Calculus, once again

David A. SANTOS
dsantos@ccp.edu

September 2, 2006 Version

Contents

Preface

For many years I have been lucky enough to have students ask for more: more challenging problems, more illuminating proofs to different theorems, a deeper look at various topics, etc. To those students I normally recommend the books in the bibliography. Some of the same students have complained of not finding the books or wanting to buy them, but being impecunious, not being able to afford to buy them. Hence I have decided to make this compilation.

Here we take a semi-rigorous tour through Calculus. We don't construct the real numbers, but we examine closer the real number axioms and some of the basic theorems of Calculus. We also consider some Olympiad-level problems whose solution can be obtained through Calculus.

The reader is assumed to be familiar with proofs using mathematical induction, proofs by contradiction, and the mechanics of differentiation and integration.

David A. SANTOS
dsantos@ccp.edu

Chapter 1

Preliminaries

Why bother? We will use the language of set theory throughout these notes. There are various elementary results that pop up in later proofs, among them, the De Morgan Laws and the Monotonicity Reversing of Complementation Rule.

The concept of a *function* lies at the core of mathematics. We will give a brief overview here of some basic properties of functions.

1.1 Sets

This section contains some of the set notation to be used throughout these notes. The one-directional arrow $\implies$ reads "implies" and the two-directional arrow $\iff$ reads "if and only if."

1 Definition We will accept the notion of *set* as a primitive notion, that is, a notion that cannot be defined in terms of more elementary notions. By a *set* we will understand a well-defined collection of objects, which we will call the *elements* of the set. If the element x belongs to the set S we will write $x \in S$, and in the contrary case we will write $x \notin S$.[1] The *cardinality* of a set is the number of elements the set has. It can either be finite or infinite. We will denote the cardinality of the set S by $\mathbf{card}(S)$.

☞ *Some sets are used so often that merit special notation. We will denote by*

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

the set of natural numbers, by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$[2]

by $\mathbb{Q}$ *the set of rational numbers*[3], *by* $\mathbb{R}$ *the real numbers, and by* $\mathbb{C}$ *the set of complex numbers. We will occasionally also use* $\alpha\mathbb{Z} = \{\ldots, -3\alpha, -2\alpha, -\alpha, 0, \alpha, 2\alpha, 3\alpha, \ldots\}$, *etc.*

We will also denote the empty set, that is, the set having no elements by $\varnothing$.

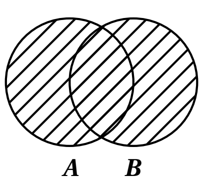


Figure 1.1: $A \cup B$

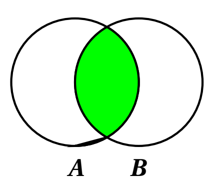


Figure 1.2: $A \cap B$

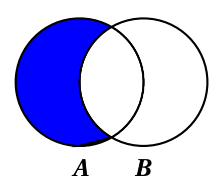


Figure 1.3: $A \setminus B$

[1] Georg Cantor(1845-1918), the creator of set theory, said "*A set is any collection into a whole of definite, distinguishable objects, called* **elements**, *of our intuition or thought.*"
[2] $\mathbb{Z}$ for the German word *Zählen* meaning "integer."
[3] $\mathbb{Q}$ for "quotients."

2 Definition The *union* of two sets A and B is the set

$$A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}.$$

This is read "A union B." See figure 1.1. The *intersection* of two sets A and B is

$$A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}.$$

This is read "A intersection B." See figure 1.2. The *set difference* of two sets A and B is

$$A \setminus B = \{x : (x \in A) \text{ and} (x \not\in B)\}.$$

This is read "A set minus B." See figure 1.3.

3 Definition Two sets A and B are *disjoint* if $A \cap B = \varnothing$.

4 Example Write $A \cup B$ as the disjoint union of three sets.

Solution: *Observe that*

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A),$$

and that the sets on the dextral side are disjoint.

5 Definition A *subset* B of a set A is a subcollection of A, and we denote this by $B \subseteqq A$. [4] This means that $x \in B \implies x \in A$.

$\varnothing$ and A are always subsets of any set A.

Observe that

$$A = B \iff (A \subseteq B) \quad \text{and} \quad (B \subseteq A).$$

We use this observation on the next theorem.

6 THEOREM (De Morgan Laws) Let A, B, C be sets. Then

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C), \qquad A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Proof: *We have*

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \quad \text{and} \quad x \not\in (B \quad \text{or} \quad C) \\ &\iff (x \in A) \quad \text{and} \quad ((x \not\in B) \quad \text{and} \quad (x \not\in C)) \\ &\iff (x \in A \quad \text{and} \quad x \not\in B) \quad \text{and} \quad (x \in A \quad \text{and} \quad x \not\in C) \\ &\iff (x \in A \setminus B) \quad \text{and} \quad (x \in A \setminus C) \\ &\iff x \in (A \setminus B) \cap (A \setminus C). \end{aligned}$$

Also,

$$\begin{aligned} x \in A \setminus (B \cap C) &\iff x \in A \quad \text{and} \quad x \not\in (B \quad \text{and} \quad C) \\ &\iff (x \in A) \quad \text{and} \quad ((x \not\in B) \quad \text{or} \quad (x \not\in C)) \\ &\iff (x \in A \quad \text{and} \quad x \not\in B) \quad \text{or} \quad (x \in A \quad \text{and} \quad x \not\in C) \\ &\iff (x \in A \setminus B) \quad \text{or} \quad (x \in A \setminus C) \\ &\iff x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

❑

7 THEOREM (Monotonicity Reversing of Complementation) Let A, B, X be sets. Then

$$A \subseteqq B \iff X \setminus B \subseteqq X \setminus A.$$

[4] There seems not to be an agreement here by authors. Some use the notation $\subset$ or $\subseteq$ instead of $\subseteqq$. Some see in the notation $\subset$ the exclusion of equality. In these notes, we will always use the notation $\subseteqq$, and if we wished to exclude equality we will write $\subsetneqq$.

Proof: *We have*

$$\begin{aligned} A \subseteq B &\iff (x \in A) \implies (x \in B) \\ &\iff (x \notin B) \implies (x \notin A) \\ &\iff (x \in X \quad \text{and} \quad x \notin B) \implies (x \in X \quad \text{and} \quad x \notin A) \\ &\iff X \setminus B \subseteq X \setminus A. \end{aligned}$$

❑

8 Definition Let $A_1, A_2, \ldots, A_n$, be sets. The *Cartesian Product* of these n sets is defined and denoted by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_k \in A_k\},$$

that is, the set of all ordered n-tuples whose elements belong to the given sets.

In the particular case when all the A_k are equal to a set A, we write

$$A_1 \times A_2 \times \cdots \times A_n = A^n.$$

If $a \in A$ and $b \in A$ we write $(a, b) \in A^2$.

9 Example The Cartesian product is not necessarily commutative. For example, $(\sqrt{2}, 1) \in \mathbb{R} \times \mathbb{Z}$ but $(\sqrt{2}, 1) \notin \mathbb{Z} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{Z}$ has an element that $\mathbb{Z} \times \mathbb{R}$ does not, $\mathbb{R} \times \mathbb{Z} \neq \mathbb{Z} \times \mathbb{R}$.

10 Example Prove that if $X \times X = Y \times Y$ then $X = Y$.

Solution: *Let $x \in X$. Then $(x, x) \in X \times X$, which gives $(x, x) \in Y \times Y$, so $y \in Y$. Hence $X \subseteq Y$.*

Similarly, if $y \in Y$ then $(y, y) \in Y \times Y$, which gives $(y, y) \in X \times X$, so $y \in X$. Hence $Y \subseteq X$.

Thus $X \subseteq Y$ and $Y \subseteq X$ gives $X = Y$.

Homework

11 Problem For a fixed $n \in \mathbb{N}$ put $A_n = \{nk : k \in \mathbb{N}\}$.

1. Find $A_2 \cap A_3$.
2. Find $\bigcap_{n=1}^{\infty} A_n$.
3. Find $\bigcup_{n=1}^{\infty} A_n$.

12 Problem Prove the following properties of the empty set:

$$A \cap \varnothing = \varnothing, \quad A \cup \varnothing = A.$$

13 Problem Prove the following commutative laws:

$$A \cap B = B \cap A, \quad A \cup B = B \cup A.$$

14 Problem Prove by means of set inclusion the following distributive law:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C).$$

15 Problem Prove the following associative laws:

$$A \cap (B \cap C) = (A \cap B) \cap C, \quad A \cup (B \cup C) = (A \cup B) \cup C.$$

16 Problem Prove that

$$A \cap B = A \iff A \subseteq B.$$

17 Problem Prove that

$$A \cup B = A \iff B \subseteq A.$$

18 Problem Prove that

$$A \subseteq B \implies A \cap C \subseteq B \cap C.$$

19 Problem Prove that

$$A \subseteq B \quad \text{and} \quad C \subseteq B \implies A \cup C \subseteq B.$$

20 Problem Prove the following distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

21 Problem Is there any difference between the sets $\varnothing$, $\{\varnothing\}$ and $\{\{\varnothing\}\}$? Explain.

22 Problem Is the Cartesian product associative? Explain.

23 Problem Let A, B, and C be sets. Shew that

$$A \times (B \setminus C) = (A \times B) \setminus (A \times C).$$

24 Problem Prove that a set with $N \in \mathbb{N}$ elements has exactly 2^N subsets.

1.2 Numerical Functions

25 Definition By a *(numerical) function* $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$ we mean the collection of the following ingredients:

❶ a *name* for the function. Usually we use the letter f.

❷ a set of real number inputs called the *domain* of the function. The domain of f is denoted by $\mathbf{Dom}(f) \subseteqq \mathbb{R}$.

❸ an *input parameter* , also called *independent variable* or *dummy variable*. We usually denote a typical input by the letter x.

❹ a set of possible real number outputs of the function, called the *target set* of the function. The target set of f is denoted by $\mathbf{Target}(f) \subseteqq \mathbb{R}$.

❺ an *assignment rule* or *formula*, assigning to **every input** a **unique** output. This assignment rule for f is usually denoted by $x \mapsto f(x)$. The output of x under f is also referred to as the *image of* x *under* f, and is denoted by $f(x)$.

The notation[5]

$$f : \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$$

read "the function f, with domain $\mathbf{Dom}(f)$, target set $\mathbf{Target}(f)$, and assignment rule f mapping x to $f(x)$" conveys all the above ingredients.

Oftentimes we will only need to mention the assignment rule of a function, without mentioning its domain or target set. In such instances we will sloppily say "the function f" or more commonly, "the function $x \mapsto f(x)$", e.g., the square function $x \mapsto x^2$.[6]

26 Definition The *image* $\mathbf{Im}(f)$ of a function f is its set of actual outputs. In other words,

$$\mathbf{Im}(f) = \{f(a) : a \in \mathbf{Dom}(f)\}.$$

Observe that we always have $\mathbf{Im}(f) \subseteq \mathbf{Target}(f)$. For a set A, we also define

$$f(A) = \{f(a) : a \in A\}.$$

27 Theorem Let $f : X \to Y$ be a function and let $A \subseteqq X$, $A' \subseteqq X$. Then

1. $A \subseteqq A' \implies f(A) \subseteqq f(A')$
2. $f(A \cup A') = f(A) \cup f(A')$
3. $f(A \cap A') \subseteqq f(A) \cap f(A')$
4. $f(A) \setminus f(A') \subseteqq f(A \setminus A')$

Proof:

1. $x \in A \implies x \in A'$ *and hence* $f(x) \in f(A) \implies f(x) \in f(A') \implies f(A) \subseteqq f(A')$
2. *Since* $A \subseteqq A \cup A'$ *and* $A' \subseteqq A \cup A'$, *we have* $f(A) \subseteqq f(A \cup A')$ *and* $f(A') \subseteqq f(A \cup A')$, *by part (1) and thus* $f(A) \subseteqq f(A') \subseteqq f(A \cup A')$. *Moreover, if* $y \in f(A \cup A')$, *then* $\exists x \in A \cup A'$ *such that* $y = f(x)$. *Then either* $x \in A$ *and so* $f(x) \in f(A)$ *or* $x \in A'$ *and so* $f x \in f(A')$. *Either way,* $f(x) \in f(A) \cup f(A')$ *and*
$$y \in f(A \cup A') \implies y \in f(A) \cup f(A') \implies f(A \cup A') \subseteqq f(A) \cup f(A').$$
Hence
$$f(A \cup A') = f(A) \cup f(A').$$
3. *Let* $y \in f(A \cap A')$. *Then* $\exists x \in A \cap A'$ *such that* $f(x) = y$. *Thus we have both* $x \in A \implies f(x) \in f(A)$ *and* $x \in A' \implies f(x) \in f(A')$. *Therefore* $f(x) \in f(A) \cap f(A')$ *and we conclude that* $f(A \cap A') \subseteqq f(A) \cap f(A')$.
4. *Let* $y \in f(A) \setminus f(A')$. *Then* $y \in f(A)$ *and* $y \notin f(A')$. *Thus* $\exists x \in A$ *such that* $f(x) = y$. *Since* $y \notin f(A')$, *then* $x \notin A'$. *Therefore* $x \in A \setminus A'$ *and finally,* $y \in f(A \setminus A')$. *This means that* $f(A) \setminus f(A') \subseteqq f(A \setminus A')$ *as claimed.*

❑

[5] Notice the difference in the arrows. The straight arrow $\longrightarrow$ is used to mean that a certain set is associated with another set, whereas the arrow $\mapsto$ (read "maps to") is used to denote that an input becomes a certain output.

[6] This corresponds to the even sloppier American usage "the function $f(x) = x^2$."

1.2.1 Injective and Surjective Functions

28 Definition A function is *injective* or *one-to-one* whenever two different values of its domain generate two different values in its image. A function is *surjective* or *onto* if every element of its target set is hit, that is, the target set is the same as the image of the function. A function is *bijective* if it is both injective and surjective.

29 Example The function

$$a: \begin{array}{lll} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is neither injective nor surjective.

The function

$$b: \begin{array}{lll} \mathbb{R} & \to & \left[0\,;+\infty\right[\\ x & \mapsto & x^2 \end{array}$$

is surjective but not injective.

The function

$$c: \begin{array}{lll} \left[0\,;+\infty\right[& \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is injective but not surjective.

The function

$$d: \begin{array}{lll} \left[0\,;+\infty\right[& \to & \left[0\,;+\infty\right[\\ x & \mapsto & x^2 \end{array}$$

is a bijection.

A bijection between two sets essentially tells us that the two sets have the same size. We will make this statement more precise now for finite sets.

30 THEOREM Let $f: A \to B$ be a function, and let A and B be finite. If f is injective, then $\mathbf{card}(A) \leq \mathbf{card}(B)$. If f is surjective then $\mathbf{card}(B) \leq \mathbf{card}(A)$. If f is bijective, then $\mathbf{card}(A) = \mathbf{card}(B)$.

Proof: *Put* $n = \mathbf{card}(A)$, $A = \{x_1, x_2, \ldots, x_n\}$ *and* $m = \mathbf{card}(B)$, $B = \{y_1, y_2, \ldots, y_m\}$.

If f were injective then $f(x_1), f(x_2), \ldots, f(x_n)$ are all distinct, and among the y_k. Hence $n \leq m$.

If f were surjective then each y_k is hit, and for each, there is an x_i with $f(x_i) = y_k$. Thus there are at least m different images, and so $n \geq m$. ❑

1.2.2 Algebra of Functions

31 Definition Let $f: \mathbf{Dom}(f) \to \mathbf{Target}(f)$ and $g: \mathbf{Dom}(g) \to \mathbf{Target}(g)$. Then $\mathbf{Dom}(f \pm g) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g)$ and the sum (respectively, difference) function $f + g$ (respectively, $f - g$) is given by

$$f \pm g: \begin{array}{lll} \mathbf{Dom}(f) \cap \mathbf{Dom}(g) & \to & \mathbf{Target}(f \pm g) \\ x & \mapsto & f(x) \pm g(x) \end{array}.$$

In other words, if x belongs both to the domain of f and g, then

$$(f \pm g)(x) = f(x) \pm g(x).$$

32 Definition Let $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$ and $g : \mathbf{Dom}(g) \to \mathbf{Target}(g)$. Then $\mathbf{Dom}(fg) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g)$ and the product function fg is given by

$$fg : \begin{array}{ccc} \mathbf{Dom}(f) \cap \mathbf{Dom}(g) & \to & \mathbf{Target}(fg) \\ x & \mapsto & f(x) \cdot g(x) \end{array}.$$

In other words, if x belongs both to the domain of f and g, then

$$(fg)(x) = f(x) \cdot g(x).$$

33 Definition Let $g : \mathbf{Dom}(g) \to \mathbf{Target}(g)$ be a function. The *support* of g, denoted by $\mathbf{supp}(g)$ is the set of elements in $\mathbf{Dom}(g)$ where g does not vanish, that is

$$\mathbf{supp}(g) = \{x \in \mathbf{Dom}(g) : g(x) \neq 0\}.$$

34 Definition Let $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$ and $g : \mathbf{Dom}(g) \to \mathbf{Target}(f)$. Then $\mathbf{Dom}\left(\frac{f}{g}\right) = \mathbf{Dom}(f) \cap \mathbf{supp}(g)$ and the quotient function $\frac{f}{g}$ is given by

$$\frac{f}{g} : \begin{array}{ccc} \mathbf{Dom}(f) \cap \mathbf{supp}(g) & \to & \mathbf{Target}(f/g) \\ x & \mapsto & \dfrac{f(x)}{g(x)} \end{array}.$$

In other words, if x belongs both to the domain of f and g and $g(x) \neq 0$, then $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

35 Definition Let $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$, $g : \mathbf{Dom}(g) \to \mathbf{Target}(g)$ and let $U = \{x \in \mathbf{Dom}(g) : g(x) \in \mathbf{Dom}(f)\}$. We define the *composition* function of f and g as

$$f \circ g : \begin{array}{ccc} U & \to & \mathbf{Target}(f \circ g) \\ x & \mapsto & f(g(x)) \end{array}. \tag{1.1}$$

We read $f \circ g$ as "f *composed with* g."

1.2.3 Inverse Image

36 Definition Let X and Y be subsets of $\mathbb{R}$ and let $f : X \to Y$ be a function. Let $B \subseteq Y$. The *inverse image of* B *by* f is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

If $B = \{b\}$ consists of only one element, we write, abusing notation, $f^{-1}(\{b\}) = f^{-1}(b)$. It is clear that $f^{-1}(Y) = X$ and $f^{-1}(\varnothing) = \varnothing$.

37 Example Let

$$f : \begin{array}{ccc} \{-2, -1, 0, 1, 3\} & \to & \{0, 1, 4, 5, 9\} \\ x & \mapsto & x^2 \end{array}.$$

Then $f^{-1}(\{0,1\}) = \{0, -1, 1\}$, $f^{-1}(1) = \{-1, 1\}$, $f^{-1}(5) = \varnothing$, $f^{-1}(4) = 2$, $f^{-1}(0) = 0$, etc. Notice that we have abused notation in all but the first example.

38 THEOREM Let $f : X \to Y$ be a function and let $B \subseteq Y$, $B' \subseteq Y$. Then

1. $B \subseteq B' \implies f^{-1}(B) \subseteq f^{-1}(B')$
2. $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$
3. $f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B')$
4. $f^{-1}(B) \setminus f(B') = f^{-1}(B \setminus B')$

Proof:

1. *Assume* $x \in f^{-1}(B)$. *Then there is* $y \in B \subseteq B'$ *such that* $f(x) = y$. *But* y *is also in* B' *so* $x \in f^{-1}(B')$. *Thus* $f^{-1}(B) \subseteq f^{-1}(B')$.
2. *Since* $B \subseteq B \cup B'$ *and* $B' \subseteq B \cup B'$, *we have* $f^{-1}(B) \subseteq f^{-1}(B \cup B')$ *and* $f^{-1}(B') \subseteq f^{-1}(B \cup B')$, *by part (1). Thus* $f^{-1}(B) \cup f^{-1}(B') \subseteq f^{-1}(B \cup B')$. *Now, let* $x \in f^{-1}(B \cup B')$. *There is* $y \in B \cup B'$ *such that* $f(x) = y$. *Either* $y \in B$ *and so* $y \in B \implies x \in f^{-1}(B)$ *or* $y \in B'$ *and so* $y \in B \implies x \in f^{-1}(B')$. *Either way,* $x \in f^{-1}(B) \cup f^{-1}(B')$. *Thus* $f^{-1}(B \cup B') \subseteq f^{-1}(B) \cup f^{-1}(B')$. *We conclude that* $f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B')$.
3. *Let* $x \in f^{-1}(B \cap B')$. *Then* $\exists y \in B \cap B'$ *such that* $f(x) = y$. *Thus we have both* $y \in B \implies x \in f^{-1}(B)$ *and* $y \in B' \implies x \in f^{-1}(B')$. *Therefore* $x \in f^{-1}(B) \cap f^{-1}(B')$ *and we conclude that* $f^{-1}(B \cap B') \subseteq f^{-1}(B) \cap f^{-1}(B')$. *Now, let* $x \in f^{-1}(B) \cap f^{-1}(B')$. *Then* $x \in f^{-1}(B)$ *and* $x \in f^{-1}(B')$. *Then* $f(x) \in B$ *and* $f(x) \in B'$. *Thus* $f(x) \in B \cap B'$ *and so* $x \in f^{-1}(B \cap B')$. *Hence* $f^{-1}(B) \cap f^{-1}(B') \subseteq f^{-1}(B \cap B')$ *also, and we conclude that* $f^{-1}(B) \cap f^{-1}(B') = f^{-1}(B \cap B')$.
4. *Let* $x \in f^{-1}(B) \setminus f^{-1}(B')$. *Then* $x \in f^{-1}(B)$ *and* $x \notin f^{-1}(B')$. *Thus* $f(x) \in B$ *and* $f(x) \notin B'$. *Thus* $f(x) \in B \setminus B'$ *and therefore* $x \in f^{-1}(B \setminus B')$, *giving* $f^{-1}(B) \setminus f^{-1}(B') \subseteq f^{-1}(B \setminus B')$. *Now, let* $x \in f^{-1}(B \setminus B')$. *Then* $f(x) \in B \setminus B'$, *which means that* $f(x) \in B$ *but* $f(x) \notin B'$. *Thus* $x \in f^{-1}(B)$ *but* $x \notin f^{-1}(B')$, *which gives* $x \in f^{-1}(B) \setminus f^{-1}(B')$ *and so* $f^{-1}(B \setminus B') \subseteq f^{-1}(B) \setminus f^{-1}(B')$. *This establishes the desired equality.*

❑

39 THEOREM Let $f : X \to Y$ be a function. Let $A \times B \subseteq X \times Y$. Then

1. $A \subseteq (f^{-1} \circ f)(A)$
2. $(f \circ f^{-1})(B) \subseteq B$

Proof: *We have*

1. *Let* $x \in A$. *Then* $\exists y \in Y$ *such that* $y = f(x)$. *Thus* $y \in f(A)$. *Therefore* $x \in f^{-1}(f(A))$.
2. $y \in (f \circ f^{-1})(B)$. *Then* $\exists x \in f^{-1}(B)$ *such that* $f(x) = y$. *Thus* $x \in f^{-1}(y)$. *Hence* $f(x) \in B$. *Therefore* $y \in B$.

