cast iron pipe, 18-inch inside diameter, 0.50-inch wall thickness
- Aluminum pipe, 4-inch inside diameter, 0.10-inch wall thickness
- Asbestos cement, 11.56-inch inside diameter, 1.26-inch wall thickness
- Class 125 PVC, 6.22-inch inside diameter, 0.20-inch wall thickness

Note the variation in a for this wide range of pipe sizes and materials.

8.6 For the PVC pipe of Problem 8.5, compute the percent change in cross-sectional area caused by a sudden velocity change of 10 ft/s .

* Problems 8.1, 8.5, and 8.7-8.11 are adapted from Elementary Fluid Mechanics, by R. L. Street, G. Z. Watters, and J. K. Vennard, Ed. 7, Copyright 1996 by John Wiley & Sons, Inc. Reprinted by permission.

8.7 The wall of a steel pipe 8000 ft long, with a 6 ft inside diameter and a wall thickness of 0.50 inches, is stressed at 7000 lb/in². Water is flowing at 5 ft/s.

- (a) For Case (a) restraint and sudden valve closure, what amount of water enters the pipe after valve closure?
- (b) How is this volume distributed among radial stretching, longitudinal stretching, and water compression?

8.8 Calculate the wave speed in a water-filled copper tube that is installed without longitudinal restraint. The tube has a 0.375 inch inside diameter, is 75 ft long and has a wall thickness of 0.03 inches. The steady state pressure in the tube is 73 lb/in².

8.9 A Class 51 ductile iron pipe conveys water between two reservoirs. The outside diameter is 15.30 inches and, for this class of pipe, the wall thickness is 0.36 inches. Assuming the pipe can be considered to be thin-walled, compute the wave speed for all three types of restraint.

8.10 For the pipe of Problem 8.9, compute the percent change in pipe volume that occurs as the result of a sudden stoppage of a 10 ft/s flow of water. Use Case (b) restraint.

What is the percent change in the density of the water?

8.11 A plastic supply pipe in a building water system is anchored at both ends and has expansion joints along its entire length. The line is 800 ft long, 6.00 inches inside diameter with a 0.200 inch wall thickness. The modulus of elasticity for this material is 500,000 lb/in². Water in the pipe normally flows at 10 ft/s, and the system valves are designed to close very quickly.

- (a) If $\mu = 0.5$ for this material, what is the wave speed?
- (b) If the steady state pressure in the pipe is about 100 lb/in², estimate the maximum pressure that could occur in the system under the worst water hammer conditions?
- (c) What would be the stresses in the pipe walls under these conditions?

8.12 An aluminum irrigation pipeline laid on level ground consists of a series of 20-ft pieces connected rigidly together. The two ends of the pipeline are anchored solidly to the ground. If the pipe is 8 inches in inside diameter with 0.10-inch walls, calculate the wave speed in the pipe.

8.13 Class 100 PVC pipe has a nominal diameter of 8 in but has an actual inside diameter of 8.205 in and a wall thickness of 0.210 in.

- (a) Calculate the wave speed in this pipe for all three types of restraint using the thin-walled pipe formulas.
- (b) Also calculate the percent change in cross-sectional area of this pipe for Case (b) restraint when a flow of velocity of 8 ft/s is suddenly brought to rest.

8.14 For the pipe of Problem 8.13, calculate the wave speed for Case (b) restraint using the thick-walled pipe formulas.

8.15 In Problem 8.1 the wave speeds for the three restraint conditions were computed, assuming thin-walled pipe, as 4190 ft/s, 4210 ft/s, and 4170 ft/s for Cases (a), (b), and (c), respectively.

- (a) Use the thick-walled pipe formulas to recompute the wave speeds, and compute the percent error caused by using the thin-walled pipe formulas.
- (b) For sudden stoppage of a 10 ft/s flow, what is the error in pressure head increase when the thin-walled pipe formulas are used?
- (c) Is the more accurate figure conservative?

8.16 In Problem 8.3 the wave speeds were computed with the thin-walled pipe formulas for T-30 Transite pipe as 2830 ft/s, 2870 ft/s, and 2790 ft/s for Cases (a), (b), and (c), respectively. Recompute the three wave speeds using the thick-walled pipe formulas and find the percent change. Will the use of more accurate wave speeds result in more conservative pressure head changes?

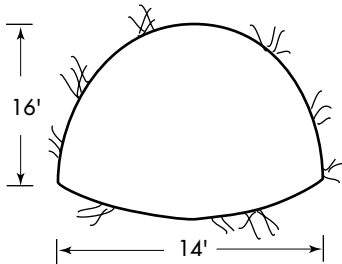
8.17 A Class 200 PVC pipe has an inside diameter of 3.146 inches and an outside diameter of 3.500 inches. The pipe sections are to be connected by the bell-and-spigot method using ring gaskets. When placed in a trench, the pipeline will be anchored at the ends and at all bends by concrete anchor blocks. Compute the wave speed for the pipeline.