❑

1.2.4 Inverse Function

40 Definition Let $A \times B \subseteq \mathbb{R}^2$. A function $F : A \to B$ is said to be *invertible* if there exists a function F^{-1} (called the *inverse* of F) such that $F \circ F^{-1} = \mathbf{Id}_B$ and $F^{-1} \circ F = \mathbf{Id}_A$. Here $\mathbf{Id}_S$ is the identity on the set S function with rule $\mathbf{Id}_S(x) = x$.

The central question is now: given a function $F : A \to B$, when is $F^{-1} : B \to A$ a function? The answer is given in the next theorem.

41 THEOREM Let $A \times B \subseteq \mathbb{R}^2$. A function $f : A \to B$ is invertible if and only if it is a bijection. That is, $f^{-1} : B \to A$ is a function if and only if f is bijective.

Proof: *Assume first that* f *is invertible. Then there is a function* $f^{-1} : B \to A$ *such that*

$$f \circ f^{-1} = \mathbf{Id}_B \quad \text{and} \quad f^{-1} \circ f = \mathbf{Id}_A. \tag{1.2}$$

Let us prove that f *is injective and surjective. Let* s, t *be in the domain of* f *and such that* $f(s) = f(t)$. *Applying* f^{-1} *to both sides of this equality we get* $(f^{-1} \circ f)(s) = (f^{-1} \circ f)(t)$. *By the definition of inverse function,* $(f^{-1} \circ f)(s) = s$ *and* $(f^{-1} \circ f)(t) = t$. *Thus* $s = t$. *Hence* $f(s) = f(t) \implies s = t$ *implying that* f *is injective. To prove that* f *is surjective we must shew that for every* $b \in f(A)$ $\exists a \in A$ *such that* $f(a) = b$. *We take* $a = f^{-1}(b)$ *(observe that* $f^{-1}(b) \in A$*). Then* $f(a) = f(f^{-1}(b)) = (f \circ f^{-1})(b) = b$ *by definition of inverse function. This shews that* f *is surjective. We conclude that if* f *is invertible then it is also a bijection.*

Assume now that f *is a bijection. For every* $b \in B$ *there exists a unique* a *such that* $f(a) = b$. *This makes the rule* $g : B \to A$ *given by* $g(b) = a$ *a function. It is clear that* $g \circ f = \mathbf{Id}_A$ *and* $f \circ g = \mathbf{Id}_B$. *We may thus take* $f^{-1} = g$. *This concludes the proof.* ❑

Homework

42 Problem Find all functions with domain $\{a,b\}$ and target set $\{c,d\}$.

43 Problem Let A, B be finite sets with $\mathbf{card}(A) = n$ and $\mathbf{card}(B) = m$. Prove that

- The number of functions from A to B is m^n.
- If $n \leq m$, the number of injective functions from A to B is $m(m-1)(m-2)\cdots(m-n+1)$. If $n > m$ there are no injective functions from A to B.

44 Problem Let A and B be two finite sets with $\mathbf{card}(A) = n$ and $\mathbf{card}(B) = m$. If $n < m$ prove that there are no surjections from A to B. If $n \geq m$ prove that the number of surjective functions from A to B is

$$m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \binom{m}{3}(m-3)^n + \cdots + (-1)^{m-1}\binom{m}{m-1}(1)^n.$$

45 Problem Let $h : \mathbb{R} \to \mathbb{R}$ be given by $h(1-x) = 2x$. Find $h(3x)$.

46 Problem Consider the polynomial

$$(1 - x^2 + x^4)^{2003} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{8012} x^{8012}.$$

Find

1. a_0
2. $a_0 + a_1 + a_2 + \cdots + a_{8012}$
3. $a_0 - a_1 + a_2 - a_3 + \cdots - a_{8011} + a_{8012}$
4. $a_0 + a_2 + a_4 + \cdots + a_{8010} + a_{8012}$
5. $a_1 + a_3 + \cdots + a_{8009} + a_{8011}$

47 Problem Let $f : \mathbb{R} \to \mathbb{R}$, be a function such that $\forall x \in]0;+\infty[$,

$$[f(x^3 + 1)]^{\sqrt{x}} = 5,$$

find the value of

$$\left[f\left(\frac{27 + y^3}{y^3}\right)\right]^{\sqrt{\frac{27}{y}}}$$

for $y \in]0;+\infty[$.

48 Problem Let f satisfy $f(n+1) = (-1)^{n+1} n - 2f(n), n \geq 1$ If $f(1) = f(1001)$ find

$$f(1) + f(2) + f(3) + \cdots + f(1000).$$

49 Problem If $f(a)f(b) = f(a+b)\ \forall\ a, b \in \mathbb{R}$ and $f(x) > 0\ \forall\ x \in \mathbb{R}$, find $f(0)$. Also, find $f(-a)$ and $f(2a)$ in terms of $f(a)$.

50 Problem Prove that $f : \begin{array}{ccc} \mathbb{R}\setminus\{-1\} & \to & \mathbb{R}\setminus\{1\} \\ x & \mapsto & \dfrac{x-1}{x+1} \end{array}$ is a bijection and find f^{-1}.

51 Problem Let $f^{[1]}(x) = f(x) = x+1, f^{[n+1]} = f \circ f^{[n]}, n \geq 1.$ Find a closed formula for $f^{[n]}$

52 Problem Let $f, g : [0;1] \to \mathbb{R}$ be functions. Demonstrate that there exist $(a,b) \in [0;1]^2$ such that $\frac{1}{4} \leq |f(a) + g(b) - ab|$.

53 Problem Demonstrate that there is no function $f : \mathbb{R}\setminus\{1/2\} \to \mathbb{R}$ such that

$$x \in \mathbb{R}\setminus\{1/2\} \implies f(x)\left(f\left(\frac{x-1}{2x-1}\right)\right) = x^2 + x + 1$$

54 Problem Find all functions $f : \mathbb{R}\setminus\{-1,0\} \to \mathbb{R}$ such that

$$x \in \mathbb{R}\setminus\{-1,0\} \implies f(x) + f\left(\frac{-1}{x+1}\right) = 3x + 2.$$

55 Problem Let $f^{[1]}(x) = f(x) = 2x, f^{[n+1]} = f \circ f^{[n]}, n \geq 1.$ Find a closed formula for $f^{[n]}$

56 Problem Find all functions $g : \mathbb{R} \to \mathbb{R}$ that satisfy $g(x+y) + g(x-y) = 2x^2 + 2y^2$.

57 Problem Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy $f(xy) = yf(x)$.

58 Problem Find all functions $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ for which

$$f(x) + 2f\left(\frac{1}{x}\right) = x.$$

59 Problem Find all functions $f : \mathbb{R}\setminus\{-1\} \to \mathbb{R}$ such that

$$(f(x))^2 \cdot f\left(\frac{1-x}{1+x}\right) = 64x.$$

60 Problem Let $f^{[1]} = f$ be given by $f(x) = \dfrac{1}{1-x}$. Find
(i) $f^{[2]}(x) = (f \circ f)(x)$,
(ii) $f^{[3]}(x) = (f \circ f \circ f)(x)$, and
(iii) $f^{[69]} = \underbrace{(f \circ f \circ \cdots f \circ f)}_{69 \text{ compositions with itself}}(x)$.

61 Problem Let $f : A \to B$ and $g : B \to C$ be functions. Shew that (i) if $g \circ f$ is injective, then f is injective. (ii) if $g \circ f$ is surjective, then g is surjective.

1.3 Countability

62 Definition A set X is countable if either it is finite or if there is a bijection $f : X \to \mathbb{N}$, that is, the set X has as many elements as $\mathbb{N}$.

Any countable set can be thus enumerated a sequence

$$x_1, x_2, x_3, \ldots.$$

Thus the strictly positive integers can be enumerated as customarily:

$$1, 2, 3, \ldots.$$

Another possible enumeration[7] is the following

$$3, 5, 7, 9, \ldots, \quad , 2\cdot 3, 2\cdot 5, 2\cdot 7, 2\cdot 9, \ldots \quad , 2^2\cdot 3, 2^2\cdot 5, 2^2\cdot 7, 2^2\cdot 9, \ldots \quad , \ldots 2^4, 2^3, 2^2, 2, 1,$$

[7] Which is relevant in chaos theory, for *Sarkovkii's Theorem*.

that is, we start with the odd integers in increasing order, then $\mathbf{2}$ times the odd integers, $\mathbf{2^2}$ times the odd integers, etc., and at the end we put the powers of $\mathbf{2}$ in decreasing order.

63 Lemma Any subset $X \subseteqq \mathbb{N}$ is countable.

> **Proof:** *If X is finite, then there is nothing to prove. If X is infinite, we can arrange the elements of X increasing order, say,*
>
> $$x_1 < x_2 < x_3 < \cdots .$$
>
> *We then map the smallest element $x_1 \in S$ to 1, the next smallest x_2 to 2, etc.* ❑

☞ *Hence, even though $2\mathbb{N} \subsetneqq \mathbb{N}$, the sets $2\mathbb{N}$ and $\mathbb{N}$ have the same number of elements. This can also be seen by noticing that $f : \mathbb{N} \to 2\mathbb{N}$ given by $x_n = 2n$ is a bijection.*

64 Lemma A set X is countable if and only if there is an injection $f : X \to \mathbb{N}$.

> **Proof:** *The assertion is evident if X is finite. Hence assume X is infinite. If $f : X \to \mathbb{N}$ is an injection then $f(X)$ is an infinite subset of $\mathbb{N}$. Hence there is a bijection $g : f(X) \to \mathbb{N}$ by virtue of Lemma 63. Thus $(g \circ f) : X \to \mathbb{N}$ is a bijection.* ❑

☞ *An obvious consequence of the above lemma is that if X' is countable and there is an injection $f : X \to X'$ then X is countable.*

65 Theorem $\mathbb{Z}$ is countable.

> **Proof:** *One can take, as a bijection between the two sets, for example, $f : \mathbb{Z} \to \mathbb{N}$,*
>
> $$f(x) = \begin{cases} 2x+1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0. \end{cases}$$
>
> ❑

66 Theorem $\mathbb{Q}$ is countable.

> **Proof:** *Consider $f : \mathbb{Q} \to \mathbb{N}$ given*
>
> $$f\left(\frac{a}{b}\right) = 2^{|a|}3^{b}5^{1+\mathrm{signum}(a)},$$
>
> *where $\frac{a}{b}$ is in least terms, and $b > 0$. By the uniqueness of the prime factorisation of an integer, f is an injection.*
> ❑

☞ *The above theorem means that there as many rational numbers as natural numbers. Thus the rationals can be enumerated as*

$$q_1, q_2, q_3, \ldots,$$

67 Theorem (Cantor's Diagonal Argument) $\mathbb{R}$ is uncountable.

> **Proof:** *Assume $\mathbb{R}$ were countable so that its complete set of elements may be enumerated, say, as in the list*
>
> $$r_1 = n_1.d_{11}d_{12}d_{13}\ldots$$
> $$r_2 = n_2.d_{21}d_{22}d_{13}\ldots$$
> $$r_3 = n_3.d_{31}d_{32}d_{33}\ldots,$$
>
> *where we have used decimal notation. Define the new real $r = 0.d_1d_2d_3\ldots$ by $d_i = 0$ if $d_{ii} \neq 0$ and $d_i = 1$ if $d_{ii} = 0$. This is real number (as it is a decimal), but it differs from r_i in the i^{th} decimal place. It follows that the list is incomplete and the reals are uncountable.* ❑

68 Theorem The interval $\left]-1\,;1\right[$ is uncountable.

> **Proof:** *Observe that the map $f : \left]-1\,;1\right[\to \mathbb{R}$ given by $f(x) = \tan\frac{\pi x}{2}$ is a bijection.* ❑

Homework

69 Problem Prove that there as many numbers in $[0;1]$ as in any interval $[a;b]$ with $a < b$.

70 Problem Prove that there as many numbers in $\left]-\infty;+\infty\right[$ as in $\left]0;+\infty\right[$.

1.4 Groups and Fields

Here we observe the rules of the game for the operations of addition and multiplication in $\mathbb{R}$.

71 Definition Let S, T be sets. A *binary operation* is a function

$$\otimes: \begin{array}{ccc} S \times S & \to & T \\ (a, b) & \mapsto & \otimes(a, b) \end{array}.$$

We usually use the "infix" notation $a \otimes b$ rather than the "prefix" notation $\otimes(a, b)$. If $S = T$ then we say that the binary operation is *internal* or *closed* and if $S \neq T$ then we say that it is *external*.

72 Example Ordinary addition is a closed binary operation on the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. Ordinary subtraction is a binary operation on these sets. It is not closed on $\mathbb{N}$, since for example $1 - 2 = -1 \not\in \mathbb{N}$, but it is closed in the remaining sets.

73 Example The operation $\otimes : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $a \otimes b = 1 + a \cdot b$, where $\cdot$ is the ordinary multiplication of real numbers is commutative but not associative. To see commutativity we have

$$a \otimes b = 1 + ab = 1 + ba = b \otimes a.$$

Now,

$$1 \otimes (1 \otimes 2) = 1 \otimes (1 + 1 \cdot 2) = 1 \otimes (3) = 1 + 1 \cdot 3 = 4, \qquad \text{but} \qquad (1 \otimes 1) \otimes 2 = (1 + 1 \cdot 1) \otimes 2 = 2 \otimes 2 = 1 + 2 \cdot 2 = 5,$$

so the operation is not associative.

74 Definition Let G be a non-empty set and $\otimes$ be a binary operation on $G \times G$. Then $\langle G, \otimes \rangle$ is called a *group* if the following axioms hold:

G1: $\otimes$ is closed, that is,

$$\forall (a, b) \in G^2, \quad a \otimes b \in G,$$

G2: $\otimes$ is associative, that is,

$$\forall (a, b, c) \in G^3, \quad a \otimes (b \otimes c) = (a \otimes b) \otimes c,$$

G3: G has an identity element, that is

$$\exists e \in G \text{ such that } \forall a \in G, \; e \otimes a = a \otimes e = a,$$

G4: Every element of G is invertible, that is

$$\forall a \in G, \quad \exists a^{-1} \in G \text{ such that } a \otimes a^{-1} = a^{-1} \otimes a = e.$$

From now on, we drop the sign $\otimes$ and rather use juxtaposition for the underlying binary operation in a given group. Thus we will say a "group G" rather than the more precise "a group $\langle G, \otimes \rangle$."

75 Definition A group G is *abelian* if its binary operation is commutative, that is, $\forall (a, b) \in G^2, a \otimes b = b \otimes a$.

76 Example $\langle \mathbb{Z}, + \rangle$, $\langle \mathbb{Q}, + \rangle$, $\langle \mathbb{R}, + \rangle$, $\langle \mathbb{C}, + \rangle$ are all abelian groups under addition. The identity element is 0 and the inverse of a is $-a$.

77 Example $\langle \mathbb{Q} \setminus \{0\}, \cdot \rangle$, $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$, $\langle \mathbb{C} \setminus \{0\}, \cdot \rangle$ are all abelian groups under multiplication. The identity element is 1 and the inverse of a is $\dfrac{1}{a}$.

78 Example $\langle \mathbb{Z}\setminus\{0\}, \cdot\rangle$ is not a group. For example the element 2 does not have a multiplicative inverse.

79 Example Let $V_4 = \{e, a, b, c\}$ and define $\otimes$ by the table below.

$\otimes$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

It is an easy exercise to check that V_4 is an abelian group, called the *Klein Viergruppe*.

80 THEOREM Let G be a group. Then

1. There is only one identity element, the identity element is unique.
2. The inverse of each element is unique.
3. $\forall (a, b) \in G^2$ we have
$$(ab)^{-1} = b^{-1}a^{-1}.$$

Proof:

1. *Let e and e' be identity elements. Since e is an identity, $e = ee'$. Since e' is an identity, $e' = ee'$. This gives $e = ee' = e'$.*
2. *Let b and b' be inverses of a. Then $e = ab$ and $b'a = e$. This gives*
$$b = eb = (b'a)b = b'(ab) = b'e = b'.$$
3. *We have*
$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = a(e)a^{-1} = aa^{-1} = e.$$
Thus $b^{-1}a^{-1}$ works as a right inverse for ab. A similar calculation shews also that it works as a left inverse. Since inverses are unique, we must have
$$(ab)^{-1} = b^{-1}a^{-1}.$$

This completes the proof. ❑

81 Definition Let $n \in \mathbb{Z}$ and let G be a group. If $a \in G$, we define
$$a^0 = e,$$
$$a^{|n|} = \underbrace{a \cdot a \cdots a}_{|n|\ a's},$$
and
$$a^{-|n|} = \underbrace{a^{-1} \cdot a^{-1} \cdots a^{-1}}_{|n|\ a^{-1}'s}.$$

☞ *If $(m, n) \in \mathbb{Z}^2$, then by associativity*
$$(a^n)(a^m) = (a^m)(a^n) = a^{m+n}.$$

82 Definition Let F be a set having at least two elements 0_F and 1_F $(0_F \neq 1_F)$ together with two binary operations $\cdot$ (field multiplication) and $+$ (field addition). A *field* $\langle F, \cdot, +\rangle$ is a triplet such that $\langle F, +\rangle$ is an abelian group with identity 0_F, $\langle F\setminus\{0_F\}, \cdot\rangle$ is an abelian group with identity 1_F and the operations $\cdot$ and $+$ satisfy
$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),$$
that is, field multiplication distributes over field addition.

☞ *We will continue our practice of denoting multiplication by juxtaposition, hence the · sign will be dropped.*

83 Example $\langle \mathbb{Q}, \cdot, + \rangle$, $\langle \mathbb{R}, \cdot, + \rangle$, and $\langle \mathbb{C}, \cdot, + \rangle$ are all fields. The multiplicative identity in each case is 1 and the additive identity is 0.

Homework

84 Problem Is the set of real irrational numbers closed under addition? Under multiplication?

85 Problem Let
$$S = \{x \in \mathbb{Z} : \exists (a,b) \in \mathbb{Z}^2, x = a^3 + b^3 + c^3 - 3abc\}.$$
Prove that S is closed under multiplication, that is, if $x \in S$ and $y \in S$ then $xy \in S$.

86 Problem (Putnam, 1971) Let S be a set and let $\circ$ be a binary operation on S satisfying the two laws
$$(\forall x \in S)(x \circ x = x),$$
and
$$(\forall (x,y,z) \in S^3)((x \circ y) \circ z = (y \circ z) \circ x).$$
Shew that $\circ$ is commutative.

87 Problem (Putnam, 1972) Let $\mathscr{S}$ be a set and let $*$ be a binary operation of $\mathscr{S}$ satisfying the laws $\forall (x,y) \in \mathscr{S}^2$
$$x * (x * y) = y, \tag{1.3}$$
$$(y * x) * x = y. \tag{1.4}$$
Shew that $*$ is commutative, but not necessarily associative.

88 Problem On $\mathbb{Q} \cap]-1;1[$ define the binary operation $\otimes$ by
$$a \otimes b = \frac{a+b}{1+ab},$$
where juxtaposition means ordinary multiplication and $+$ is the ordinary addition of real numbers. Prove that $\langle \mathbb{Q} \cap]-1;1[, \otimes \rangle$ is an abelian group by following these steps.

1. Prove that $\otimes$ is a closed binary operation on $\mathbb{Q} \cap]-1;1[$.
2. Prove that $\otimes$ is both commutative and associative.
3. Find an element $e \in \mathbb{Q} \cap]-1;1[$ such that $(\forall a \in \mathbb{Q} \cap]-1;1[)\ (e \otimes a = a)$.
4. Given e as above and an arbitrary element $a \in \mathbb{Q} \cap]-1;1[$, solve the equation $a \otimes b = e$ for b.

89 Problem Let G be a group satisfying $(\forall a \in G)$
$$a^2 = e.$$
Prove that G is an abelian group.

90 Problem Let G be a group where $(\forall (a,b) \in G^2)$
$$((ab)^3 = a^3b^3) \quad \text{and} \quad ((ab)^5 = a^5b^5).$$
Shew that G is abelian.

91 Problem Suppose that in a group G there exists a pair $(a,b) \in G^2$ satisfying
$$(ab)^k = a^k b^k$$
for three consecutive integers $k = i, i+1, i+2$. Prove that $ab = ba$.

1.5 Addition and Multiplication in $\mathbb{R}$

Since $\mathbb{R}$ is a field, it satisfies the following list of axioms, which we list for future reference.

92 Axiom (Arithmetical Axioms of $\mathbb{R}$) $\langle \mathbb{R}, \cdot, + \rangle$—that is, the set of real numbers endowed with multiplication $\cdot$ and addition $+$—is a field. This entails that $+$ and $\cdot$ verify the following properties.

R1: $+$ and $\cdot$ are closed binary operations, that is,
$$\forall (a,b) \in \mathbb{R}^2, \quad a+b \in \mathbb{R}, \qquad a \cdot b \in \mathbb{R},$$

R2: $+$ and $\cdot$ are associative binary operations, that is,
$$\forall (a,b,c) \in \mathbb{R}^3, \quad a+(b+c) = (a+b)+c, \quad a\cdot(b\cdot c) = (a\cdot b)\cdot c$$

R3: $+$ and $\cdot$ are commutative binary operations, that is,
$$\forall (a,b) \in \mathbb{R}^2, \quad a+b = b+a, \quad a \cdot b = b \cdot a,$$

R4: $\mathbb{R}$ has an additive identity element 0, and a multiplicative identity element 1, with $0 \neq 1$, such that
$$\forall a \in \mathbb{R}, \quad 0+a = a+0 = a, \quad 1\cdot a = a \cdot 1 = a,$$

R5: Every element of $\mathbb{R}$ has an additive inverse, and every element of $\mathbb{R} \setminus \{0\}$ has a multiplicative inverse, that is,
$$\forall a \in \mathbb{R}, \quad \exists (-a) \in \mathbb{R} \text{ such that } a + (-a) = (-a) + a = 0,$$
$$\forall b \in \mathbb{R} \setminus \{0\}, \quad \exists b^{-1} \in \mathbb{R} \setminus \{0\} \text{ such that } b \cdot b^{-1} = b^{-1} \cdot b = 1,$$

R6: $+$ and $\cdot$ satisfy the following distributive law:

$$\forall (a,b,c,) \in \mathbb{R}^3, \quad a\cdot(b+c) = a\cdot b + a\cdot c.$$

Since $+$ and $\cdot$ are associative in $\mathbb{R}$, we may write a sum $a_1 + a_2 + \cdots + a_n$ or a product $a_1 a_2 \cdots a_n$ of real numbers without risking ambiguity. We often use the following shortcut notation.

93 Definition For real numbers a_i we define

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \qquad \text{and} \qquad a_1 a_2 \cdots a_n = \prod_{k=1}^{n} a_k.$$

By convention $\sum_{k\in\varnothing} a_k = 0$ *and* $\prod_{k\in\varnothing} a_k = 1$.

94 THEOREM (Lagrange's Identity) Let a_k, b_k be real numbers. Then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) - \sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2.$$

Proof: *For* $j = k$, $a_k b_j - a_j b_k = 0$, *so we may relax the inequality in the last sum. We have*

$$\begin{aligned}
\sum_{1\le k<j\le n} (a_k b_j - a_j b_k)^2 &= \sum_{1\le k\le j\le n} (a_k^2 b_j^2 - 2a_k b_k a_j b_j + a_j^2 b_k^2) \\
&= \sum_{1\le k\le j\le n} a_k^2 b_j^2 - 2\sum_{1\le k\le j\le n} a_k b_k a_j b_j + \sum_{1\le k\le j\le n} a_j^2 b_k^2 \\
&= \sum_{k=1}^{n}\sum_{j=1}^{n} a_k^2 b_j^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2,
\end{aligned}$$

proving the theorem. ❑

Recall that the factorial symbol **!** is defined by

$$0! = 1; \qquad k! = k(k-1)! \quad \text{if} \quad k \ge 1.$$

95 Definition (Binomial Coefficients) Let $n \in \mathbb{N}$ We define $\binom{n}{0} = 1 = \binom{n}{n}$ and for $1 \le k \le n$,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$ we take $\binom{n}{k} = 0$.

96 LEMMA (Pascal's Identity) For $n \ge 1$ and $1 \le k \le n$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof: *We have*

$$\begin{aligned}
\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)}\left(\frac{1}{k} + \frac{1}{n-k}\right) \\
&= \frac{(n-1)!}{(k-1)!(n-1-k)}\left(\frac{n}{k(n-k)}\right) \\
&= \frac{n!}{k!(n-k)!} = \binom{n}{k}.
\end{aligned}$$

❏

Using Pascal's Identity we obtain *Pascal's Triangle*.

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

When the numerical values are substituted, the triangle then looks like this.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

We see from Pascal's Triangle that binomial coefficients are symmetric. This symmetry is easily justified by the identity $\binom{n}{k} = \binom{n}{n-k}$. We also notice that the binomial coefficients tend to increase until they reach the middle, and that then they decrease symmetrically.

97 THEOREM (Binomial Theorem) For $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: *The theorem is obvious for* $n = 0$ *(defining* $(x+y)^0 = 1$*),* $n = 1$ *(as* $(x+y)^1 = x+y$*), and* $n = 2$ *(as* $(x+y)^2 = x^2 + 2xy + y^2$*). Assume* $n \geq 3$*. The induction hypothesis is that* $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$*. Then we* *have*

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right) \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k+1} \\
&= x^{n+1} + \sum_{k=0}^{n-1}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n-k+1} + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n}{k-1}x^k y^{n-k+1} + \sum_{k=1}^{n}\binom{n}{k}x^k y^{n-k+1} + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1}+\binom{n}{k}\right)x^k y^{n-k+1} + y^{n+1} \\
&= x^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n-k+1} + y^{n+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n-k+1},
\end{aligned}$$

proving the theorem. ❑

98 Lemma If $a \in \mathbb{R}$, $a \neq 1$ and $n \in \mathbb{N} \setminus \{0\}$, then

$$1 + a + a^2 + \cdots a^{n-1} = \frac{1-a^n}{1-a}.$$

Proof: *For, put* $S = 1 + a + a^2 + \cdots + a^{n-1}$. *Then* $aS = a + a^2 + \cdots + a^{n-1} + a^n$. *Thus*

$$S - aS = (1 + a + a^2 + \cdots + a^{n-1}) - (a + a^2 + \cdots + a^{n-1} + a^n) = 1 - a^n,$$

and from $(1-a)S = S - aS = 1 - a^n$ *we obtain the result.* ❑

99 Theorem Let n be a strictly positive integer. Then

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}).$$

Proof: *By making the substitution* $a = \frac{x}{y}$ *in Lemma 98 we see that*

$$1 + \frac{x}{y} + \left(\frac{x}{y}\right)^2 + \cdots + \left(\frac{x}{y}\right)^{n-1} = \frac{1-\left(\frac{x}{y}\right)^n}{1-\frac{x}{y}}$$

we obtain

$$\left(1-\frac{x}{y}\right)\left(1 + \frac{x}{y} + \left(\frac{x}{y}\right)^2 + \cdots + \left(\frac{x}{y}\right)^{n-1}\right) = 1 - \left(\frac{x}{y}\right)^n,$$

or equivalently,

$$\left(1-\frac{x}{y}\right)\left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \cdots + \frac{x^{n-1}}{y^{n-1}}\right) = 1 - \frac{x^n}{y^n}.$$

Multiplying by y^n *both sides,*

$$y\left(1-\frac{x}{y}\right)y^{n-1}\left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \cdots + \frac{x^{n-1}}{y^{n-1}}\right) = y^n\left(1 - \frac{x^n}{y^n}\right),$$

which is

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}),$$

yielding the result. ❑

100 THEOREM $1+2+\cdots+n = \dfrac{n(n+1)}{2}$.

First Proof: *Observe that*

$$k^2-(k-1)^2 = 2k-1.$$

From this

$$\begin{array}{lcl} 1^2-0^2 & = & 2\cdot 1-1 \\ 2^2-1^2 & = & 2\cdot 2-1 \\ 3^2-2^2 & = & 2\cdot 3-1 \\ \vdots & \vdots & \vdots \\ n^2-(n-1)^2 & = & 2\cdot n-1 \end{array}$$

Adding both columns,

$$n^2-0^2 = 2(1+2+3+\cdots+n)-n.$$

Solving for the sum,

$$1+2+3+\cdots+n = n^2/2+n/2 = \frac{n(n+1)}{2}.$$

❑

Second Proof: *We may utilise Gauss' trick: If*

$$A_n = 1+2+3+\cdots+n$$

then

$$A_n = n+(n-1)+\cdots+1.$$

Adding these two quantities,

$$\begin{array}{ccccccccc} A_n & = & 1 & + & 2 & + & \cdots & + & n \\ A_n & = & n & + & (n-1) & + & \cdots & + & 1 \\ \hline 2A_n & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) \\ & = & n(n+1), & & & & & & \end{array}$$

since there are n *summands. This gives* $A_n = \dfrac{n(n+1)}{2}$*, that is,*

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Applying Gauss's trick to the general arithmetic sum

$$(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d)$$

we obtain

$$(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \frac{n(2a+(n-1)d)}{2} \qquad (1.5)$$

❑

101 THEOREM $1^2+2^2+3^2+\cdots+n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof: *Observe that*

$$k^3-(k-1)^3 = 3k^2-3k+1.$$

Hence

$$\begin{array}{lll} 1^3-0^3 & = & 3\cdot 1^2-3\cdot 1+1 \\ 2^3-1^3 & = & 3\cdot 2^2-3\cdot 2+1 \\ 3^3-2^3 & = & 3\cdot 3^2-3\cdot 3+1 \\ \vdots & \vdots & \vdots \\ n^3-(n-1)^3 & = & 3\cdot n^2-3\cdot n+1 \end{array}$$

Adding both columns,

$$n^3-0^3=3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+3+\cdots+n)+n.$$

From the preceding example $1+2+3+\cdots+n=\cdot n^2/2+n/2=\dfrac{n(n+1)}{2}$ *so*

$$n^3-0^3=3(1^2+2^2+3^2+\cdots+n^2)-\frac{3}{2}\cdot n(n+1)+n.$$

Solving for the sum,

$$1^2+2^2+3^2+\cdots+n^2=\frac{n^3}{3}+\frac{1}{2}\cdot n(n+1)-\frac{n}{3}.$$

After simplifying we obtain

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

❑

Homework

102 Problem Prove that for $n\geq 1$,

$$2^n=\sum_{k=0}^{n}\binom{n}{k};\qquad 0=\sum_{k=0}^{n}(-1)^k\binom{n}{k},\qquad 2^{n-1}=\sum_{\substack{0\leq k\leq n\\ k\text{ even}}}\binom{n}{k}=\sum_{\substack{1\leq k\leq n\\ k\text{ odd}}}\binom{n}{k}.$$

103 Problem Given that 1002004008016032 has a prime factor $p>250000$, find it.

104 Problem Prove that $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$.

105 Problem Let a,b,c be real numbers. Prove that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

106 Problem Prove that

$$\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}.$$

107 Problem Prove that

$$\binom{n}{k}=\frac{n}{k}\cdot\frac{n-1}{k-1}\cdot\binom{n-2}{k-2}.$$

108 Problem Prove that

$$\sum_{k=1}^{n}k\binom{n}{k}p^k(1-p)^{n-k}=np.$$

109 Problem Prove that

$$\sum_{k=2}^{n}k(k-1)\binom{n}{k}p^k(1-p)^{n-k}=n(n-1)p^2.$$

110 Problem Demonstrate that

$$\sum_{k=0}^{n}(k-np)^2\binom{n}{k}p^k(1-p)^{n-k}=np(1-p).$$

111 Problem Let $x\in\mathbb{R}\setminus\{1\}$ and let $n\in\mathbb{N}\setminus\{0\}$. Prove that

$$\sum_{k=0}^{n}\frac{2^k}{x^{2^k}+1}=\frac{1}{x-1}-\frac{2^{n+1}}{x^{2^{n+1}}+1}.$$

112 Problem Consider the n^k k-tuples $(a_1,a_2,\ldots,a_k)$ which can be formed by taking $a_i\in\{1,2,\ldots,n\}$, repetitions allowed. Demonstrate that

$$\sum_{a_i\in\{1,2,\ldots,n\}}\min(a_1,a_2,\ldots,a_k)=1^k+2^k+\cdots+n^k.$$

1.6 Order Axioms

Vocabulary Alert! We will call a number x positive *if $x\geq 0$ and* strictly positive *if $x>0$. Similarly, we will call a number y* negative *if $y\leq 0$ and* strictly negative *if $y<0$. This usage differs from most Anglo-American books, who prefer such terms as* non-negative *and* non-positive.

We assume $\mathbb{R}$ endowed with a relation $>$ which satisfies the following axioms.

113 Axiom (Trichotomy Law) $\forall(x,y)\in\mathbb{R}^2$ exactly one of the following holds:

$$x>y,\quad x=y,\quad\text{or}\quad y>x.$$

114 Axiom (Transitivity of Order) $\forall(x,y,z)\in\mathbb{R}^3$,

$$\text{if}\quad x>y\quad\text{and}\quad y>z\quad\text{then}\quad x>z.$$

115 Axiom (Preservation of Inequalities by Addition) $\forall(x,y,z)\in\mathbb{R}^3$,

$$\text{if}\quad x>y\quad\text{then}\quad x+z>y+z.$$

116 Axiom (Preservation of Inequalities by Positive Factors) $\forall(x,y,z)\in\mathbb{R}^3$,

$$\text{if}\quad x>y\quad\text{and}\quad z>0\quad\text{then}\quad xz>yz.$$

☞ $x<y$ *means that* $y>x$. $x\leq y$ *means that either* $y>x$ *or* $y=x$, *etc.*

117 Theorem The square of any real number is positive, that is, $\forall a\in\mathbb{R}$, $a^2\geq 0$. In fact, if $a\neq 0$ then $a^2>0$.

Proof: *If* $a=0$, *then* $0^2=0$ *and there is nothing to prove. Assume now that* $a\neq 0$. *By trichotomy, either* $a>0$ *or* $a<0$. *Assume first that* $a>0$. *Applying Axiom 116 with* $x=z=a$ *and* $y=0$ *we have*

$$aa>a0\implies a^2>0,$$

so the theorem is proved if $a>0$.

If $a<0$ *then* $-a>0$ *and we apply the result just obtained:*

$$-a>0\implies(-a)^2>0\implies 1\cdot a^2>0\implies a^2>0,$$

so the result is true regardless the sign of a. ❑

Theorem 117 will prove to be extremely powerful and will be the basis for many of the classical inequalities that follow.

118 Theorem If $(x,y)\in\mathbb{R}^2$,

$$x>y\iff x-y>0.$$

Proof: *This is a direct consequence of Axiom 115 upon taking* $z=-y$. ❑

119 Theorem If $(x,y,a,b)\in\mathbb{R}^4$,

$$x>y\quad\text{and}\quad a\geq b\implies x+a>y+b.$$

Proof: *We have*

$$x>y\implies x+a>y+a,\quad y+a\geq y+b,$$

by Axiom 115 and so by Axiom 114 $x+a>y+b$. ❑

120 Theorem If $(x,y,a,b)\in\mathbb{R}^4$,

$$x>y>0\quad\text{and}\quad a\geq b>0\implies xa>yb.$$

Proof: *Indeed*

$$x>y\implies xa>ya,\quad ya\geq yb,$$

by Axiom 116 and so by Axiom 114 $xa>yb$. ❑

121 Theorem $1>0$.

Proof: *By definition of* $\mathbb{R}$ *being a field* $0\neq 1$. *Assume that* $1<0$ *then* $1^2>0$ *by Theorem 117. But* $1^2=1$ *and so* $1>0$, *a contradiction to our original assumption.* ❑

122 THEOREM $x > 0 \implies -x < 0$ and $x^{-1} > 0$.

Proof: *Indeed,* $-1 < 0$ *since* $-1 \neq 0$ *and assuming* $-1 > 0$ *would give* $0 = -1 + 1 > 1$*, which contradicts Theorem* 121*. Thus*

$$-x = -1 \cdot x < 0.$$

Similarly, assuming $x^{-1} < 0$ *would give* $1 = x^{-1}x < 0$. ❑

123 THEOREM $x > 1 \implies x^{-1} < 1$.

Proof: *Since* $x^{-1} \neq 1$*, assuming* $x^{-1} > 1$ *would give* $1 = xx^{-1} > 1 \cdot 1 = 1$*, a contradiction.* ❑

1.6.1 Absolute Value

124 Definition (The Signum (Sign) Function) Let x be a real number. We define $\mathbf{signum}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$

125 LEMMA The signum function is multiplicative, that is, if $(x, y) \in \mathbb{R}^2$ then $\mathbf{signum}(x \cdot y) = \mathbf{signum}(x)\,\mathbf{signum}(y)$.

Proof: *Immediate from the definition of signum.* ❑

126 Definition (Absolute Value) Let $x \in \mathbb{R}$. The *absolute value* of x is defined and denoted by

$$|x| = \mathbf{signum}(x)\,x.$$

127 THEOREM Let $x \in \mathbb{R}$. Then

1. $|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$
2. $|x| \geq 0$,
3. $|x| = \max(x, -x)$,
4. $|-x| = |x|$,
5. $-|x| \leq x \leq |x|$.
6. $\sqrt{x^2} = |x|$
7. $|x|^2 = |x^2| = x^2$
8. $x = \mathbf{signum}(x)\,|x|$

Proof: *These are immediate from the definition of* $|x|$. ❑

128 THEOREM $(\forall (x, y) \in \mathbb{R}^2)$,

$$|xy| = |x|\,|y|.$$

Proof: *We have*

$$|xy| = \mathbf{signum}(xy)\,xy = (\mathbf{signum}(x)\,x)(\mathbf{signum}(y)\,y) = |x|\,|y|,$$

where we have used Lemma 125. ❑

129 THEOREM Let $t \geq 0$. Then

$$|x| \leq t \iff -t \leq x \leq t.$$

Proof: *Either* $|x| = x$ *or* $|x| = -x$. *If* $|x| = x$,

$$|x| \le t \iff x \le t \iff -t \le 0 \le x \le t.$$

If $|x| = -x$,

$$|x| \le t \iff -x \le t \iff -t \le x \le 0 \le t.$$

❑

130 THEOREM If $(x, y) \in \mathbb{R}^2$, $\max(x, y) = \dfrac{x+y+|x-y|}{2}$ and $\min(x, y) = \dfrac{x+y-|x-y|}{2}$.

Proof: *Observe that* $\max(x, y) + \min(x, y) = x + y$, *since one of these quantities must be the maximum and the other the minimum, or else, they are both equal.*

Now, either $|x - y| = x - y$, *and so* $x \ge y$, *meaning that* $\max(x, y) - \min(x, y) = x - y$, *or* $|x - y| = -(x - y) = y - x$, *which means that* $y \ge x$ *and so* $\max(x, y) - \min(x, y) = y - x$. *In either case we get* $\max(x, y) - \min(x, y) = |x - y|$. *Solving now the system of equations*

$$\begin{aligned} \max(x, y) + \min(x, y) &= x + y \\ \max(x, y) - \min(x, y) &= |x - y|, \end{aligned}$$

for $\max(x, y)$ *and* $\min(x, y)$ *gives the result.* ❑

Homework

131 Problem Let x, y be real numbers. Then

$$0 \le x < y \iff x^2 < y^2.$$

132 Problem Let $t \ge 0$. Prove that

$$|x| \ge t \iff (x \ge t) \quad \text{or} \quad (x \le -t).$$

133 Problem Let $(x, y) \in \mathbb{R}^2$. Prove that $\max(x, y) = -\min(-x, -y)$.

134 Problem Let x, y, z be real numbers. Prove that

$$\max(x, y, z) = x + y + z - \min(x, y) - \min(y, z) - \min(z, x) + \min(x, y, z).$$

135 Problem Let $a < b$. Demonstrate that

$$|x - a| < |x - b| \iff x < \frac{a+b}{2}.$$

1.7 Classical Inequalities

1.7.1 Triangle Inequality

136 THEOREM (Triangle Inequality) Let $(a, b) \in \mathbb{R}^2$. Then

$$\boxed{|a + b| \le |a| + |b|.} \tag{1.6}$$

Proof: *From* 5 *in Theorem* 127, *by addition,*

$$-|a| \le a \le |a|$$

to

$$-|b| \le b \le |b|$$

we obtain

$$-(|a| + |b|) \le a + b \le (|a| + |b|),$$

whence the theorem follows by applying Theorem 129. ❑

By induction, we obtain the following generalisation to n terms.

137 COROLLARY Let $x_1, x_2, \ldots, x_n$ be real numbers. Then

$$|x_1 + x_2 + \cdots + x_n| \le |x_1| + |x_2| + \cdots + |x_n|.$$

Proof: *We apply Theorem 136 $n-1$ times*

$$\begin{aligned} |x_1 + x_2 + \cdots + x_n| &\le |x_1| + |x_2 + \cdots x_{n-1} + x_n| \\ &\le |x_1| + |x_2| + |x_3 + \cdots x_{n-1} + x_n| \\ &\vdots \\ &\le |x_1| + |x_2| + \cdots + |x_{n-1} + x_n| \\ &\le |x_1| + |x_2| + \cdots + |x_{n-1}| + |x_n|. \end{aligned}$$

❑

138 COROLLARY Let $(a, b) \in \mathbb{R}^2$. Then

$$\boxed{||a| - |b|| \le |a - b|}. \tag{1.7}$$

Proof: *We have*

$$|a| = |a - b + b| \le |a - b| + |b|,$$

giving

$$|a| - |b| \le |a - b|.$$

Similarly,

$$|b| = |b - a + a| \le |b - a| + |a| = |a - b| + |a|,$$

gives

$$|b| - |a| \le |a - b| \implies -|a - b| \le |a| - |b|.$$

Thus

$$-|a - b| \le |a| - |b| \le |a - b|,$$

and we now apply Theorem 129. ❑

139 THEOREM Let $b_i > 0$ for $1 \le i \le n$. Then

$$\min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) \le \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \le \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right).$$

Proof: *For every k,* $\quad 1 \le k \le n$,

$$\min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) \le \frac{a_k}{b_k} \le \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) \implies b_k \min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) \le a_k \le b_k \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right).$$

Adding all these inequalities for $1 \le k \le n$,

$$(b_1 + b_2 + \cdots + b_n) \min\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right) \le a_1 + a_2 + \cdots + a_n \le (b_1 + b_2 + \cdots + b_n) \max\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right),$$

from where the result is obtained.❑

1.7.2 Bernoulli's Inequality

140 THEOREM If $0 \le a < b$, $n \ge 1 \in \mathbb{N}$

$$na^{n-1} < \frac{b^n - a^n}{b - a} < nb^{n-1}.$$

Proof: *By Theorem 99,*

$$\begin{aligned}\frac{b^n - a^n}{b-a} &= b^{n-1} + b^{n-2}a + b^{n-3}a^2 + \cdots + b^2a^{n-3} + ba^{n-2} + a^{n-1} \\ &< b^{n-1} + b^{n-1} + \cdots + b^{n-1} + b^{n-1} \\ &= nb^{n-1},\end{aligned}$$

from where the dextral inequality follows. The sinistral inequality can be established similarly. ❑

141 THEOREM (Bernoulli's Inequality) If $x > -1, x \neq 0,$ and if $n \in \mathbb{N} \setminus \{0\}$ then

$$(1+x)^n > 1 + nx.$$

Proof: *Set $b = 1 + x, a = 1$ in Theorem 140 and use the sinistral inequality.* ❑

☞ *If $x > 0$ then Bernoulli's Inequality is an easy consequence of the Binomial Theorem, as*

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots > 1 + \binom{n}{1}x = 1 + nx.$$

1.7.3 Rearrangement Inequality

142 Definition Given a set of real numbers $\{x_1, x_2, \ldots, x_n\}$ denote by

$$\check{x}_1 \geq \check{x}_2 \geq \cdots \geq \check{x}_n$$

the decreasing rearrangement of the x_i and denote by

$$\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_n$$

the increasing rearrangement of the x_i.

143 Definition Given two sequences of real numbers $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ of the same length n, we say that they are *similarly sorted* if they are both increasing or both decreasing, and *differently sorted* if one is increasing and the other decreasing..

144 Example The sequences $1 \leq 2 \leq \cdots \leq n$ and $1^2 \leq 2^2 \leq \cdots \leq n^2$ are similarly sorted, and the sequences $\frac{1}{1^2} \geq \frac{1}{2^2} \geq \cdots \geq \frac{1}{n^2}$ and $1^3 \leq 2^3 \leq \cdots \leq n^3$ are differently sorted.

145 THEOREM (Rearrangement Inequality) Given sets of real numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ we have

$$\sum_{1 \leq k \leq n} \check{a}_k \hat{b}_k \leq \sum_{1 \leq k \leq n} a_k b_k \leq \sum_{1 \leq k \leq n} \hat{a}_k \hat{b}_k.$$

Thus the sum $\sum_{1 \leq k \leq n} a_k b_k$ is minimised when the sequences are differently sorted, and maximised when the sequences are similarly sorted.

☞ *Observe that*

$$\sum_{1 \leq k \leq n} \check{a}_k \hat{b}_k = \sum_{1 \leq k \leq n} \hat{a}_k \check{b}_k \quad \text{and} \quad \sum_{1 \leq k \leq n} \hat{a}_k \hat{b}_k = \sum_{1 \leq k \leq n} \check{a}_k \check{b}_k.$$

Proof: *Let $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ be a reordering of $\{1, 2, \ldots, n\}$. If there are two sub-indices i, j, such that the sequences pull in opposite directions, say, $a_i > a_j$ and $b_{\sigma(i)} < b_{\sigma(j)}$, then consider the sums*

$$\begin{aligned} S &= a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \cdots + a_i b_{\sigma(i)} + \cdots + a_j b_{\sigma(j)} + \cdots + a_n b_{\sigma(n)} \\ S' &= a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + \cdots + a_i b_{\sigma(j)} + \cdots + a_j b_{\sigma(i)} + \cdots + a_n b_{\sigma(n)} \end{aligned}$$

Then

$$S' - S = (a_i - a_j)(b_{\sigma(j)} - b_{\sigma(i)}) > 0.$$

This last inequality shews that the closer the a's and the b's are to pulling in the same direction the larger the sum becomes. This proves the result. ❑

1.7.4 Arithmetic Mean-Geometric Mean Inequality

146 THEOREM (Arithmetic Mean-Geometric Mean Inequality) Let $a_1, \ldots, a_n$ be positive real numbers. Then their geometric mean is at most their arithmetic mean, that is,

$$\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n},$$

with equality if and only if $a_1 = \cdots = a_n$.