8.18 An unlined power tunnel is to be excavated through limestone between a diversion dam and hydroelectric power plant penstocks. If the tunnel is approximately circular in cross section and 14.5 ft in diameter, what wave speed is appropriate for use in water hammer calculations?

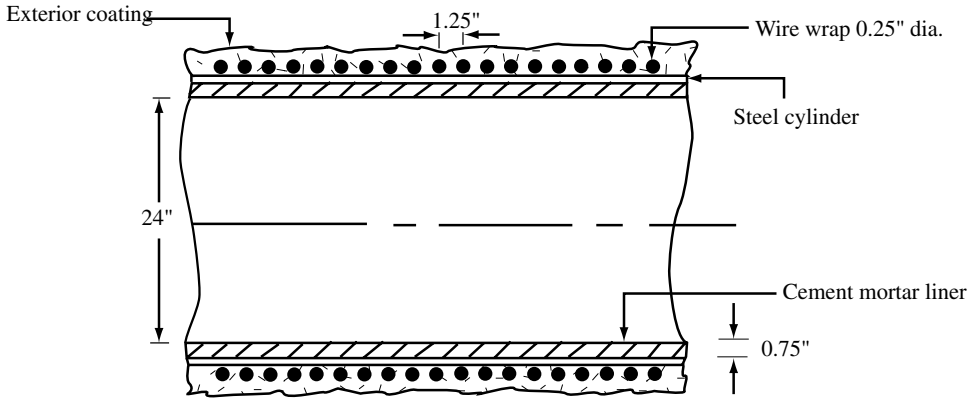
8.19 A 12-inch-diameter hydraulic conduit is drilled through a massive part of a concrete gravity dam. Estimate the wave speed in this situation for concrete with a 28-day strength of 4000 lb/in².

8.20 A circular tunnel which is 18 ft in diameter is being cut through Sierra granite as part of a hydroelectric power development. For purposes of a water hammer study, compute the wave speed that would apply in this circumstance.

8.21 The power tunnel for a hydroelectric power plant is unlined and cut through quartzite. The tunnel has the dimensions shown below. Estimate the wave speed to be used for water hammer analysis.

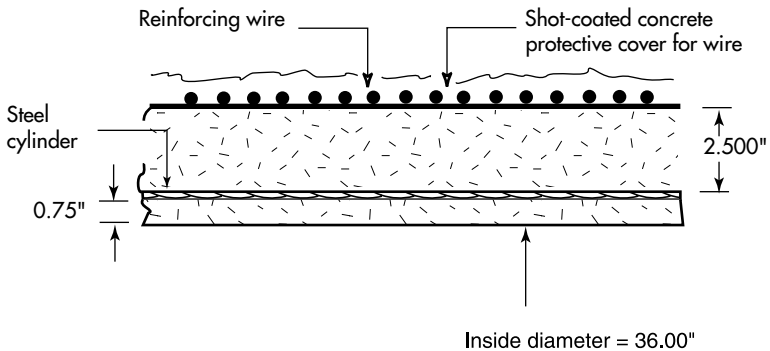


8.22 Pretensioned concrete cylinder pipe is used in a long water-supply transmission line. The pipe shown below has 24-in inside diameter, and the steel cylinder is 14 gage (0.0747 in). The steel wrapping wire is 0.25 in in diameter and spaced on 1.25-in centers. The cement-mortar liner is 0.75 in thick and has a 28-day strength of 4500 lb/in². The exterior coating is 0.75 in thick over the wire and is for the protection of the wire only. Calculate the wave speed in the pipe.



8.23 A 36-in inside diameter reinforced concrete pipe has steel reinforcing rod wrapped around the pipe midway within a 4-in wall. The steel rod has an average area of 0.80 in^2 per lineal foot of pipe, and each wrap is spaced 3 in on center. Assuming the gasket joints act as expansion joints, estimate the wave speed in the pipe, both including and excluding the concrete. The 28-day strength of the concrete is 6000 lb/in^2 .

8.24 A water project under design will be using embedded-cylinder, prestressed concrete pipe. The cylinder is to be fabricated from 10 gage (0.1345 in) steel, and the 7/16-in wire will be wrapped on 1.00-in centers. A sketch of a longitudinal section of the pipe is shown below. All of the concrete in the pipe will have a 28-day strength of 5000 lb/in^2 . Recommend a wave speed to be used in a water hammer analysis.



8.25 A pretensioned 48-inch inside diameter reinforced concrete pipe similar in configuration to that of Problem 8.22 is manufactured using a 10 gage (0.1345 inch) steel cylinder wrapped with 0.244-inch diameter steel wire on 1.00-inch centers. The total wall thickness of the pipe is 5.0 inches with a 3.00-inch cement-mortar lining. The cement liner has a 28-day strength of 6000 lb/in^2 . Assuming the gasket joints act as expansion joints, calculate the wave speed for the pipe.