We will provide multiple proofs of this important inequality. Some other proofs will be found in latter chapters.

First Proof: *Our first proof uses the Rearrangement Inequality (Theorem 145) in a rather clever way. We may assume that the a_k are strictly positive. Put*

$$x_1 = \frac{a_1}{(a_1 a_2 \cdots a_n)^{1/n}}, \quad x_2 = \frac{a_1 a_2}{(a_1 a_2 \cdots a_n)^{2/n}}, \quad \ldots, \quad x_n = \frac{a_1 a_2 \cdots a_n}{(a_1 a_2 \cdots a_n)^{n/n}} = 1,$$

and

$$y_1 = \frac{1}{x_1}, \quad y_2 = \frac{1}{x_2}, \quad \ldots, \quad y_n = \frac{1}{x_n} = 1.$$

Observe that for $2 \leq k \leq n$,

$$x_k y_{k-1} = \frac{a_1 a_2 \cdots a_k}{(a_1 a_2 \cdots a_n)^{k/n}} \cdot \frac{(a_1 a_2 \cdots a_n)^{(k-1)/n}}{a_1 a_2 \cdots a_{k-1}} = \frac{a_k}{(a_1 a_2 \cdots a_n)^{1/n}}.$$

The x_k and y_k are differently sorted, so by virtue of the Rearrangement Inequality we gather

$$\begin{aligned} 1 + 1 + \cdots + 1 &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \\ &\leq x_1 y_n + x_2 y_1 + \cdots + x_n y_{n-1} \\ &= \frac{a_1}{(a_1 a_2 \cdots a_n)^{1/n}} + \frac{a_2}{(a_1 a_2 \cdots a_n)^{1/n}} + \cdots + \frac{a_n}{(a_1 a_2 \cdots a_n)^{1/n}}, \end{aligned}$$

or

$$n \leq \frac{a_1 + a_2 + \cdots + a_n}{(a_1 a_2 \cdots a_n)^{1/n}},$$

from where we obtain the result. ❑

Second Proof: *This second proof is a clever induction argument due to Cauchy. It proves the inequality first for powers of 2 and then interpolates for numbers between consecutive powers of 2.*

Since the square of a real number is always positive, we have, for positive real numbers a, b

$$(\sqrt{a} - \sqrt{b})^2 \geq 0 \implies \sqrt{ab} \leq \frac{a+b}{2},$$

proving the inequality for $k = 2$. Observe that equality happens if and only if $a = b$. Assume now that the inequality is valid for $k = 2^{n-1} > 2$. This means that for any positive real numbers $x_1, x_2, \ldots, x_{2^{n-1}}$ we have

$$\left(x_1 x_2 \cdots x_{2^{n-1}}\right)^{1/2^{n-1}} \leq \frac{x_1 + x_2 + \cdots + x_{2^{n-1}}}{2^{n-1}}. \tag{1.8}$$

Let us prove the inequality for $2k = 2^n$. Consider any any positive real numbers $y_1, y_2, \ldots, y_{2^n}$. Notice that there are $2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$ integers in the interval $\left[2^{n-1} + 1\,; 2^n\right]$. We have

$$\begin{aligned} \left(y_1 y_2 \cdots y_{2^n}\right)^{1/2^n} &= \sqrt{\left(y_1 y_2 \cdots y_{2^{n-1}}\right)^{1/2^{n-1}} \left(y_{2^{n-1}+1} \cdots y_{2^n}\right)^{1/2^{n-1}}} \\ &\leq \frac{\left(y_1 y_2 \cdots y_{2^{n-1}}\right)^{1/2^{n-1}} + \left(y_{2^{n-1}+1} \cdots y_{2^n}\right)^{1/2^{n-1}}}{2} \\ &\leq \frac{\dfrac{y_1 + y_2 + \cdots + y_{2^{n-1}}}{2^{n-1}} + \dfrac{y_{2^{n-1}+1} + \cdots + y_{2^n}}{2^{n-1}}}{2} \\ &= \frac{y_1 + \cdots + y_{2^n}}{2^n}, \end{aligned}$$

where the first inequality follows by the Case $n = 2$ *and the second by the induction hypothesis (1.8). The theorem is thus proved for powers of* 2.

Assume now that $2^{n-1} < k < 2^n$, *and consider the* k *positive real numbers* $a_1, a_2, \ldots, a_k$. *The trick is to pad this collection of real numbers up to the next highest power of* 2, *the added real numbers being the average of the existing ones. Hence consider the* 2^n *real numbers*

$$a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_{2^n}$$

with $a_{k+1} = \ldots = a_{2^n} = \dfrac{a_1 + a_2 + \cdots + a_k}{k}$. *Since we have already proved the theorem for* 2^n *we have*

$$\left(a_1 a_2 \cdots a_k \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^{2^n - k}\right)^{1/2^n} \leq \frac{a_1 + a_2 + \cdots + a_k + (2^n - k)\left(\dfrac{a_1 + a_2 + \cdots + a_k}{k}\right)}{2^n},$$

whence

$$(a_1 a_2 \cdots a_k)^{1/2^n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^{1-k/2^n} \leq \frac{k\dfrac{a_1 + a_2 + \cdots + a_k}{k} + (2^n - k)\left(\dfrac{a_1 + a_2 + \cdots + a_k}{k}\right)}{2^n},$$

which implies

$$(a_1 a_2 \cdots a_k)^{1/2^n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^{1-k/2^n} \leq \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right),$$

Solving for $\dfrac{a_1 + a_2 + \cdots + a_k}{k}$ *gives the desired inequality.* ❑

Third Proof: *As in the second proof, the Case* $k = 2$ *is easily established. Put*

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}, \qquad G_k = (a_1 a_2 \cdots a_k)^{1/k}.$$

Observe that

$$a_{k+1} = (k+1)A_{k+1} - kA_k.$$

The inductive hypothesis is that $A_k \geq G_k$ *and we must shew that* $A_{k+1} \geq G_{k+1}$. *Put*

$$A = \frac{a_{k+1} + (k-1)A_{k+1}}{k}, \qquad G = \left(a_{k+1} A_{k+1}^{k-1}\right)^{1/k}.$$

By the inductive hypothesis $A \geq G$. *Now,*

$$\frac{A + A_k}{2} = \frac{\dfrac{(k+1)A_{k+1} - kA_k + (k-1)A_{k+1}}{k} + A_k}{2} = A_{k+1}.$$

Hence

$$\begin{aligned} A_{k+1} &= \frac{A + A_k}{2} \\ &\geq (AA_k)^{1/2} \\ &\geq (GG_k)^{1/2}. \\ &= \left(G_{k+1}^{k+1} A_{k+1}^{k-1}\right)^{1/2k} \end{aligned}$$

We have established that

$$A_{k+1} \geq \left(G_{k+1}^{k+1} A_{k+1}^{k-1}\right)^{1/2k} \implies A_{k+1} \geq G_{k+1},$$

completing the induction. ❑

Fourth Proof: *We will make a series of substitutions that preserve the sum*

$$a_1 + a_2 + \cdots + a_n$$

while strictly increasing the product

$$a_1 a_2 \cdots a_n.$$

At the end, the a_i will all be equal and the arithmetic mean A of the numbers will be equal to their geometric mean G. If the a_i where all $> A$ then $\dfrac{a_1 + a_2 + \cdots + a_n}{n} > \dfrac{nA}{n} = A$, impossible. Similarly, the a_i cannot be all $< A$. Hence there must exist two indices say i, j, such that $a_i < A < a_j$. Put $a_i' = A$, $a_j' = a_i + a_j - A$. Observe that $a_i + a_j = a_i' + a_j'$, so replacing the original a's with the primed a's does not alter the arithmetic mean. On the other hand,

$$a_i' a_j' = A(a_i + a_j - A) = a_i a_j + (a_j - A)(A - a_i) > a_i a_j$$

since $a_j - A > 0$ and $A - a_i > 0$.

This change has replaced one of the a's by a quantity equal to the arithmetic mean, has not changed the arithmetic mean, and made the geometric mean larger. Since there at most n a's to be replaced, the procedure must eventually terminate when all the a's are equal (to their arithmetic mean). Strict inequality then holds when at least two of the a's are unequal. ❑

1.7.5 Cauchy-Bunyakovsky-Schwarz Inequality

147 THEOREM (Cauchy-Bunyakovsky-Schwarz Inequality) Let x_k, y_k be real numbers, $1 \le k \le n$. Then

$$\left| \sum_{k=1}^{n} x_k y_k \right| \le \left(\sum_{k=1}^{n} x_k^2 \right)^{1/2} \left(\sum_{k=1}^{n} y_k^2 \right)^{1/2},$$

with equality if and only if

$$(a_1, a_2, \ldots, a_n) = t(b_1, b_2, \ldots, b_n)$$

for some real constant t.

First Proof: *The inequality follows at once from Lagrange's Identity*

$$\left(\sum_{k=1}^{n} x_k y_k \right)^2 = \left(\sum_{k=1}^{n} x_k^2 \right) \left(\sum_{k=1}^{n} y_k^2 \right) - \sum_{1 \le k < j \le n} (x_k y_j - x_j y_k)^2$$

(Theorem 94), since $\sum_{1 \le k < j \le n} (x_k y_j - x_j y_k)^2 \ge 0$. ❑

Second Proof: *Put* $a = \sum_{k=1}^{n} x_k^2$, $b = \sum_{k=1}^{n} x_k y_k$, *and* $c = \sum_{k=1}^{n} y_k^2$. *Consider the quadratic polynomial*

$$at^2 + bt + c = t^2 \sum_{k=1}^{n} x_k^2 - 2t \sum_{k=1}^{n} x_k y_k + \sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} (t x_k - y_k)^2 \ge 0,$$

where the inequality follows because a sum of squares of real numbers is being summed. Thus this quadratic polynomial is positive for all real t, so it must have complex roots. Its discriminant $b^2 - 4ac$ must be non-positive, from where we gather

$$4 \left(\sum_{k=1}^{n} x_k y_k \right)^2 \le 4 \left(\sum_{k=1}^{n} x_k^2 \right) \left(\sum_{k=1}^{n} y_k^2 \right),$$

which gives the inequality ❑

For our third proof of the CBS Inequality we need the following lemma.

148 Lemma For $(a,b,x,y) \in \mathbb{R}^4$ with $x > 0$ and $y > 0$ the following inequality holds:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}.$$

Equality holds if and only if $\frac{a}{x} = \frac{b}{y}$.

Proof: *Since the square of a real number is always positive, we have*

$$\begin{aligned}(ay-bx)^2 \geq 0 &\implies a^2y^2 - 2abxy + b^2x^2 \geq 0\\ &\implies a^2y(x+y) + b^2x(x+y) \geq (a+b)^2xy\\ &\implies \frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}.\end{aligned}$$

Equality holds if and only if the first inequality is 0.❑

☞ *Iterating the result on Lemma 148,*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\cdots+a_n)^2}{b_1+b_2+\cdots+b_n},$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Third Proof: *By the preceding remark, we have*

$$\begin{aligned}x_1^2 + x_2^2 + \cdots + x_n^2 &= \frac{x_1^2y_1^2}{y_1^2} + \frac{x_2^2y_2^2}{y_2^2} + \cdots + \frac{x_n^2y_n^2}{y_n^2}\\ &\geq \frac{(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2}{y_1^2 + y_2^2 + \cdots + y_n^2},\end{aligned}$$

and upon rearranging, CBS is once again obtained.❑

1.7.6 Minkowski's Inequality

149 Theorem (Minkowski's Inequality) Let x_k, y_k be any real numbers. Then

$$\left(\sum_{k=1}^n (x_k+y_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^n x_k^2\right)^{1/2} + \left(\sum_{k=1}^n y_k^2\right)^{1/2}.$$

Proof: *We have*

$$\begin{aligned}\sum_{k=1}^n (x_k+y_k)^2 &= \sum_{k=1}^n x_k^2 + 2\sum_{k=1}^n x_ky_k + \sum_{k=1}^n y_k^2\\ &\leq \sum_{k=1}^n x_k^2 + 2\left(\sum_{k=1}^n x_k^2\right)^{1/2}\left(\sum_{k=1}^n y_k^2\right)^{1/2} + \sum_{k=1}^n y_k^2\\ &= \left(\left(\sum_{k=1}^n x_k^2\right)^{1/2} + \left(\sum_{k=1}^n y_k^2\right)^{1/2}\right)^2,\end{aligned}$$

where the inequality follows from the CBS Inequality.❑

Homework

150 Problem Let $(a,b,c,d) \in \mathbb{R}^4$. Prove that

$$||a-c| - |b-c|| \leq |a-b| \leq |a-c| + |b-c|.$$

151 Problem Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = x_1^3 + x_2^3 + \cdots + x_n^3 = x_1^4 + x_2^4 + \cdots + x_n^4.$$

Prove that $x_k \in \{0, 1\}$.

152 Problem Let $n \geq 2$ an integer. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1.$$

Prove that $x_1 = x_2 = \cdots = x_n$.

153 Problem If $b>0$ and $B>0$ prove that

$$\frac{a}{b}<\frac{A}{B}\implies\frac{a}{b}<\frac{a+A}{b+B}<\frac{A}{B}.$$

Further, if p and q are positive integers such that

$$\frac{7}{10}<\frac{p}{q}<\frac{11}{15},$$

what is the least value of q?

154 Problem Prove that if $r\geq s\geq t$ then

$$r^2-s^2+t^2\geq(r-s+t)^2.$$

155 Problem Assume that $a_k,b_k,c_k,k=1,\ldots,n$, are positive real numbers. Shew that

$$\left(\sum_{k=1}^{n}a_kb_kc_k\right)^4\leq\left(\sum_{k=1}^{n}a_k^4\right)\left(\sum_{k=1}^{n}b_k^4\right)\left(\sum_{k=1}^{n}c_k^2\right)^2.$$

156 Problem Prove that for integer $n>1$,

$$n!<\left(\frac{n+1}{2}\right)^n.$$

157 Problem Prove that for integer $n>2$,

$$n^{n/2}<n!.$$

158 Problem Prove that for all integers $n\geq 0$ the inequality $n(n-1)<2^{n+1}$ is verified.

159 Problem Prove that $\forall(a,b,c)\in\mathbb{R}^3$,

$$a^2+b^2+c^2\geq ab+bc+ca.$$

160 Problem Prove that $\forall(a,b,c)\in\mathbb{R}^3$, with $a\geq 0$, $b\geq 0$, $c\geq 0$, the following inequalities hold:

$$a^3+b^3+c^3\geq\max(a^2b+b^2c+c^2a,a^2c+b^2a+c^2b),$$

$$a^3+b^3+c^3\geq 3abc,$$

$$a^3+b^3+c^3\geq\frac{1}{2}\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right).$$

161 Problem (Chebyshev's Inequality) Given sets of real numbers $\{a_1,a_2,\ldots,a_n\}$ and $\{b_1,b_2,\ldots,b_n\}$ prove that

$$\frac{1}{n}\sum_{1\leq k\leq n}\check{a}_k\hat{b}_k\leq\left(\frac{1}{n}\sum_{1\leq k\leq n}a_k\right)\left(\frac{1}{n}\sum_{1\leq k\leq n}b_k\right)\leq\frac{1}{n}\sum_{1\leq k\leq n}\hat{a}_k\hat{b}_k.$$

162 Problem If $x>0$, from

$$\sqrt{x+1}-\sqrt{x}=\frac{1}{\sqrt{x+1}+\sqrt{x}},$$

prove that

$$\frac{1}{2\sqrt{x+1}}<\sqrt{x+1}-\sqrt{x}<\frac{1}{2\sqrt{x}}.$$

Use this to prove that if $n>1$ is a positive integer, then

$$2\sqrt{n+1}-2<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$

163 Problem If $0<a\leq b$, shew that

$$\frac{1}{8}\cdot\frac{(b-a)^2}{b}\leq\frac{a+b}{2}-\sqrt{ab}\leq\frac{1}{8}\cdot\frac{(b-a)^2}{a}$$

164 Problem Shew that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000}<\frac{1}{100}.$$

165 Problem Prove that for all $x>0$,

$$\sum_{k=1}^{n}\frac{1}{(x+k)^2}<\frac{1}{x}-\frac{1}{x+n}.$$

166 Problem Let $x_i\in\mathbb{R}$ such that $\sum_{i=1}|x_i|=1$ and $\sum_{i=1}x_i=0$. Prove that

$$\left|\sum_{i=1}^{n}\frac{x_i}{i}\right|\leq\frac{1}{2}\left(1-\frac{1}{n}\right).$$

167 Problem Let n be a strictly positive integer. Let $x_i\geq 0$. Prove that

$$\prod_{k=1}^{n}(1+x_k)\geq 1+\sum_{k=1}^{n}x_k.$$

When does equality hold?

168 Problem (Nesbitt's Inequality) Let a,b,c be strictly positive real numbers. Then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

169 Problem Let $a>0$. Use mathematical induction to prove that

$$\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}<\frac{1+\sqrt{4a+1}}{2},$$

where the left member contains an arbitrary number of radicals.

170 Problem Let a,b,c be positive real numbers. Prove that

$$(a+b)(b+c)(c+a)\geq 8abc.$$

171 Problem (IMO, 1978) Let a_k be a sequence of pairwise distinct positive integers. Prove that

$$\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{1}{k}.$$

172 Problem (Harmonic Mean-Geometric Mean Inequality) Let $x_i>0$ for $1\leq i\leq n$. Then

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\leq(x_1x_2\cdots x_n)^{1/n},$$

with equality iff $x_1=x_2=\cdots=x_n$.

173 Problem (Arithmetic Mean-Quadratic Mean Inequality) Let $x_i\geq 0$ for $1\leq i\leq n$. Then

$$\frac{x_1+x_2+\cdots+x_n}{n}\leq\left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}\right)^{1/2},$$

with equality iff $x_1=x_2=\cdots=x_n$.

174 Problem Given a set of real numbers $\{a_1,a_2,\ldots,a_n\}$ prove that there is an index $m\in\{0,1,\ldots,n\}$ such that

$$\left|\sum_{1\leq k\leq m}a_k-\sum_{m<k\leq n}a_k\right|\leq\max_{1\leq k\leq n}|a_k|.$$

If $m=0$ the first sum is to be taken as 0 and if $m=n$ the second one will be taken as 0.

175 Problem Give a purely geometric proof of Minkowski's Inequality for $n=2$. That is, prove that if $(a,b),(c,d)\in\mathbb{R}^2$, then

$$\sqrt{(a+c)^2+(b+d)^2}\leq\sqrt{a^2+b^2}+\sqrt{c^2+d^2}.$$

Equality occurs if and only if $ad=bc$.

176 Problem Let $x_k\in\left[0\,;1\right]$ for $1\leq k\leq n$. Demonstrate that

$$\min\left(\prod_{k=1}^{n}x_k,\prod_{k=1}^{n}(1-x_k)\right)\leq\frac{1}{2^n}.$$

177 Problem If $n>0$ is an integer and if $a_k>0$, $1\leq k\leq n$ are real numbers, demonstrate that

$$\left(\sum_{k=1}^{n}\frac{a_k}{k}\right)^2\leq\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{a_ja_k}{j+k-1}.$$

178 Problem Let n be a strictly positive integer, let $a_k \geq 0$, $1 \leq k \leq n$ be real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$, and let b_k, $1 \leq k \leq n$ be real numbers. Assume that for all indices $k \in \{1, 2, \ldots, n\}$,

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i.$$

Prove that

$$\sum_{i=1}^{n} a_i^2 \leq \sum_{i=1}^{n} b_i^2$$

179 Problem Let $n \geq 2$ an integer and let a_k, $1 \leq k \leq n$ be real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$. Prove that there is an index $k \in \{1, 2, \ldots, n\}$ such that

$$(a_{k+1} - a_k)^2 \leq \frac{12}{n(n^2-1)}(a_1^2 + a_2^2 + \cdots a_n^2).$$

180 Problem (AIME 1991) Let $P = \{a_1, a_2, \ldots, a_n\}$ be a collection of points with

$$0 < a_1 < a_2 < \cdots < a_n < 17.$$

Consider

$$S_n = \min_P \sum_{k=1}^{n} \sqrt{(2k-1)^2 + a_k^2},$$

where the minimum runs over all such partitions P. Shew that exactly one of $S_2, S_3, \ldots, S_n, \ldots$ is an integer, and find which one it is.

1.8 Completeness Axiom

Why bother? We saw that both $\mathbb{Q}$ and $\mathbb{R}$ are fields, and hence they both satisfy the same arithmetical axioms. Why the need then for $\mathbb{R}$? In this section we will study a property of $\mathbb{R}$ that is not shared with $\mathbb{Q}$, that of completeness. It essentially means that there are no 'holes' on the real line.

181 Definition A number u is an *upper bound* for a set of numbers $A \subseteq \mathbb{R}$ if for all $a \in A$ we have $a \leq u$. The smallest such upper bound is called the *supremum or least upper bound* of the set A, and is denoted by $\sup A$. If $\sup A \in A$ then we say that A has a *maximum* and we denote it by $\max A (= \sup A)$. Similarly, a number l is a *lower bound* for a set of numbers $B \subseteq \mathbb{R}$ if for all $b \in B$ we have $l \leq b$. The largest such lower bound is called the *infimum or greatest lower bound* of the set B, and is denoted by $\inf B$. If $\sup B \in B$ then we say that B has a *minimum* and we denote it by $\inf B (= \inf B)$.

☞ *We define* $\inf(\mathbb{R}) = -\infty$, $\sup(\mathbb{R}) = +\infty$, $\inf(\varnothing) = +\infty$ *and* $\sup(\varnothing) = -\infty$.

182 Definition A set of numbers A is said to be *complete* if every non-empty subset of A which is bounded above has a supremum lying in A.

183 Axiom (Completeness of $\mathbb{R}$) Any non-empty set of real numbers which is bounded above has a supremum. Any non-empty set of real numbers which is bounded below has a infimum.

184 Theorem (Approximation Property of the Supremum and Infimum) Let $A \neq \varnothing$ be a set of real numbers possessing a supremum $\sup A$. Then

$$\forall \varepsilon > 0 \quad \exists a \in A \quad \text{such that} \quad \sup A - \varepsilon \leq a.$$

Let $B \neq \varnothing$ be a set of real numbers possessing an infimum $\inf B$. Then

$$\forall \varepsilon > 0 \quad \exists b \in B \quad \text{such that} \quad \inf A + \varepsilon \geq b.$$

Proof: *If* $\forall a \in A$, $\sup A - \varepsilon > a$ *then* $\sup A - \varepsilon$ *would be an upper bound smaller than the least upper bound, a contradiction to the definition of* $\sup A$. *Hence there must be a rogue* $a \in A$ *such that* $\sup A - \varepsilon \leq a$.

If $\forall b \in A$, $\inf B + \varepsilon < b$ *then* $\inf B + \varepsilon$ *would be a lower bound greater than the greatest lower bound, a contradiction to the definition of* $\inf B$. *Hence there must be a rogue* $b \in B$ *such that* $\inf B + \varepsilon \geq b$.

❑

☞ *The above result should be intuitively clear.* $\sup A$ *sits on the fence, just to the right of* A*, so that going just a bit to the left should put* $\sup A - \varepsilon$ *within* A*, etc.*

185 Theorem (Monotonicity Property of the Supremum and Infimum) Let $\varnothing \subsetneqq A \subseteq B \subsetneqq \mathbb{R}$ and suppose that both A and B have a supremum and an infimum. Then $\sup A \leq \sup B$ and $\inf B \leq \inf A$.

Proof: *Assume* B *is bounded above with supremum* $\sup B$. *Suppose* $x \in A$. *Then* $x \in B$ *and so* $x \leq \sup B$. *Thus* $\sup B$ *is an upper bound for the elements of* A*, and so* A *and so by definition,* $\sup A \leq \sup B$.

Assume B is bounded below with infimum $\inf B$. Suppose $x \in A$. Then $x \in B$ and so $x \geq \inf B$. Thus $\inf B$ is a lower bound for the elements of A and so by definition, $\inf A \geq \inf B$. ❑

186 LEMMA Let a, b be real numbers and assume that for all numbers $\varepsilon > 0$ the following inequality holds:

$$a < b + \varepsilon.$$

Then $a \leq b$.

Proof: *Assume contrariwise that $a > b$. Hence $\frac{a-b}{2} > 0$. Since the inequality $a < b + \varepsilon$ holds for every $\varepsilon > 0$ in particular it holds for $\varepsilon = \frac{a-b}{2}$. This implies that*

$$a < b + \frac{a-b}{2} \quad \text{or} \quad a < b.$$

Thus starting with the assumption that $a > b$ we reach the incompatible conclusion that $a < b$. The original assumption must be wrong. We therefore conclude that $a \leq b$. ❑

187 THEOREM (Additive Property of the Supremum) Let $\varnothing \subsetneqq A \subseteq \mathbb{R}$, and $B \subseteqq \mathbb{R}$. Put

$$A + B = \{x + y : (x, y) \in A \times B\}$$

and suppose that both A and B have a supremum. Then $A + B$ has also a supremum and

$$\sup(A + B) = \sup A + \sup B.$$

Proof: *If $t \in A + B$ then $t = x + y$ with $(x, y) \in A \times B$. Then $t = x + y \leq \sup A + \sup B$, and so $\sup A + \sup B$ is an upper bound for $A + B$. By the Completeness Axiom, $A + B$ is bounded. Thus $\sup(A + B) \leq \sup A + \sup B$.*

We now prove that $\sup A + \sup B \leq \sup(A + B)$. By the approximation property, $\forall \varepsilon > 0\ \exists a \in A$ and $b \in B$ such that $\sup A - \frac{\varepsilon}{2} < a$ and $\sup B - \frac{\varepsilon}{2} < b$. Observe that $a + b \in A + B$ and so $a + b \leq \sup(A + B)$. Then

$$\sup A + \sup B - \varepsilon < a + b \leq \sup(A + B),$$

and by Lemma 186 we must have

$$\sup A + \sup B \leq \sup(A + B).$$

This completes the proof.❑

188 THEOREM (Archimedean Property of the Real Numbers) If $(x, y) \in \mathbb{R}^2$ with $x > 0$, then there exists a natural number n such that $nx > y$.

Proof: *Consider the set*

$$A = \{nx : n \in \mathbb{N}\}.$$

Since $1 \cdot x \in A$, A is non-empty. If $\forall n \in \mathbb{N}$ we had $nx \leq y$, then A would be bounded above by y. By the Completeness Axiom, A would have a supremum $\sup A$. Thus $\forall n \in \mathbb{N}$, $\quad nx \leq \sup A$. Since $(n+1)x \in A$, we would also have

$$(n+1)x \leq \sup A \implies nx \leq \sup A - x.$$

This means that $\sup A - x$ is an upper bound for A which is smaller than its supremum, a contradiction Thus there must be an n for which $nx > y$. ❑

189 COROLLARY $\mathbb{N}$ is unbounded above.

Proof: *This follows by taking $x = 1$ in Theorem 188.* ❑

The Completeness Axioms tells us, essentially, that there are no "holes" in the real numbers. We will see that this property distinguishes the reals from the rational numbers.

190 LEMMA [Hipassos of Metapontum] $\sqrt{2}$ is irrational.

Proof: *Assume there is $s \in \mathbb{Q}$ such that $s^2 = 2$. We can find integers $m, n \neq 0$ such that $s = \dfrac{m}{n}$. The crucial part of the argument is that we can choose m, n such that this fraction be in least terms, and hence, m, n must have opposite parity. Now, $m^2 s^2 = n^2$, that is $2m^2 = n^2$. This means that n^2 is even. But then n itself must be even, since the product of two odd numbers is odd. Thus $n = 2a$ for some non-zero integer a (since $n \neq 0$). This means that $2m^2 = (2a)^2 = 4a^2 \implies m^2 = 2a^2$. This means once again that m is even. But then we have a contradiction, since m and n were of opposite parity.* ❑

191 THEOREM $\mathbb{Q}$ is not complete.

Proof: *We must shew that there is a non-empty set of rational numbers which is bounded above but that does not have a supremum in $\mathbb{Q}$. Consider the set $A = \{r \in \mathbb{Q} : r^2 \leq 2\}$ of rational numbers. This set is bounded above by $u = 2$. For assume that there were a rogue element of A, say r_0 such that $r_0 > 2$. Then $r_0^2 > 4$ and so r_0 would not belong to A, a contradiction. Thus $r \leq 2$ for every $r \in A$ and so A is bounded above. Suppose that A had a supremum s, which must satisfy $s \leq 2$. Now, by Lemma 190 we cannot have $s^2 = 2$ and thus $s^2 < 2$. By Theorem 188 there is an integer n such that $2 - s^2 > \dfrac{1}{10^n}$. Put $t = s + \dfrac{1}{10^{n-1}}$, a rational number and observe that since $s \leq 2$ one has*

$$t^2 = s^2 + \frac{2s}{10^{n-1}} + \frac{1}{10^{2n-2}} < s^2 + \frac{2s}{10^{n-1}} + \frac{1}{10^{n-1}} \leq s^2 + \frac{5}{10^{n-1}} < s^2 + \frac{1}{10^n} < 2.$$

Thus $t \in A$ and $t > s$, that is t is an element of A larger than its least upper bound, a contradiction. Hence A does not have a least upper bound. ❑

1.8.1 Greatest Integer Function

192 THEOREM Given $y \in \mathbb{R}$ there exists a unique integer n such that

$$n \leq y < n + 1.$$

Proof: *By Theorem 188, the set $\{n \in \mathbb{Z} : n \leq y\}$ is non-empty and bounded above. We put*

$$\lfloor y \rfloor = \sup\{n \in \mathbb{Z} : n \leq y\}.$$

❑

$\forall x \in \mathbb{R}, \quad \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1.$

193 Definition The unique integer in Theorem 192 is called the *floor* of x and is denoted by $\lfloor x \rfloor$.

The greatest integer function enjoys the following properties:

194 THEOREM Let $\alpha, \beta \in \mathbb{R}, a \in \mathbb{Z}, n \in \mathbb{N}$. Then

1. $\lfloor \alpha + a \rfloor = \lfloor \alpha \rfloor + a$
2. $\left\lfloor \dfrac{\alpha}{n} \right\rfloor = \left\lfloor \dfrac{\lfloor \alpha \rfloor}{n} \right\rfloor$
3. $\lfloor \alpha \rfloor + \lfloor \beta \rfloor \leq \lfloor \alpha + \beta \rfloor \leq \lfloor \alpha \rfloor + \lfloor \beta \rfloor + 1$

Proof:

1. *Let $m = \lfloor \alpha + a \rfloor$. Then $m \leq \alpha + a < m + 1$. Hence $m - a \leq \alpha < m - a + 1$. This means that $m - a = \lfloor \alpha \rfloor$, which is what we wanted.*

2. *Write α/n as $\alpha/n = \lfloor\alpha/n\rfloor + \theta, 0 \le \theta < 1$. Since $n\lfloor\alpha/n\rfloor$ is an integer, we deduce by (1) that*

$$\lfloor\alpha\rfloor = \lfloor n\lfloor\alpha/n\rfloor + n\theta\rfloor = n\lfloor\alpha/n\rfloor + \lfloor n\theta\rfloor.$$

Now, $0 \le \lfloor n\theta\rfloor \le n\theta < n$, and so $0 \le \lfloor n\theta\rfloor/n < 1$. If we let $\Theta = \lfloor n\theta\rfloor/n$, we obtain

$$\frac{\lfloor\alpha\rfloor}{n} = \lfloor\frac{\alpha}{n}\rfloor + \Theta, \quad 0 \le \Theta < 1.$$

This yields the required result.

3. *From the inequalities $\alpha - 1 < \lfloor\alpha\rfloor \le \alpha, \beta - 1 < \lfloor\beta\rfloor \le \beta$ we get $\alpha + \beta - 2 < \lfloor\alpha\rfloor + \lfloor\beta\rfloor \le \alpha + \beta$. Since $\lfloor\alpha\rfloor + \lfloor\beta\rfloor$ is an integer less than or equal to $\alpha + \beta$, it must be less than or equal to the integral part of $\alpha + \beta$, i.e. $\lfloor\alpha + \beta\rfloor$. We obtain thus $\lfloor\alpha\rfloor + \lfloor\beta\rfloor \le \lfloor\alpha + \beta\rfloor$. Also, $\alpha + \beta$ is less than the integer $\lfloor\alpha\rfloor + \lfloor\beta\rfloor + 2$, so its integer part $\lfloor\alpha + \beta\rfloor$ must be less than $\lfloor\alpha\rfloor + \lfloor\beta\rfloor + 2$, but $\lfloor\alpha + \beta\rfloor < \lfloor\alpha\rfloor + \lfloor\beta\rfloor + 2$ yields $\lfloor\alpha + \beta\rfloor \le \lfloor\alpha\rfloor + \lfloor\beta\rfloor + 1$. This proves the inequalities.*

❑

195 Definition The *ceiling of a real number* x is the unique integer $\lceil x\rceil$ satisfying the inequalities

$$\lceil x\rceil - 1 < x \le \lceil x\rceil.$$

196 Definition The *fractional part of a real number* x is defined and denoted by

$$\{x\} = x - \lfloor x\rfloor.$$

Observe that $0 \le \{x\} < 1$.

Homework

197 Problem Let A and B be non-empty sets of real numbers. Put

$$-A = \{-x : x \in A\}, \qquad A - B = \{a - b : (a, b) \in A \times B\}.$$

Prove that

1. If A is bounded above, then $-A$ is bounded below and $\sup A = -\inf(-A)$.
2. If A and B are bounded above then $A \cup B$ is also bounded above and $\sup(A \cup B) = \max(\sup A, \sup B)$.
3. If A is bounded above and B is bounded below, then $A - B$ is bounded above and $\sup(A - B) = \sup A - \inf B$.

198 Problem Assume that A is a subset of the strictly positive real numbers. Prove that if A is bounded above, then the set $A^{-1} = \{\frac{1}{x} : x \in A\}$ is bounded below and that $\sup A = \frac{1}{\inf A^{-1}}$.

199 Problem Let $n \ge 2$ be an integer. Prove that

$$\max_{0 \le x_1 \le x_2 \le \cdots \le x_n \le 1} \left(\sum_{1 \le i < j \le n} (x_j - x_i) \right) = \lfloor \frac{n^2}{4} \rfloor.$$

200 Problem Find a non-zero polynomial $P(x, y)$ such that

$$P(\lfloor 2t\rfloor, \lfloor 3t\rfloor) = 0$$

for all real t.

201 Problem Prove that the integers

$$\lfloor \left(1 + \sqrt{2}\right)^n \rfloor$$

with n a positive integer, are alternately even or odd.

202 Problem Let $x \in \mathbb{R}$ and let n be a strictly positive integer. Prove that

$$\lfloor nx\rfloor = \sum_{k=1}^{n-1} \lfloor x + \frac{k}{n} \rfloor.$$

203 Problem (Putnam 1948) If n is a positive integer, demonstrate that

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor.$$

204 Problem Find a formula for the n-th non-square.

205 Problem Prove that if a, b are strictly positive integers then

$$\frac{a^2}{b^2} < 2 \implies \frac{(a+2b)^2}{(a+b)^2} < 2.$$

Prove, moreover, that

$$\frac{(a+2b)^2}{(a+b)^2} - 2 < 2 - \frac{a^2}{b^2}.$$

This means that $\frac{(a+2b)^2}{(a+b)^2}$ is closer to 2 than $\frac{a^2}{b^2}$ is.

206 Problem Shew that $\forall x > 0$, x is farther from $\sqrt{5}$ than $\frac{2x+5}{x+2}$ is.

207 Problem (Existence of n-th Roots) Let $a > 0$ and let $n \in \mathbb{R}$, $n \ge 2$. Prove that there is a unique $b \in \mathbb{R}$ such that $b^n = a$.

Chapter 2

Topology of $\mathbb{R}$

2.1 Intervals

Why bother? In this section we give a more precise definition of what an interval is, and establish the interesting property that between any two real numbers there is always a rational number.

208 Definition An *interval* I is a subset of the real numbers with the following property: if $s \in I$ and $t \in I$, and if $s < x < t$, then $x \in I$. In other words, intervals are those subsets of real numbers with the property that every number between two elements is also contained in the set. Since there are infinitely many decimals between two different real numbers, intervals with distinct endpoints contain infinitely many members.

The empty set $\varnothing$ is trivially an interval.

We will now establish that there are nine types of intervals.

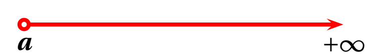
Interval Notation	Set Notation	Graphical Representation
$[a;b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$[1]	
$]a;b[$	$\{x \in \mathbb{R} : a < x < b\}$	
$[a;b[$	$\{x \in \mathbb{R} : a \leq x < b\}$	
$]a;b]$	$\{x \in \mathbb{R} : a < x \leq b\}$	
$]a;+\infty[$	$\{x \in \mathbb{R} : x > a\}$	
$[a;+\infty[$	$\{x \in \mathbb{R} : x \geq a\}$	
$]-\infty;b[$	$\{x \in \mathbb{R} : x < b\}$	
$]-\infty;b]$	$\{x \in \mathbb{R} : x \leq b\}$	
$]-\infty;+\infty[$	$\mathbb{R}$	

Table 2.1: Types of Intervals. Observe that we indicate that the endpoints are included by means of shading the dots at the endpoints and that the endpoints are excluded by not shading the dots at the endpoints.

If $x \in \mathbb{R}$, then $\{x\} = [x;x]$.

209 THEOREM The only kinds of intervals are those sets shewn in Table 2.1, and conversely, all sets shewn in this table are intervals.

Proof: *The converse is easily established, so assume that* $I \subseteqq \mathbb{R}$ *possesses the property that* $\forall (a,b) \in I^2,\quad \left[a\,;b\right] \subseteqq I$. *Since* $\varnothing$ *is an interval one may assume that* $I \neq \varnothing$. *Let* $a \in I$ *be a fixed element of* I *and put* $M_a = \{x \in I : x \leq a\} = \left]-\infty\,;a\right] \cap I$ *and* $N_a = \{x \in I : x \geq a\} = \left[a\,;+\infty\right[\cap I$.

If N_a *is not bounded above, then* $\forall b \in \left[a\,;+\infty\right[,\quad \exists c \in N_a$ *such that* $b \leq c$. *Since* $a \leq b \leq c$, *this entails that* $b \in N_a$. *Thus* $N_a = \left[a\,;+\infty\right[$.

If N_a *is bounded above, then it has supremum* $s = \sup(N_a)$ *and* $N_a \subseteqq \left[a\,;s\right]$. *By Theorem 184,* $\forall b \in \left[a\,;s\right[,\quad c \in N_a$ *such that* $b \leq c$, *and since* $a \leq b \leq c$, *this entails that* $b \in N_a$. *Thus*

$$\left[a\,;s\right[\subseteqq N_a \subseteqq \left[a\,;s\right],$$

and so $N_a = \left[a\,;s\right[$ *or* $N_a = \left[a\,;s\right]$.

Thus N_a *is one among three possible forms:* $\left[a\,;+\infty\right[$, $\left[a\,;s\right]$, *or* $\left[a\,;s\right[$. *Applying a similar reasoning, one obtains gathers that* M_a *is of one of the forms* $\left]-\infty\,;a\right]$, $\left]l\,;a\right]$, *or* $\left[l\,;a\right]$, *where* $l = \inf(M_a)$. *Since* $I = M_a \cup N_a$, *there are* 3 *choices for* M_a *and* 3 *for* N_a, *hence there are* $3 \cdot 3 = 9$ *choices for* I. *The result is established.* ❑

210 Example Determine $\displaystyle\bigcap_{k=1}^{\infty}\left[1-\frac{1}{2^k}\,;1+\frac{1}{k}\right]$.

Solution: *Observe that the intervals are, in sequence,*

$$\left[\frac{1}{2}\,;2\right];\quad \left[\frac{3}{4}\,;\frac{3}{2}\right];\quad \left[\frac{7}{8}\,;\frac{4}{3}\right];\quad \ldots.$$

We claim that $\displaystyle\bigcap_{k=1}^{\infty}\left[1-\frac{1}{2^k}\,;1+\frac{1}{k}\right] = 1$. *For we see that*

$$\forall k \geq 1,\quad \frac{1}{2} \leq 1-\frac{1}{2^k} < 1 < 1+\frac{1}{k} \leq 2,$$

so 1 *is in every interval. Could this intersection contain a number smaller than* 1*? No, for if* $\frac{1}{2} \leq a < 1$, *then we can take* k *large enough so that*

$$a < 1-\frac{1}{2^k},$$

for example

$$a < 1-\frac{1}{2^k} \implies k > -\log_2(1-a),$$

so taking $k \geq \lfloor -\log_2(1-a) \rfloor + 1$ *will work. Could the intersection contain a number* b *larger than* 1*? No, for if* $1 < b < 2$, *then we can take* k *large enough so that*

$$1+\frac{1}{k} < b,$$

for example

$$1+\frac{1}{k} < b \implies k > \frac{1}{b-1},$$

so taking $k \geq \left\lfloor \frac{1}{b-1} \right\rfloor + 1$ *will work. Hence the only number in the intersection is* 1.

2.2 Dense Sets

211 Definition A set $B \subseteq \mathbb{R}$ is *dense in* $A \subseteq \mathbb{R}$ if $\forall (a_1, a_2) \in A^2, \quad a_1 < a_2, \quad \exists b \in B$ such that $a_1 < b < a_2$, that is, between any two different elements of A one can always find an element of B.

212 THEOREM $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: *Let x, y be real numbers with $x < y$. Since there are infinitely many positive integers, there must be a positive integer n such that $n > \frac{1}{y-x}$ by the Archimedean Property of $\mathbb{R}$. Consider the rational number $r = \frac{m}{n}$, where m is the least natural number with $m > nx$. This means that*

$$m > nx \geq m - 1.$$

We claim that $x < \frac{m}{n} < y$. The first inequality is clear, since by choice $x < \frac{m}{n}$. For the second inequality observe that, again

$$nx \geq m-1 \text{ and } y - x > \frac{1}{n} \implies x > \frac{m}{n} - \frac{1}{n} \text{and } y > x + \frac{1}{n} \implies y > \frac{m}{n} - \frac{1}{n} + \frac{1}{n} = \frac{m}{n}.$$

Thus $\frac{m}{n}$ is a rational number between x and y. ❑

213 THEOREM $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: *Let $a < b$ be two real numbers. By Theorem 212, there is a rational number r with $\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$. But then $a < \sqrt{2}r < b$, and the number $\sqrt{2}r$ is an irrational number.* ❑

214 THEOREM (Dirichlet) For any real number θ and any integer $Q \geq 1$, there exist integers a and q, $1 \leq q \leq Q$, such that

$$\left|\theta - \frac{a}{q}\right| \leq \frac{1}{qQ}.$$

Proof: *For $1 \leq n \leq Q$, let*

$$I_n = \left[\frac{n-1}{Q}; \frac{n}{Q}\right[.$$

Thus these Q intervals partition the interval $[0;1[$. The $Q+1$ numbers

$$\{0\theta\}, \{1\theta\}, \{2\theta\}, \ldots, \{Q\theta\}$$

lie in $[0;1[$. Hence by the pigeonhole principle there is an n such that I_n contains at least two of these numbers, say

$$\{q_1\theta\} \in I_n, \quad \{q_2\theta\} \in I_n, \quad 0 \leq q_1 < q_2 \leq Q.$$

Put $q = q_2 - q_1$, $a = [q_2\theta] - [q_1\theta]$. Since $\{q_1\theta\} \in I_n, \{q_2\theta\} \in I_n$ we must have

$$\left|\{q_2\theta\} - \{q_1\theta\}\right| < \frac{1}{Q}.$$

But

$$\{q_2\theta\} - \{q_1\theta\} = q_2\theta - [q_2\theta] - q_1\theta + [q_1\theta] = q\theta - a,$$

whence the result. ❑

215 COROLLARY If θ is irrational prove that there exist infinitely many rational numbers $\frac{a}{q}$, $\gcd(a, q) = 1$, such that θ lies in the open intervals $\left]\frac{a}{q} - \frac{1}{q^2}; \frac{a}{q} + \frac{1}{q^2}\right[$.

Proof: *Suppose that* $\left|\theta - \frac{a_r}{q_r}\right| < \frac{1}{q_r^2}$ *for* $1 \le r \le R$. *Since the differences* $\theta - \frac{a_r}{q_r}$ *are non-zero, we may choose* Q *so large in Theorem 214 that none of these rational numbers is a solution of* $\left|\theta - \frac{a}{q}\right| < \frac{1}{qQ}$. *Since this latter inequality does have a solution, the* R *given rational approximations do not exhaust the set of solutions of* $\left|\theta - \frac{a}{q}\right| < \frac{1}{q^2}$.
❑

Homework

216 Problem Determine $\bigcap_{1\le k\le 500} \left[k\,;1001-k\right]$.

217 Problem Determine $\bigcup_{k=1}^{\infty} \left[1\,;1+\frac{1}{k}\right]$.

218 Problem Determine $\bigcup_{k=1}^{\infty} \left[-k\,;k\right]$.

219 Problem Determine $\bigcap_{k=1}^{\infty} \left[1\,;1+\frac{1}{k}\right]$.

220 Problem Determine $\bigcap_{k=1}^{\infty} \left[k\,;+\infty\right[$.

221 Problem Determine $\bigcap_{k=1}^{\infty} \left]1\,;1+\frac{1}{k}\right]$.

222 Problem Let $I = \left[a\,;b\right]$, and $I' = \left[a'\,;b'\right]$ be closed intervals in $\mathbb{R}$. Prove that $I \subsetneqq I'$ if and only if $a' \le a$ and $b \le b'$.

223 Problem Let
$$\mathbb{Q} + \sqrt{2}\mathbb{Q} = \{a + \sqrt{2}b : (a,b) \in \mathbb{Q}^2\}$$
and define addition on this set as
$$(a+\sqrt{2}b) + (c+\sqrt{2}d) = (a+c) + \sqrt{2}(b+d),$$
and multiplication as
$$(a+\sqrt{2}b)(c+\sqrt{2}d) = (ac+2bd) + \sqrt{2}(ad+bc).$$
Then $\langle \mathbb{Q}+\sqrt{2}\mathbb{Q}, \cdot, +\rangle$ is a field.

224 Problem Put $D = \{x : x = q^2 \quad \text{or} \quad x = -q^2, \quad q \in \mathbb{Q}\}$. Prove that D is dense in $\mathbb{R}$.

225 Problem A *dyadic rational* is a rational number of the form $\frac{m}{2^n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Prove that the set of dyadic rationals is dense in $\mathbb{R}$.

2.3 Open and Closed Sets

Why bother? Many of the properties that we will study in these notes generalise to sets other than $\mathbb{R}$. To better understand what is it from the features of $\mathbb{R}$ that is essential for a generalisation, the language of topology is used.

226 Definition The *open ball* $\mathscr{B}_{x_0}(r)$ *centred at* $x = x_0$ *and radius* $\varepsilon > 0$ is the set
$$\mathscr{B}_{x_0}(\varepsilon) = \left]x_0 - \varepsilon\,; x_0 + \varepsilon\right[.$$

227 Definition A set $\mathscr{N}_{x_0} \subsetneqq \mathbb{R}$ is an *open neighbourhood of a point* x_0 if $\exists \varepsilon > 0$ such that $\mathscr{B}_{x_0}(r) \subsetneqq \mathscr{N}_{x_0}$, that is, there is a sufficiently small open ball containing x_0 completely contained in $\mathscr{N}_{x_0}$.

228 Definition A set $U \subsetneqq \mathbb{R}$ is said to be *open in* $\mathbb{R}$ if $\forall\ x \in U$ there is an open neighbourhood $\mathscr{N}_{x_0}$ such that $\mathscr{N}_{x_0} \subsetneqq U$. A set $F \subseteq \mathbb{R}$ is said to be *closed in* $\mathbb{R}$ if its complement $U = \mathbb{R} \setminus F$ is open in $\mathbb{R}$.

229 Theorem Every open ball is open.

Proof: *Let* $\mathscr{B}_{x_0}(r)$ *with* $r > 0$ *be an open ball and let* $x \in \mathscr{B}_{x_0}(r)$. *We must shew that there is a sufficiently small neighbourhood of* x *completely within* $\mathscr{B}_{x_0}(r)$. *That is, we search for* $\varepsilon > 0$ *such that* $y \in \mathscr{B}_x(\varepsilon) \implies y \in \mathscr{B}_{x_0}(r)$. *Now,*
$$y \in \mathscr{B}_x(\varepsilon) \implies y \in \mathscr{B}_{x_0}(r) \iff |y - x| < \varepsilon \implies |y - x_0| < r.$$
By the Triangle Inequality
$$|y - x_0| \le |y - x| + |x - x_0| < \varepsilon + |x - x_0|.$$
So, as long as
$$\varepsilon + |x - x_0| < r,$$

we will be within $\mathscr{B}_{x_0}(r)$. *One can take*

$$\varepsilon = \frac{r - |x - x_0|}{2}.$$

❑

230 Example The open intervals $\left]a\,;b\right[$, $\left]a\,;+\infty\right[$, $\left]-\infty\,;b\right[$, $\left]-\infty\,;+\infty\right[$, are open in $\mathbb{R}$.

The closed intervals $\{a\}$, $\left[a\,;b\right]$, $\left[a\,;+\infty\right[$, $\left]-\infty\,;b\right]$, $\left]-\infty\,;+\infty\right[=\mathbb{R}$, are closed in $\mathbb{R}$.

The sets $\varnothing$ and $\mathbb{R}$ are simultaneously open and closed in $\mathbb{R}$.

The intervals $\left]a\,;b\right]$ and $\left[a\,;b\right[$ are neither open nor closed in $\mathbb{R}$.

231 Theorem The union of any (finite or infinite) number of open sets in $\mathbb{R}$ is open in $\mathbb{R}$. The union of a finite number of closed in $\mathbb{R}$ sets is closed in $\mathbb{R}$.

The intersection of a finite number of open sets in $\mathbb{R}$ is open in $\mathbb{R}$. The intersection of any (finite or infinite) number of closed sets in $\mathbb{R}$ is closed in $\mathbb{R}$.

Proof: *Let* $U_1, U_2, \ldots,$ *be a sequence of open sets in* $\mathbb{R}$ *(some may be empty) and consider* $x \in \bigcup_{n=1}^{\infty} U_n$. *There is an index* N *such that* $x \in U_N$. *Since* U_N *is open in* $\mathbb{R}$, *there is an open neighbourhood of* x $\left]x-\varepsilon\,;x+\varepsilon\right[\subsetneqq U_N$, *for* $\varepsilon > 0$ *small enough. But then*

$$\left]x-\varepsilon\,;x+\varepsilon\right[\subsetneqq U_N \subsetneqq \bigcup_{n=1}^{\infty} U_n,$$

and so given an arbitrary point of the union, there is a small enough open neighbourhood enclosing the point and within the union, meaning that the union is open.

If $\bigcap_{n=1}^{\infty} F_n$ *is an arbitrary intersection of closed sets, then there are open sets* $U_n = \mathbb{R} \setminus F_n$. *By the De Morgan Laws,*

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} (\mathbb{R} \setminus U_n) = \mathbb{R} \setminus \bigcup_{n=1}^{\infty} U_n,$$

and since $\bigcup_{n=1}^{\infty} U_n$ *is open by the above paragraph,* $\bigcap_{n=1}^{\infty} F_n$ *is the complement of an open set, that is, it is closed.*

Let $U_1, U_2, \ldots, U_L$ *be a sequence of open sets in* $\mathbb{R}$ *and consider* $x \in \bigcap_{n=1}^{L} U_n$. *Then* x *belongs to each of the* U_k *and so there are* $\varepsilon_k > 0$ *such that* $x \in \left]x-\varepsilon_k\,;x+\varepsilon_k\right[\subsetneqq U_k$. *Let* $\varepsilon = \min_{1\leq k\leq L} \varepsilon_k$ *be the smallest one of such. But then for all* k,

$$\left]x-\varepsilon\,;x+\varepsilon\right[\subsetneqq \left]x-\varepsilon_k\,;x+\varepsilon_k\right[\subsetneqq U_k, \implies \left]x-\varepsilon\,;x+\varepsilon\right[\subsetneqq \bigcap_{n=1}^{L} U_n,$$

and so given an arbitrary point of the intersection, there is a small enough open neighbourhood enclosing the point and within the intersection, meaning that the intersection is open.

Using the De Morgan Laws and the preceding paragraph, the remaining statement can be proved. ❑

232 Example The intersection of an infinite number of open sets may not be open. For example

$$\bigcap_{k=1}^{\infty} \left]1 - \frac{1}{n+1}\,;2 - \frac{1}{n+1}\right[= \left[1\,;2\right[,$$

which is neither open nor closed.

233 Theorem (Characterisation of the Open Sets of $\mathbb{R}$) A set $A \subsetneqq \mathbb{R}$ is open if an only if it is the countable union of open sets of $\mathbb{R}$.

2.4 Interior, Boundary, and Closure of a Set

234 Definition Let $A \subseteqq \mathbb{R}$. The *interior* of A is defined and denoted by

$$\mathring{A} = \bigcup_{\substack{\Omega \subseteqq A \\ \Omega \text{ open}}} \Omega,$$

that is, the largest open set inside A. The points of $\mathring{A}$ are called the *interior points of* A.

235 Definition Let $A \subseteqq \mathbb{R}$. The *closure* of A is defined and denoted by

$$\overline{A} = \bigcup_{\substack{\Omega \supseteqq A \\ \Omega \text{ closed}}} \Omega,$$

that is, the smallest closed set containing A. The points of $\overline{A}$ are called the *adherence points of* A.

One always has $\mathring{A} \subseteqq A \subseteqq \overline{A}$. *A set* U *is open if and only if* $U = \mathring{U}$. *A set* F *is closed if and only if* $F = \overline{F}$.

236 Definition Let $A \subseteqq \mathbb{R}$. The *boundary* of A is defined and denoted by

$$\mathbf{Bdy}(A) = \overline{A} - \mathring{A}.$$

The elements of $\mathbf{Bdy}(A)$ are called the *boundary points* of A.

237 Example We have

1. $\widehat{\left]0\,;1\right]}^{\circ} = \left]0\,;1\right[, \quad \overline{\left]0\,;1\right]} = \left[0\,;1\right], \quad \mathbf{Bdy}\left(\left]0\,;1\right]\right) = \{0,1\}$
2. $\mathring{\{0,1\}} = \varnothing, \quad \overline{\{0,1\}} = \{0,1\}, \quad \mathbf{Bdy}(\{0,1\}) = \{0,1\}$
3. $\mathring{\mathbb{Q}} = \varnothing, \quad \overline{\mathbb{Q}} = \mathbb{R}, \quad \mathbf{Bdy}(\mathbb{Q}) = \mathbb{R}$

238 Theorem Let $A \subseteqq \mathbb{R}$. Then

$$\mathbb{R} \setminus \mathring{A} = \overline{\mathbb{R} \setminus A}, \qquad \mathbb{R} \setminus \overline{A} = \mathring{\widehat{\mathbb{R} \setminus A}}.$$

Proof: *The theorem follows from the De Morgan Laws, as*

$$\mathbb{R} \setminus \mathring{A} = \mathbb{R} \setminus \bigcup_{\substack{\Omega \subseteqq A \\ \Omega \text{ open}}} \Omega = \bigcap_{\substack{\Omega \subseteqq A \\ \Omega \text{ open}}} (\mathbb{R} \setminus \Omega) = \bigcap_{\substack{\mathbb{R}\setminus A \subseteqq \mathbb{R}\setminus\Omega \\ \Omega \text{ open}}} (\mathbb{R} \setminus \Omega) = \bigcap_{\substack{\mathbb{R}\setminus A \subseteqq F \\ F \text{ closed}}} F = \overline{\mathbb{R} \setminus A},$$

and

$$\mathbb{R} \setminus \overline{A} = \mathbb{R} \setminus \bigcap_{\substack{F \supseteqq A \\ F \text{ closed}}} F = \bigcup_{\substack{F \supseteqq A \\ F \text{ closed}}} (\mathbb{R} \setminus F) = \bigcup_{\substack{\mathbb{R}\setminus A \supseteqq \mathbb{R}\setminus F \\ F \text{ closed}}} (\mathbb{R} \setminus F) = \bigcap_{\substack{\mathbb{R}\setminus A \supseteqq \Omega \\ \Omega \text{ open}}} \Omega = \mathring{\widehat{\mathbb{R} \setminus A}}.$$

❑

239 Theorem $x \in \overline{A} \iff \forall \mathscr{N}_x, \quad \mathscr{N}_x \cap A \neq \varnothing$. That is, x is an adherent point if and only if every neighbourhood of x has a nonempty intersection with A.

Proof: *Assume* $x \in \overline{A}$ *and let* $r > 0$. *If* $\left]x - r\,; x + r\right[\cap A = \varnothing$, *then* $\left]x - r\,; x + r\right[\subseteqq \mathbb{R} \setminus A$. *Since* $\left]x - r\,; x + r\right[$ *is open, we have—in particular—* $\left]x - r\,; x + r\right[\subseteqq \mathring{\widehat{\mathbb{R} \setminus A}} = \mathbb{R} \setminus \overline{A}$ *by Theorem 238. This means that* $x \notin \overline{A}$, *a contradiction.*

Conversely, assume that for all neighbourhoods $\mathcal{N}_x$ of x we have $\mathcal{N}_x \cap A \neq \varnothing$. If $x \notin \overline{A}$ then $x \in \mathbb{R} \setminus \overline{A} = \overset{\circ}{\widehat{\mathbb{R} \setminus A}}$. Since $\overset{\circ}{\widehat{\mathbb{R} \setminus A}}$ is open there is an $r' > 0$ such that $\left]x - r' \,; x + r'\right[\subseteq \overset{\circ}{\widehat{\mathbb{R} \setminus A}} \subseteq \mathbb{R} \setminus A$. But then $\left]x - r' \,; x + r'\right[\cap A = \varnothing$, a contradiction. ❑

240 THEOREM Let $\varnothing \subsetneqq A \subseteq \mathbb{R}$ be bounded above. Then $\sup A \in \overline{A}$. If, moreover, A is closed then $\sup(A) \in A$.

Proof: *Let $r > 0$. By Theorem 184, there exists $a \in A$ such that $\sup(A) - r < a$, which gives $\left|\sup(A) - a\right| < r$. This shews that $\left]\sup A - r \,; \sup A + r\right[\cap A \neq \varnothing$ regardless of how small $r > 0$ might be and, hence, $\sup(A) \in \overline{A}$ by Theorem 239. If A is closed, then $A = \overline{A}$.* ❑

241 Definition Let $A \subseteq \mathbb{R}$. A point $x \in A$ is called an *isolated point of A* if there exists an $r > 0$ such that $\mathscr{B}_x(r) \cap A = \{x\}$. The set of isolated points of A is denoted by A^*.

A point $y \in \mathbb{R}$ is called an *accumulation point of A in $\mathbb{R}$* if

$$\forall \mathcal{N}_x, \quad (\mathcal{N}_x \setminus \{x\}) \cap A \neq \varnothing,$$

that is, if any neighbourhood of x meets A at a point different than x. The set of accumulation points of A is called the *derived set of A* and is denoted by $\mathbf{Acc}(A)$.

242 Example We have

1. 0 is an isolated point of the set $A = \{0\} \cup \left[1\,; 2\right]$.
2. Every point of the set $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is isolated. This is because we may take $r = \frac{1}{2^{n+2}}$ in the definition of isolated point, and then $\left]\frac{1}{n} - \frac{1}{2^{n+2}} \,; \frac{1}{n} + \frac{1}{2^{n+2}}\right[\cap A = \left\{\frac{1}{n}\right\}$. Observe that $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$ and $\frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)}$ and that $2^{n+2} > \max(n(n+1), n(n-1))$.
3. 0 is an accumulation point of $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

243 THEOREM x is an accumulation point of A if and only if every neighbourhood of x in $\mathbb{R}$ has an infinite number of points of A.

Proof: *Suppose $x \in A'$. Suppose a neighbourhood of x had only finitely many elements of A, say $\{y_1, y_2, \ldots, y_n\}$. Take $2r = \min_{1 \leq k \leq n} |y_k - x|$. Then $\left(\left]x - r \,; x + r\right[\setminus \{x\}\right) \cap A = \varnothing$ contradicting the fact that every neighbourhood of x meets A at a point different from x.*

Conversely if every neighbourhood of x in $\mathbb{R}$ has an infinite number of points of A, then a fortiori, any intersection of such a neighbourhood with A will contain a point different from x, and so $x \in \mathbf{Acc}(A)$. ❑

244 THEOREM A set is closed if and only if it contains all its accumulation points.

Proof: *If A is closed then $\mathbb{R} \setminus A$ is open. If $c \in \mathbb{R} \setminus A$ then there exists $r > 0$ such that $\left]c - r \,; c + r\right[\subseteq \mathbb{R} \setminus A$, a neighbourhood that clearly does not contain any points of A, which means $c \notin \mathbf{Acc}(A)$.*

Conversely, suppose a set $\mathbf{Acc}(A) \subseteq A$. We will prove that $\mathbb{R} \setminus A$ is open. If $x \in \mathbb{R} \setminus A$, then a fortiori, *$x \notin \mathbf{Acc}(A)$. This means that there is an $r > 0$ such that $\left]x - r \,; x + r\right[\cap A = \varnothing$. Hence $\left]x - r \,; x + r\right[\subseteq \mathbb{R} \setminus A$, and so $\mathbb{R} \setminus A$ is open.* ❑

One has

$$A^* \subseteq A, \quad \overline{A} - A \subseteq \mathbf{Acc}(A), \quad A^* \cap \mathbf{Acc}(A) = \varnothing, \quad A^* \cup \mathbf{Acc}(A) = \overline{A}.$$

2.5 Connected Sets

245 Definition A set $X \subseteq \mathbb{R}$ is connected if, given open sets U, V of $\mathbb{R}$ with $U \cup V = X$, $U \cap V = \varnothing$, either $U = \varnothing$ or $V = \varnothing$.

246 Theorem If $X \subseteq \mathbb{R}$ is connected, and if $(a, c) \in X^2$, $b \in \mathbb{R}$, are such that $a < b < c$ then $b \in X$.

247 Corollary The only connected sets of $\mathbb{R}$ are the intervals. In particular, $\mathbb{R}$ is connected.

2.6 Compact Sets

248 Definition A sequence of open sets $U_1, U_2, \ldots$ is said to be an *open cover* for $A \subseteq \mathbb{R}$ if $A \subseteq \bigcup_{n=1}^{\infty} U_n$. $U_1, U_2, \ldots$ has a *subcover* $U_{k_1}, U_{k_2}, \ldots$ of A if $A \subseteq \bigcup_{n=1}^{\infty} U_{k_n}$.

249 Definition A set of real numbers is said to be *compact in* $\mathbb{R}$ if every open cover of the set has a finite subcover.[2]

250 Example Since $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} \left]n-1\,; n+1\right[$, the sequence of intervals $\left]n-1\,; n+1\right[$, $\quad n \in \mathbb{Z}$ is a cover for $\mathbb{R}$.

251 Theorem Let a, b be real numbers with $a \leq b$. The closed interval $\left[a\,; b\right]$ is compact in $\mathbb{R}$.

Proof: *Let $U_1, U_2, \ldots$ be an open cover for $\left[a\,; b\right]$. Let E be the collection of all $x \in \left[a\,; b\right]$ such that $\left[a\,; x\right]$ has a finite subcover from the U_i. We will shew that $b \in E$.*

Since $a \in \bigcup_{i=1}^{\infty} U_i$, there exists U_r such that $a \in U_r$. Thus $\{a\} = \left[a\,; a\right] \subseteq U_r$ and so $E \neq \varnothing$. Clearly, b is an upper bound for E. By the Completeness Axiom, $\sup E$ exists. We will shew that $b = \sup E$.

By Theorem 240, $\sup E \in \left[a\,; b\right] \subseteq \bigcup_{i=1}^{\infty} U_i$, hence there exists U_s such that $\sup E \in U_s$. Since U_s is open, there exists $\varepsilon > 0$ such that $\left]\sup E - \varepsilon\,; \sup E + \varepsilon\right[\subseteq U_s$. By Theorem 184 there is $x \in E$ such that $\sup E - \varepsilon < x \leq \sup E$. Thus there is a finite subcover from the U_i, say, $U_{p_1}\ U_{p_2}, \ldots, U_{p_n}$ such that $\left[a\,; x\right] \subseteq \bigcup_{i=1}^{n} U_{k_i}$.

We thus have

$$\left[a\,; \sup E\right] \subseteq \left[a\,; x\right] \bigcup \left]\sup E - \varepsilon\,; \sup E + \varepsilon\right[\subseteq \left(\bigcup_{i=1}^{n} U_{k_i}\right) \cup U_s,$$

a finite subcover. This means that $\sup E \in E$.

Suppose now that $\sup E < b$, and consider $y = \sup E + \frac{1}{2}\min(b - \sup E, \varepsilon)$. Then

$$\sup E < y, \qquad \left[a\,; y\right] = \left[a\,; \sup E\right] \cup \left[\sup E\,; y\right] \subseteq \left(\bigcup_{i=1}^{n} U_{k_i}\right) \cup U_s,$$

whence $y \in E$, contradicting the definition of $\sup E$. This proves that $\sup E = b$ and finishes the proof of the theorem. ❑

252 Theorem (Heine-Borel) A set A of $\mathbb{R}$ is closed and bounded if and only if it is compact.

[2] This definition is appropriate for $\mathbb{R}$ but it is not valid in general. However, it very handy for one-variable calculus, hence we will retain it.

Proof: *Let A be closed and bounded in $\mathbb{R}$, and let $U_1, U_2, \ldots,$ be an open cover for A. There exist $(a,b) \in \mathbb{R}^2$, $a \leq b$, such that $A \subseteqq \left[a\,;b\right]$. Since*

$$\left[a\,;b\right] \subseteqq (\mathbb{R} \setminus A) \cup \bigcup_{i=1}^{\infty} U_i,$$

by Theorem 251 there is a finite subcover of the U_i, say, U_{k_i} such that

$$\left[a\,;b\right] \subseteqq (\mathbb{R} \setminus A) \cup \bigcup_{i=1}^{\infty} U_{k_i}.$$

Therefore

$$A = A \cap \left[a\,;b\right] \subseteqq \left[a\,;b\right] \subseteqq \bigcup_{i=1}^{\infty} U_{k_i},$$

and so A admits an open subcover.

Conversely, suppose that every open cover of A admits a finite subcover. The open cover $\left]-n\,;n\right[, n \in \mathbb{R}$ of A must admit a finite subcover by our assumption, hence there is $N \in \mathbb{N}$ such that $A \subseteqq \left]-N\,;N\right[$, meaning that A is bounded. Let us shew now that $\mathbb{R} \setminus A$ is open.

Let $x \in \mathbb{R} \setminus A$. We have

$$\bigcup_{n \geq 1} \left(\mathbb{R} \setminus \left[x - \frac{1}{n}\,; x + \frac{1}{n}\right]\right) = \mathbb{R} \setminus \bigcap_{n \geq 1} \left[x - \frac{1}{n}\,; x + \frac{1}{n}\right] = \mathbb{R} \setminus \{x\} \supseteqq A,$$

since $x \notin A$. By hypothesis there is $N \in \mathbb{N}$ and $n_1, n_2, \ldots, n_N$ such that

$$A \subseteqq \bigcup_{k=1}^{N} \left(\mathbb{R} \setminus \left[x - \frac{1}{n_k}\,; x + \frac{1}{n_k}\right]\right) \subseteqq \mathbb{R} \setminus \left[x - \frac{1}{n_m}\,; x + \frac{1}{n_m}\right],$$

where $m = \max(n_1, n_2, \ldots, n_N)$. This gives $\left[x - \frac{1}{n_m}\,; x + \frac{1}{n_m}\right] \subseteqq \mathbb{R} \setminus A$, meaning that $\mathbb{R} \setminus A$ is open, whence A is closed.

❑

253 COROLLARY (Cantor's Intersection Theorem) Let

$$\left[a_1\,;b_1\right] \supseteqq \left[a_2\,;b_2\right] \supseteqq \left[a_3\,;b_3\right] \supseteqq \ldots$$

be a sequence of non-empty, bounded, nested closed intervals. Then

$$\bigcap_{j=1}^{\infty} \left[a_j\,;b_j\right] \neq \varnothing.$$

Proof: *Assume that $\left[a_1\,;b_1\right] \cap \bigcap_{j=2}^{\infty} \left[a_j\,;b_j\right] = \varnothing$. Then*

$$\left[a_1\,;b_1\right] \subseteqq \mathbb{R} \setminus \bigcap_{j=2}^{\infty} \left[a_j\,;b_j\right] = \bigcup_{j=2}^{\infty} \left(\mathbb{R} \setminus \left[a_j\,;b_j\right]\right).$$

The $\mathbb{R} \setminus \left[a_j\,;b_j\right]$ for an open cover for $\left[a_1\,;b_1\right]$, which is closed and bounded. By Theorem 7 we have

$$\left[a_j\,;a_j\right] \subseteqq \left[a_i\,;b_i\right] \implies \mathbb{R} \setminus \left[a_i\,;b_i\right] \subseteqq \mathbb{R} \setminus \left[a_j\,;b_j\right].$$

By the Heine-Borel Theorem 252 there is a finite subcover, say

$$\left[a_1\,;b_1\right] \subseteqq \bigcup_{j=1}^{N} \left(\mathbb{R} \setminus \left[a_{n_j}\,;b_{n_j}\right]\right) \subseteqq \mathbb{R} \setminus \left[a_{n_N}\,;b_{n_N}\right].$$

But then $\left[a_{n_N}\,;b_{n_N}\right] \subseteqq \mathbb{R} \setminus \left[a_1\,;b_1\right]$, which contradicts $\left[a_{n_N}\,;b_{n_N}\right] \subseteqq \left[a_1\,;b_1\right]$, and the proof is complete.❑

254 THEOREM (Bolzano-Weierstrass) Ever bounded infinite set of $\mathbb{R}$ has at least one accumulation point.

> **Proof:** *Let A be a bounded set of $\mathbb{R}$ with $\mathbf{Acc}(A) = \varnothing$. Then $A^* = A = \overline{A}$. Notice that then every element of A is an isolated point of A, and hence,*
>
> $$\forall a \in A, \quad \exists r_a > 0, \quad \text{such that} \quad \left]a - r_a\,; a + r_a\right[\cap A = \{a\}.$$
>
> *Observe that*
>
> $$A \subseteqq \bigcup_{a \in A} \left]a - r_a\,; a + r_a\right[,$$
>
> *and so the $\left]a - r_a\,; a + r_a\right[$ form an open cover for A. Since $A = \overline{A}$, A is closed. By the Heine-Borel Theorem 252 A has a finite subcover from among the $\left]a - r_a\,; a + r_a\right[$ and so there exists an integer $N > 0$ and a_i such that*
>
> $$A \subseteqq \bigcup_{i=1}^{N} \left]a_i - r_{a_i}\,; a_i + r_{a_i}\right[.$$
>
> *Since*
>
> $$A = A \cap \bigcup_{i=1}^{N} \left]a_i - r_{a_i}\,; a_i + r_{a_i}\right[= \bigcup_{i=1}^{N} \{a_i\},$$
>
> *A has only N elements and thus it is finite.* ❑

255 THEOREM Let $X \subseteqq \mathbb{R}$. Then the following are equivalent.

1. X is compact.
2. X is closed and bounded.
3. every infinite set of X has an accumulation point.
4. every infinite sequence of X has a converging subsequence in X.

Homework

256 Problem Give an example shewing that the union of an infinite number of closed sets is not necessarily closed.

257 Problem Prove that a set $A \subseteqq \mathbb{R}$ is dense if and only if $\overline{A} = \mathbb{R}$.

258 Problem For any set $A \subseteqq \mathbb{R}$ prove that $\mathbf{Bdy}(A) = \mathbf{Bdy}(\mathbb{R} \setminus A)$.

259 Problem Let $A \neq \varnothing$ be a subset of $\mathbb{R}$. Assume that A is bounded above. Prove that $\mathbf{sup}(A) = \mathbf{sup}(\overline{A})$.

260 Problem Demonstrate that the only subsets of $\mathbb{R}$ which are simultaneously open and closed in $\mathbb{R}$ are $\varnothing$ and $\mathbb{R}$. One codifies this by saying that $\mathbb{R}$ is *connected.*

261 Problem Prove that the closed additive subgroups of the real numbers are (i) just zero; or (ii) all integral multiples of a fixed non-zero number (which may be assumed positive); or (iii) or all reals.

262 Problem Let $A \in \mathbb{R}$. Prove the following

1. $\overline{\overline{A}} = \overline{A}$
2. $\mathring{\mathring{A}} = \mathring{A}$
3. $A \subseteqq B \implies \overline{A} \subseteqq \overline{B}$
4. $A \subseteqq B \implies \mathring{A} \subseteqq \mathring{B}$
5. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
6. $\overline{A \cap B} \subseteqq \overline{A} \cap \overline{B}$
7. $\mathring{A} \cup \mathring{B} \subseteqq \overset{\circ}{\widehat{A \cup B}}$
8. $\overset{\circ}{\widehat{A \cap B}} = \mathring{A} \cap \mathring{B}$

2.7 $\overline{\overline{\mathbb{R}}}$

Why bother? The algebraic rules introduced here will simplify some computations and statements in subsequent chapters.

Geometrically, each real number can be viewed as a point on a straight line. We make the convention that we orient the real line with $\mathbf{0}$ as the origin, the positive numbers increasing towards the right from $\mathbf{0}$ and the negative numbers decreasing towards the left of $\mathbf{0}$, as in figure 2.1.

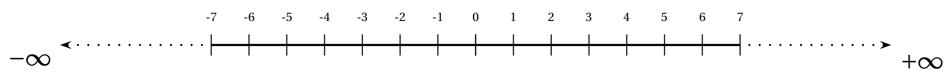


Figure 2.1: The Real Line.

We append the object $+\infty$, which is larger than any real number, and the object $-\infty$, which is smaller than any real number. Letting $x \in \mathbb{R}$, we make the following conventions.

$$(+\infty)+(+\infty) = +\infty \tag{2.1}$$

$$(-\infty)+(-\infty) = -\infty \tag{2.2}$$

$$x+(+\infty) = +\infty \tag{2.3}$$

$$x+(-\infty) = -\infty \tag{2.4}$$

$$x(+\infty) = +\infty \quad \text{if } x > 0 \tag{2.5}$$

$$x(+\infty) = -\infty \quad \text{if } x < 0 \tag{2.6}$$

$$x(-\infty) = -\infty \quad \text{if } x > 0 \tag{2.7}$$

$$x(-\infty) = +\infty \quad \text{if } x < 0 \tag{2.8}$$

$$\frac{x}{\pm\infty} = 0 \tag{2.9}$$

Observe that we leave the following undefined:

$$\frac{\pm\infty}{\pm\infty}, \quad (+\infty)+(-\infty), \quad 0(\pm\infty).$$

263 Definition We denote by $\overline{\overline{\mathbb{R}}} = \left[-\infty\,;+\infty\right]$ the set of real numbers such with the two symbols $-\infty$ and $+\infty$ appended, obeying the algebraic rules above. Observe that then every set in $\overline{\mathbb{R}}$ has a supremum (it may as well be $+\infty$ if the set is unbounded by finite numbers) and an infimum (which may be $-\infty$).

2.8 Lebesgue Measure

264 Definition Let $(a,b) \in \mathbb{R}^2$. The *measure* of the open interval $\left]a\,;b\right[$ is $b-a$. We denote this by $\mu\left(\left]a\,;b\right[\right) = b-a$. If $G = \bigcup_{k=1}^{\infty} \left]a_k\,;b_k\right[$ is a union of disjoint, bounded, open intervals, then $\mu(G) = \sum_{k=1}^{\infty}(b_k - a_k)$.

265 Definition Let $E \subseteqq \mathbb{R}$ be a bounded set. The *outer measure of E* is defined and denoted by

$$\overline{\mu}(E) = \inf_{\substack{E \subseteqq O \\ O \text{ open}}} \mu(O).$$

266 Definition A set $E \subseteqq \mathbb{R}$ is said to be *Lebesgue measurable* if $\forall \varepsilon > 0$, $\exists G \supseteqq E$ open such that $\overline{\mu}(G \setminus E) < \varepsilon$. In this case $\mu(E) = \overline{\mu}(E)$.

2.9 The Cantor Set

267 Definition (The Cantor Set) The Cantor set C is the canonical example of an uncountable set of measure zero. We construct C as follows.

Begin with the unit interval $C_0 = \left[0\,;1\right]$, and remove the middle third open segment $R_1 := \left]\frac{1}{3}\,;\frac{2}{3}\right[$. Define C_1 as

$$C_1 := C_0 \setminus R_1 = \left[0\,;\frac{1}{3}\right] \bigcup \left[\frac{2}{3}\,;1\right] \tag{2.10}$$

Iterate this process on each remaining segment, removing the open set

$$R_2 := \left]\frac{1}{9}\,;\frac{2}{9}\right[\bigcup \left]\frac{7}{9}\,;\frac{8}{9}\right[\tag{2.11}$$

to form the four-interval set

$$C_2 := C_1 \setminus R_2 = \left[0\,;\frac{1}{9}\right] \bigcup \left[\frac{2}{9}\,;\frac{1}{3}\right] \bigcup \left[\frac{2}{3}\,;\frac{7}{9}\right] \bigcup \left[\frac{8}{9}\,;1\right] \tag{2.12}$$

Continue the process, forming $C_3, C_4, \ldots$ Note that C_k has 2^k pieces.

At each step, the endpoints of each closed segment will remain in the set. See figure 2.2.

The *Cantor set* is defined as

$$C := \bigcap_{k=1}^{\infty} C_k = C_0 \setminus \bigcup_{n=1}^{\infty} R_n \tag{2.13}$$

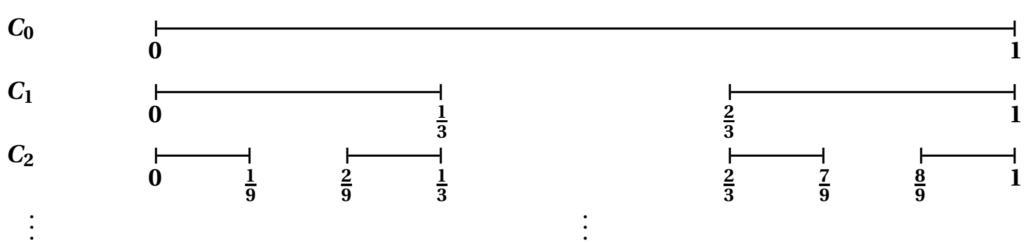


Figure 2.2: Construction of the Cantor Set.

268 THEOREM (Cardinality of the Cantor Set) The Cantor Set is uncountable.

Proof: *Starting with the two pieces of C_1, we mark the sinistral segment "0" and the dextral segment "1". We then continue to C_2, and consider only the leftmost pair. Again, mark the segments "0" and "1", and do the same for the rightmost pair. Successively then, mark the 2^{k-1} leftmost segments of C_k "0" and the 2^{k-1} rightmost segments "1." The elements of the Cantor Set are those with infinite binary expansions. Since there uncountable many such expansions, the Cantor Set in uncountable.* ❑

269 THEOREM (Measure of the Cantor Set) The Cantor Set has (Lebesgue) measure 0.

Proof: *Using the notation of Definition 267, observe that*

$$\mu(R_1) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \tag{2.14}$$

$$\mu(R_2) = \left(\frac{2}{9} - \frac{1}{9}\right) + \left(\frac{8}{9} - \frac{7}{9}\right) = \frac{2}{9} \tag{2.15}$$

$$\vdots \tag{2.16}$$

$$\mu(R_k) = \sum_{n=1}^{k} \frac{2^{n-1}}{3^n} \tag{2.17}$$

Since the R's are disjoint, the measure of their union is the sum of their measures. Taking the limit as $k \to \infty$,

$$\mu\left(\bigcup_{n=1}^{\infty} R_n\right) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1. \tag{2.18}$$

Since clearly $\mu(C_0) = 1$, we then have

$$\mu(C) = \mu\left(C_0 \setminus \bigcup_{n=1}^{\infty} R_n\right) = \mu(C_0) - \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 - 1 = 0. \tag{2.19}$$

❑

270 THEOREM The Cantor set is closed and its interior is empty.

Proof: *Each of $C_0, C_1, C_2, \ldots$, is closed, being the union of a finite number of closed intervals. Thus the Cantor Set is closed, as it is the intersection of closed sets.*

Now, let I be an open interval. Since the numbers of the form $\frac{m}{3^n}$, $(m, n) \in \mathbb{Z}$ are dense in the reals, there is exists a rational number $\frac{m}{3^n} \in I$. Expressed in ternary, this rational number has a finite expansion. If this expansion contains the digit "1", then this number does not belong to Cantor Set, and we are done. If not, since I is open, there must exist a number $k > n$ such that $\frac{m}{3^n} + \frac{1}{3^k} \in I$. By construction, the last digit of the ternary expansion of this number is also "1", and hence this number does not belong to the Cantor Set either.❑

Chapter 3

Sequences

3.1 Limit of a Sequence

Why bother? The *limit* concept is at the centre of calculus. We deal with discrete quantities first, that is, with limits of sequences.

271 Definition A *(numerical) sequence* is a function $a : \mathbb{N} \to \mathbb{R}$. We usually denote $a(n)$ by a_n.[1]

☞ *We will use the notation* $\{a_n\}_{n=k}^{l}$ *to denote the sequence* $a_k, a_{k+1} \ldots, a_l$. *For example*

$$\{a_n\}_{n=0}^{10} = \{a_0, a_1, a_2, \ldots, a_{10}\},$$

$$\{b_n\}_{n=4}^{6} = \{b_4, b_5, b_6\},$$

$$\left\{\left(1+\frac{1}{n}\right)^n\right\}_{n=1}^{+\infty} = \{2, \frac{9}{4}, \frac{64}{27}, \ldots, \},$$

etc.

272 Example The *Harmonic sequence* is

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \ldots,$$

or $a_n = \dfrac{1}{n}$ for $n \geq 1$.

273 Definition A sequence $\{a_n\}_{n=1}^{+\infty}$ is *bounded* if there exists a constant $K > 0$ such that $\forall n, |a_n| \leq K$. It is *increasing* if for all $n > 0$, $a_n \leq a_{n+1}$ and *decreasing* if for all $n \geq 0$, $a_n \geq a_{n+1}$.

3.2 Convergence of Sequences

274 Definition A sequence $\{a_n\}_{n=1}^{+\infty}$ is said to *converge* if

$$\exists L \in \mathbb{R}, \forall \varepsilon > 0, \quad \exists N > 0 \quad \text{such that} \quad \forall n \in \mathbb{N}, \quad n \geq N \implies |a_n - L| < \varepsilon.$$

In other words, eventually[2] the differences

$$|a_n - L|, |a_{n+1} - L|, |a_{n+2} - L|, \ldots$$

remain smaller that an arbitrarily prescribed small quantity. We denote the fact that the sequence $\{a_n\}_{n=1}^{+\infty}$ converges to L as $n \to +\infty$ by

$$\lim_{n \to +\infty} a_n = L, \quad \text{or by} \quad a_n \to L \quad \text{as} \quad n \to +\infty.$$

[1] It is customary to start at $n = 1$ rather than $n = 0$. We won't be too fuzzy about such complications, but we will be careful to write sense.

[2] A good word to use in informal speech "eventually" will mean "for large enough values" or in the case at hand $\forall n \geq N$ for some strictly positive integer N.

A sequence that does not converge is said to *diverge*. Thus a sequence diverges if

$$\forall L\in\mathbb{R}, \exists \varepsilon>0, \forall N\in\mathbb{N}, \exists n\in\mathbb{N} \quad \text{such that} \quad n>N \quad \text{and} \quad |a_n-L|\geq\varepsilon.$$

☞ *Given a sequence sequence* $\{a_n\}_{n=1}^{+\infty}$ *and* $L\in\mathbb{R}$,

$$a_n\to L \quad \text{as} \quad n\to+\infty \quad \text{ifandonlyif} \quad \liminf a_n=\limsup a_n=\lim a_n=L.$$

275 Definition A sequence $\{b_n\}_{n=1}^{+\infty}$ *diverges to plus infinity* if $\forall M>0, \quad \exists N>0$ such that $\forall n\geq N, \quad b_n>M$. A sequence $\{c_n\}_{n=1}^{+\infty}$ *diverges to minus infinity* if $\forall M>0, \quad \exists N>0$ such that $\forall n\geq N, \quad c_n<-M$. A sequence that diverges to plus or minus infinity is said to *properly diverge*. Otherwise it is said to *oscillate*.

276 Definition Given a sequence $\{a_n\}_{n=1}^{+\infty}$, we say that $\lim\limits_{n\to+\infty} a_n$ *exists* it is either convergent or properly divergent.

277 Example The constant sequence

$$1,1,1,1,\ldots$$

converges to 1. Similarly, if a sequence is eventually stationary, that is, constant, it converges to that constant.

278 Example Consider the sequence

$$1,\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n},\ldots,$$

We claim that $\frac{1}{n}\to 0$ as $n\to+\infty$. Suppose we wanted terms that get closer to 0 by at least $.00001=\frac{1}{10^5}$. We only need to look at the 100000-term of the sequence: $\frac{1}{100000}=\frac{1}{10^5}$. Since the terms of the sequence get smaller and smaller, any term after this one will be within $.00001$ of 0. We had to wait a long time—till after the 100000-th term—but the sequence eventually did get closer than $.00001$ to 0. The same argument works for any distance, no matter how small, so we can eventually get arbitrarily close to 0. A rigorous proof is as follows. If $\varepsilon>0$ is no matter how small, we need only to look at the terms after $N=\left\lfloor\frac{1}{\varepsilon}+1\right\rfloor$ to see that, indeed, if $n>N$, then

$$s_n=\frac{1}{n}<\frac{1}{N}=\frac{1}{\left\lfloor\frac{1}{\varepsilon}+1\right\rfloor}<\varepsilon.$$

Here we have used the inequality

$$t-1<\lfloor t\rfloor\leq t, \quad \forall t\in\mathbb{R}.$$

279 Example The sequence

$$0,1,4,9,16,\ldots,n^2,\ldots$$

diverges to $+\infty$, as the sequence gets arbitrarily large. A rigorous proof is as follows. If $M>0$ is no matter how large, then the terms after $N=\lfloor\sqrt{M}\rfloor+1$ satisfy ($n>N$)

$$t_n=n^2>N^2=(\lfloor\sqrt{M}\rfloor+1)^2>M.$$

280 Example The sequence

$$1,-1,1,-1,1,-1,\ldots,(-1)^n,\ldots$$

has no limit (diverges), as it bounces back and forth from -1 to $+1$ infinitely many times.

281 Example The sequence

$$0,-1,2,-3,4,-5,\ldots,(-1)^n n,\ldots,$$

has no limit (diverges), as it is unbounded and alternates back and forth positive and negative values..

We will now see some properties of limits of sequences.

282 THEOREM (Uniqueness of Limits) If $a_n \to L$ and $a_n \to L'$ as $n \to +\infty$ then $L = L'$.

Proof: *The statement only makes sense if both L and L' are finite, so assume so. If $L \neq L'$, take $\varepsilon = \dfrac{|L-L'|}{2} > 0$ in the definition of convergence. Now*

$$\lim_{n\to+\infty} a_n = L \implies \exists N_1 > 0, \quad \forall n \geq N_1\, |a_n - L| < \varepsilon,$$

$$\lim_{n\to+\infty} a_n = L' \implies \exists N_2 > 0, \quad \forall n \geq N_2\, \left|a_n - L'\right| < \varepsilon.$$

If $n > \max(N_1, N_2)$ then by the Triangle Inequality (Theorem 136) then

$$\left|L - L'\right| \leq |L - a_n| + \left|a_n - L'\right| < 2\varepsilon = \left|L - L'\right|,$$

a contradiction, so $L = L'$. ❑

283 THEOREM Every convergent sequence is bounded.

Proof: *Let $\{a_n\}_{n=1}^{+\infty}$ converge to L. Using $\varepsilon = 1$ in the definition of convergence, $\exists N > 0$ such that*

$$n \geq N \implies |a_n - L| < 1 \implies L - 1 < a_n < L + 1,$$

hence, eventually, a_n does not exceed $L + 1$. ❑

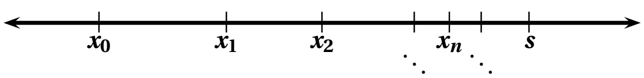


Figure 3.1: Theorem **??**.

When is it guaranteed that a sequence of real numbers has a limit? We have the following result.

284 THEOREM Every bounded increasing sequence $\{a_n\}_{n=0}^{+\infty}$ of real numbers converges to its supremum. Similarly, every bounded decreasing sequence of real numbers converges to its infimum.

Proof: *The idea of the proof is sketched in figure 3.1. By virtue of Axiom 183, the sequence has a supremum s. Every term of the sequence satisfies $a_n \leq s$. We claim that eventually all the terms of the sequence are closer to s than a preassigned small distance $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound for the sequence, there must be a term of the sequence, say a_{n_0} with $s - \varepsilon \leq a_{n_0}$ by virtue of the Approximation Property Theorem 184. Since the sequence is increasing, we then have*

$$s - \varepsilon \leq a_{n_0} \leq a_{n_0+1} \leq a_{n_0+2} \leq a_{n_0+2} \leq \ldots \leq s,$$

which means that after the n_0-th term, we get to within ε of s.

To obtain the second half of the theorem, we simply apply the first half to the sequence $\{-a_n\}_{n=0}^{+\infty}$. ❑

285 THEOREM (Order Properties of Sequences) Let $\{a_n\}_{n=1}^{+\infty}$ be a sequence of real numbers converging to the real number L. Then

1. If $a < L$ then eventually $a < a_n$.
2. If $L < b$ then eventually $a_n < b$.
3. If $a < L < b$ then eventually $a < a_n < b$.
4. If eventually $a_n \geq a$ then $L \geq a$.

5. If eventually $a_n \le b$ then $L \le b$.
6. If eventually $a \le a_n \le b$ then $a \le L \le b$.

Proof: *We apply the definition of convergence repeatedly.*

1. *Taking* $\varepsilon = L - a$ *in the definition of convergence,* $\exists N_1 > 0$ *such that*

$$\forall n \ge N_1, \quad |a_n - L| < L - a \implies \forall n \ge N_1, \quad a - L < a_n - L < L - a \implies \forall n \ge N_1, \quad a < a_n,$$

that is, eventually $a < a_n$.

2. *Taking* $\varepsilon = b - L$ *in the definition of convergence,* $\exists N_2 > 0$ *such that*

$$\forall n \ge N_2, \quad |a_n - L| < b - L \implies \forall n \ge N_2, \quad L - b < a_n - L < b - L \implies \forall n \ge N_2, \quad a_n < b,$$

that is, eventually $a_n < b$.

3. *It suffices to take* $N = \max(N_1, N_2)$ *above.*
4. *If, to the contrary,* $L > a$*, then by part (1) we will eventually have* $a_n > a$*, a contradiction.*
5. *If, to the contrary,* $L < b$*, then by part (2) we will eventually have* $a_n < b$*, a contradiction.*
6. *If either* $L < a$ *or* $b < L$ *we would obtain a contradiction to parts (4) or (5).*

❑

286 THEOREM (Sandwich Theorem) Let $\{a_n\}_{n=1}^{+\infty}$, $\{u_n\}_{n=1}^{+\infty}$, $\{v_n\}_{n=1}^{+\infty}$ be sequences of real numbers such that eventually

$$u_n \le a_n \le v_n.$$

If for $L \in \mathbb{R}$, $u_n \to L$ and $v_n \to L$ then $a_n \to L$.

Proof: *For all* $\varepsilon > 0$ *there are* $N_1 > 0$, $N_2 > 0$ *such that*

$$\forall n \ge \max(N_1, N_2), \quad |u_n - L| < \varepsilon, \quad |v_n - L| < \varepsilon \implies -\varepsilon < u_n - L < \varepsilon, \quad -\varepsilon < v_n - L < \varepsilon.$$

Thus for such n*,*

$$-\varepsilon < u_n - L \le a_n - L \le v_n - L < \varepsilon, \implies -\varepsilon < a_n - L < \varepsilon \implies |a_n - L| < \varepsilon,$$

from where $\{a_n\}_{n=1}^{+\infty}$ *converges to* L. ❑

287 THEOREM Let $\{a_n\}_{n=1}^{+\infty}$ be a sequence of real numbers such that $a_n \to L$. Then $|a_n| \to |L|$.

Proof: *From Corollary 137, we have the inequality* $||a_n| - |L|| \le |a_n - L|$ *from where the result follows.* ❑

288 THEOREM Let $\{a_n\}_{n=1}^{+\infty}$ be a sequence of real numbers such that $a_n \to 0$, and let $\{b_n\}_{n=1}^{+\infty}$ be a bounded sequence. Then $a_n b_n \to 0$.

Proof: *Eventually* $|a_n| < \varepsilon$. *Assume that eventually* $|b_n| \le U$. *Then*

$$|a_n b_n| \le U|a_n| < U\varepsilon,$$

from where the result follows. ❑

289 THEOREM If $b_n \to l \ne 0$ then b_n is eventually different from 0 and $\dfrac{1}{b_n} \to \dfrac{1}{l}$.

Proof: *By Theorem 288,* $|b_n| \to |l|$. *Using* $\varepsilon = \frac{|l|}{2} > 0$ *in the definition of convergence, we have that eventually*

$$||b_n| - |l|| < \frac{|l|}{2} \implies |l| - \frac{|l|}{2} < |b_n| < |l| + \frac{|l|}{2} \implies \frac{|l|}{2} < |b_n|,$$

That is, eventually $|b_n|$ *is strictly positive and so* $\frac{1}{b_n}$ *makes sense. Also, eventually,* $\frac{1}{|b_n|} < \frac{2}{|l|}$ *Now, for sufficiently large* n,

$$\left|\frac{1}{b_n} - \frac{1}{l}\right| = \left|\frac{l - b_n}{|b_n||l|}\right| = \frac{|b_n - l|}{|b_n||l|} < \frac{2\varepsilon}{|l||l|},$$

from where the result follows.❑

290 THEOREM (Algebraic Properties of Sequences) Let $k \in \mathbb{R}$. If $\{a_n\}_{n=1}^{+\infty}$ converges to L and $\{b_n\}_{n=1}^{+\infty}$ converges to L' then

$$\lim_{n\to+\infty}(ka_n + b_n) = kL + L', \qquad \lim_{n\to+\infty}(a_n b_n) = LL'.$$

Moreover, if $L' \neq 0$ then

$$\lim_{n\to+\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{L'}.$$

Proof: *The trick in all these proofs is the following observation: If one multiplies a bounded quantity by an arbitrarily small quantity, one gets an arbitrarily small quantity. Hence once considers the absolute value of the desired terms of the sequence from the expected limit.*

Given $\varepsilon > 0$ *there are* $N_1 > 0$ *and* $N_2 > 0$ *such that* $|a_n - L| < \varepsilon$ *and* $|b_n - L'| < \varepsilon$. *Then*

$$|(ka_n + b_n) - (kL + L')| = |(ka_n - kL) + (b_n - L')| \le |k||a_n - L| + |b_n - L'| < \varepsilon(k+1),$$

and so the sinistral side is arbitrarily close to 0, *establishing the first assertion.*

For the product, observe that by Theorem 283 there exists a constant $K > 0$ *such that* $|b_n| < K$. *Hence*

$$|a_n b_n - LL'| = |(a_n - L)b_n + L(b_n - L')| \le |a_n - L||b_n| + |L||b_n - L'| < \varepsilon K + |L|\varepsilon = \varepsilon(K + |L|),$$

and again, the sinistral side is made arbitrarily close to 0.

Finally, if $L' \neq 0$ *then by Theorem 289,* b_n *is eventually* $\neq 0$ *and* $\frac{1}{b_n} \to \frac{1}{L'}$. *We now simply apply the result we obtained for products, giving*

$$a_n b_n \to L\left(\frac{1}{L'}\right) = \frac{L}{L'}.$$

❑

Homework

291 Problem If $\forall n > 0,\ a_n > 0$ and $\{a_n\}_{n=1}^{+\infty}$ converges to L must it be the case that $L > 0$?

292 Problem Prove that if $a_n \to +\infty$ and if $\{b_n\}_{n=1}^{+\infty}$ is bounded, then $a_n + b_n \to +\infty$.

293 Problem Prove that if $a_n \to +\infty$ and $b_n \to +\infty$ is bounded, then $a_n + b_n \to +\infty$.

294 Problem Prove that if $a_n \to +\infty$ and if there exists $K > 0$ such that eventually $b_n \ge K$, then $a_n b_n \to +\infty$.

295 Problem Prove that if $a_n \to +\infty$ and $b_n \to +\infty$ is bounded, then $a_n b_n \to +\infty$.

296 Problem Prove that if $a_n \to +\infty$ and if $\{b_n\}_{n=1}^{+\infty}$ is bounded, then $a_n + b_n \to +\infty$.

297 Problem Prove that if $a_n \to +\infty$ then $\frac{1}{a_n} \to 0$.

298 Problem Prove that if $a_n \to 0$ and if eventually $a_n > 0$, then $\frac{1}{a_n} \to +\infty$.

299 Problem Prove that $\sum_{i=1}^{n} \frac{n}{n^2 + i} \to 1$ as $n \to +\infty$.

300 Problem Prove that $\frac{1}{(n!)^{1/n}} \to 0$.

301 Problem Prove that $\frac{2^n}{n!} \to 0$.

302 Problem Let $x_1, x_2, \ldots$ be a *bounded* sequence of real numbers, and put $s_n = x_1 + x_2 + \cdots + x_n$. Suppose that $\frac{s_{n^2}}{n^2} \to 0$. Prove that $\frac{s_n}{n} \to 0$.

303 Problem Prove rigorously that the sequence $\{\sin n\}_{n=0}^{+\infty}$ is divergent.

304 Problem Prove that $(n!)^{1/n} \to +\infty$ as $n \to +\infty$.

305 Problem A sequence of real numbers $a_1, a_2, \ldots$ satisfies, for all m, n, the inequality

$$|a_m + a_n - a_{m+n}| \le \frac{1}{m+n}.$$

Prove that this sequence is an arithmetic progression.

306 Problem Prove rigorously that $\sqrt{n+1} - \sqrt{n} \to 0$ as $n \to +\infty$.

307 Problem Prove that the sequence $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges to $+\infty$.

308 Problem Find

$$\lim_{K \to +\infty} \sum_{n=1}^{K} \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n})}.$$

309 Problem What reasonable meaning can be given to

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{\cdots}}}} \quad ?$$

310 Problem Prove that

$$\frac{1+2+\cdots+n}{n^2} \to \frac{1}{2}, \quad \text{as } n \to +\infty.$$

311 Problem Calculate the following limits:

1. $\lim_{n \to +\infty} \left(\frac{1^2}{n^2} + \frac{2^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2} \right)$,
2. $\lim_{n \to +\infty} \left(\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \right)$,
3. $\lim_{n \to +\infty} \left(\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n(n+1)(n+2)} \right)$.

312 Problem What reasonable meaning can be given to

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}} \quad ?$$

313 Problem Let $K \in \mathbb{N} \setminus \{0\}$, and let $a_1, \ldots, a_K, \lambda_1, \ldots, \lambda_K$ be strictly positive real numbers. Prove that

$$\lim_{n \to +\infty} \left(\sum_{k=1}^{K} \lambda_k a_k^n \right)^{1/n} = \max_{1 \le k \le K} a_k, \qquad \lim_{n \to +\infty} \left(\sum_{k=1}^{K} \lambda_k a_k^{-n} \right)^{-1/n} = \min_{1 \le k \le K} a_k.$$

314 Problem Prove that if $\left\{ \frac{a_n}{b_n} \right\}_{n=1}^{+\infty}$ is a monotonic sequence, then the $\left\{ \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \right\}_{n=1}^{+\infty}$ is also monotonic in the same sense.

315 Problem Let a, b, c be real numbers such that $b^2 - 4ac < 0$. Let $\{X_n\}_{n=1}^{+\infty}$, $\{Y_n\}_{n=1}^{+\infty}$ be sequences of real numbers such that

$$aX_n^2 + bX_nY_n + cY_n^2 \to 0, \quad \text{as } n \to +\infty.$$

Prove that $X_n \to 0$ and $Y_n \to 0$ as $n \to +\infty$.

316 Problem (Gram's Product) Prove that

$$\lim_{n \to +\infty} \prod_{k=2}^{n} \frac{k^3 - 1}{k^3 + 1} = \frac{2}{3}.$$

317 Problem Prove that the sequence $\{x_n\}_{n=1}^{+\infty}$ with $x_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ satisfies $x_n \le 2 - \frac{1}{n}$ for $n \ge 1$. Hence deduce that it converges.

318 Problem Prove the convergence of the sequence $x_n = \sum_{k=1}^{n} \frac{1}{n+k}$, $n \ge 1$.

319 Problem Prove the convergence of the sequence, $x_1 = a$, $x_2 = b$, $x_{n+1} = \frac{x_n + x_{n-1}}{2}$, $n \ge 2$ and $(a, b) \in \mathbb{R}^2$, $a \ne b$. Also, find its limit.

320 Problem Prove the convergence of the sequence, $x_1 = a$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{b}{x_n}\right)$, $n \ge 1$ and $(a, b) \in \mathbb{R}^2$, $a > 0, b > 0$. Also, find its limit.

321 Problem Prove the convergence of the sequence, $x_1 = a$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{b}{x_n}\right)$, $n \ge 1$ and $(a, b) \in \mathbb{R}^2$, $a < 0, b > 0$. Also, find its limit.

322 Problem Let $(a, b) \in \mathbb{R}^2$, $a > b > 0$. Set $a_1 = \frac{a+b}{2}$, $b_1 = \sqrt{ab}$. If for $n > 1$,

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_n = \sqrt{a_n b_n},$$

Prove that

1. $\{a_n\}_{n=1}^{+\infty}$ is monotonically decreasing,
2. $\{b_n\}_{n=1}^{+\infty}$ is monotonically inreasing,
3. both sequences converge,
4. their limits are equal.

3.3 Classical Limits of Sequences

Why bother? In this section we will obtain various classical limits. In particular, we define the constant e and obtain a few interesting results about it.

323 THEOREM Let $r \in \mathbb{R}$ be fixed. If $|r| < 1$ then $r^n \to 0$ as $n \to +\infty$. If $|r| > 1$ then $r^n \to +\infty$ as $n \to +\infty$.

Proof: *Taking* $x = \left|\frac{1}{r}\right| - 1$ *in Bernoulli's Inequality (Theorem 141), we find*

$$\left|\frac{1}{r}\right|^n > 1 + n\left(\left|\frac{1}{r}\right| - 1\right) > n\left(\left|\frac{1}{r}\right| - 1\right).$$

Therefore

$$|r|^n < \frac{|r|}{n(1-|r|)} \to 0,$$

as $n \to +\infty$, *since* $\frac{1}{n} \to 0$ *as* $n \to +\infty$.

If $|r| > 1$, *again by Bernoulli's Inequality*

$$|r|^n = (1 + |r| - 1)^n > 1 + n(|r| - 1),$$

and the dextral side can be made arbitrarily large.❑

324 THEOREM Let $|r| < 1$. Then

$$1 + r + r^2 + \cdots + r^n \to \frac{1}{1-r}, \quad \text{as} \quad n \to +\infty.$$

Proof: *If* $S_n = 1 + r + r^2 + \cdots + r^n$ *then* $rS_n = r + r^2 + r^3 + + \cdots + r^{n+1}$ *and*

$$S_n - rS_n = 1 - r^{n+1} \implies S_n = \frac{1 - r^{n+1}}{1-r}.$$

Then apply Theorem 323. ❑

☞ *An estimating trick that we will use often is the following. If* $0 < r < 1$ *then the truncated sum is smaller than the infinite sum and so, for all positive integers* k*:*

$$1 + r + r^2 + \cdots + r^k < 1 + r + r^2 + \cdots + \cdots = \frac{1}{1-r}.$$

325 THEOREM Let $a \in \mathbb{R}$, $a > 0$, be fixed. Then $a^{1/n} \to 1$ as $n \to +\infty$.

Proof: *If* $a > 1$ *then* $a^{1/n} > 1$ *and by Bernoulli's Inequality,*

$$a = (1 + (a^{1/n} - 1))^n > 1 + n(a^{1/n} - 1) \implies 0 \le a^{1/n} - 1 < \frac{a-1}{n},$$

whence $a^{1/n} - 1 \to 0$ *as* $n \to +\infty$.

If $0 < a < 1$ *then* $b = \frac{1}{a} > 1$ *and so by what we just proved,*

$$b^{1/n} \to 1 \implies \frac{1}{a^{1/n}} \to 1 \implies a^{1/n} \to 1,$$

proving the theorem.❑

326 THEOREM Let $a \in \mathbb{R}$, $a > 1$, $k \in \mathbb{N} \setminus \{0\}$, be fixed. Then $\frac{a^n}{n^k} \to +\infty$ as $n \to +\infty$.

Proof: *Observe that* $a^{1/k} > 1$. *We have, using the Binomial Theorem,*

$$\left(a^{1/k}\right)^n = \left(1 + (a^{1/k} - 1)\right)^n = \sum_{i=0}^{n} \binom{n}{i} (a^{1/k} - 1)^i.$$

Since each term of the above expansion is ≥ 0, *we gather that*

$$\left(a^{1/k}\right)^n \ge \frac{n(n-1)}{2}(a^{1/k} - 1)^2 \implies \frac{\left(a^{1/k}\right)^n}{n} \ge \frac{(n-1)}{2}(a^{1/k} - 1)^2 \implies \frac{\left(a^{1/k}\right)^n}{n} \to +\infty \implies \frac{a^n}{n^k} \to +\infty,$$

as desired. ❑

☞ *In particular* $\frac{2^n}{n} \to +\infty$ *as* $n \to +\infty$.

327 THEOREM Let $a \in \mathbb{R}$, , be fixed. Then $\frac{a^n}{n!} \to 0$ as $n \to +\infty$.

Proof: *Put* $N = \lfloor |a| \rfloor + 1$ *and let* $n \geq N$. *Then*

$$\left|\frac{a^n}{n!}\right| = \left(\frac{|a|}{1} \cdot \frac{|a|}{2} \cdots \frac{|a|}{N}\right)\left(\frac{|a|}{N+1} \cdot \frac{|a|}{N+2} \cdots \frac{|a|}{n}\right) \leq \left(\frac{|a|^N}{N!}\right)\left(1 \cdot 1 \cdots 1 \cdot \frac{|a|}{n}\right) \to 0,$$

as $n \to +\infty$. ❑

328 THEOREM The sequence

$$e_n = \left(1 + \frac{1}{n}\right)^n, n = 1, 2, \ldots$$

is a bounded increasing sequence, and hence it converges to a limit, which we call e. Also, for all strictly positive integers n, $\left(1 + \frac{1}{n}\right)^n < e$.

Proof: *By Theorem 140*

$$\frac{b^{n+1} - a^{n+1}}{b - a} \leq (n+1)b^n \implies b^n[(n+1)a - nb] < a^{n+1}.$$

Putting $a = 1 + \frac{1}{n+1}$, $b = 1 + \frac{1}{n}$ *we obtain*

$$e_n = \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} = e_{n+1},$$

whence the sequence $e_n, n = 1, 2, \ldots$ *increases. Again, by putting* $a = 1$, $b = 1 + \frac{1}{2n}$ *we obtain*

$$\left(1 + \frac{1}{2n}\right)^n < 2 \implies \left(1 + \frac{1}{2n}\right)^{2n} < 4 \implies e_{2n} < 4.$$

Since $e_n < e_{2n} < 4$ *for all* n, *the sequence is bounded above. In view of Theorem 284 the sequence converges to a limit. We call this limit* e. *It follows also from this proof and from Theorem 285 that for all strictly positive integers* n, $\left(1 + \frac{1}{n}\right)^n < e$. ❑

☞ *Another way of obtaining* $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$ *is as follows. Using the AM-GM Inequality with* $x_1 = 1, x_2 = \cdots = x_{n+1} = 1 + \frac{1}{n}$ *we have*

$$\left(1 + \frac{1}{n}\right)^{n/(n+1)} < \frac{1 + n\left(1 + \frac{1}{n}\right)}{n+1} \implies \left(1 + \frac{1}{n}\right)^{n/(n+1)} < \frac{n+2}{n+1} = \left(1 + \frac{1}{n+1}\right)$$

from where the desired inequality is obtained.

329 THEOREM The sequence $\left\{\left(1 + \frac{1}{n}\right)^{n+1}\right\}_{n=1}^{+\infty}$ is strictly decreasing and $\left(1 + \frac{1}{n}\right)^{n+1} \to e$. Also, for all strictly positive integers n, $\left(1 + \frac{1}{n}\right)^{n+1} > e$.

Proof: *By Theorem 140*

$$\frac{b^{n+1} - a^{n+1}}{b - a} \geq (n+1)a^n.$$

Putting $a = 1 + \frac{1}{n+1}$, $b = 1 + \frac{1}{n}$ *we obtain*

$$\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}\left(\frac{n^3 + 4n^2 + 4n + 1}{n(n+2)^2}\right).$$

The result will follow as long as $\left(\frac{n^3+4n^2+4n+1}{n(n+2)^2}\right) > 1$. *But*

$$n(n+2)^2 = n(n^2+4n+4) = n^3+4n^2+4n < n^3+4n^2+4n+1 \implies \frac{n^3+4n^2+4n+1}{n(n+2)^2} > 1.$$

Thus the sequence is a sequence of strictly decreasing sequence of real numbers. Putting $a = 1$, $b = 1+\frac{1}{n}$ *in* $\frac{b^{n+1}-a^{n+1}}{b-a} \geq (n+1)a^n$ *we get*

$$\left(1+\frac{1}{n}\right)^{n+1} > 1+n(n+1) > 2,$$

so the sequence is bounded below. In view of Theorem 284 the sequence converges to a limit L. *To see that* $L = e$ *observe that*

$$\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) \to e\cdot 1 = e.$$

It follows also from this proof and from Theorem 285 that for all strictly positive integers n, $\left(1+\frac{1}{n}\right)^{n+1} > e$. ❑

☞ *The inequality* $\left(1+\frac{1}{n+1}\right)^{n+2} < \left(1+\frac{1}{n}\right)^{n+1}$ *can be obtained by the Harmonic Mean-Geometric Mean Inequality by putting* $x_1 = 1, x_2 = x_2 = \cdots = x_{n+2} = 1+\frac{1}{n}$

$$\frac{n+2}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_{n+2}}} \leq (x_1x_2\cdots x_{n+2})^{1/(n+2)} \implies \frac{n+2}{1+(n+1)\left(\frac{n}{n+1}\right)} < \left(1+\frac{1}{n}\right)^{(n+1)/(n+2)}.$$

330 THEOREM $2 < e < 3$.

Proof: *By the Binomial Theorem*

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot\frac{1}{n^k}.$$

Now, for $2 \leq k \leq n$,

$$\binom{n}{k}\cdot\frac{1}{n^k} = \frac{1}{k!}\cdot\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^k} = \frac{1}{k!}\cdot(1)\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\left(1-\frac{k-1}{n}\right) \leq \frac{1}{2\cdot 3\cdots k} \leq \frac{1}{2^{k-1}}.$$

Thus

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot\frac{1}{n^k} \leq 1+1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}} < 1+1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}}+\cdots < 1+2 = 3,$$

by Theorem 324 (with $r = \frac{1}{2}$*), and so the dextral inequality is proved. The sinistral inequality follows from Theorem 328.* ❑

☞ $e = 2.7182818284590452353602874713 52\ldots$

331 THEOREM $e = \lim_{n\to+\infty}\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)$.

Proof: *Put* $y_k = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}$. *Clearly* $y_{k+1} > y_k$ *so that* $\{y_k\}_{k=1}^{+\infty}$ *is an increasing sequence. We will prove that it is bounded above with supremum* e. *By the Binomial Theorem*

$$\left(1+\frac{1}{n}\right)^n = \sum_{j=0}^{n}\binom{n}{j}\cdot\frac{1}{n^j} = 1+\binom{n}{1}\frac{1}{n}+\cdots+\binom{n}{k}\frac{1}{n^k}+\cdots+\binom{n}{n}\frac{1}{n^n} \geq 1+\binom{n}{1}\frac{1}{n}+\cdots+\binom{n}{k}\frac{1}{n^k},$$

for $0 < k < n$. *Now let* j *be fixed,* $0 < j < n$. *Taking limits as* $n \to +\infty$,

$$\binom{n}{j}\cdot\frac{1}{n^j} = \frac{1}{j!}\cdot\frac{n(n-1)(n-2)\cdots(n-k+1)}{n^j} = \frac{1}{j!}\cdot(1)\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\left(1-\frac{j-1}{n}\right) \implies \lim_{n\to+\infty}\binom{n}{j}\cdot\frac{1}{n^j} = \frac{1}{j!}.$$

Hence, taking limits as $n \to +\infty$,

$$\left(1+\frac{1}{n}\right)^n \geq 1+\binom{n}{1}\frac{1}{n}+\cdots+\binom{n}{k}\frac{1}{n^k} \implies e \geq 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}=y_k,$$

or renaming,

$$e \geq 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=y_n. \tag{3.1}$$

Moreover, since $\binom{n}{k}\cdot\frac{1}{n^k}=\frac{1}{k!}\cdot(1)\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdot\left(1-\frac{k-1}{n}\right) \leq \frac{1}{2\cdot 3\cdots k} \leq \frac{1}{k!}$, *we have*

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1+\binom{n}{1}\frac{1}{n}+\cdots+\binom{n}{k}\frac{1}{n^k}+\cdots+\binom{n}{n}\frac{1}{n^n}\\ &\leq 1+\frac{1}{1!}+\cdots+\frac{1}{k!}+\cdots+\frac{1}{n!}\\ &= y_n.\end{aligned} \tag{3.2}$$

In conclusion, from 3.1 and 3.2 we get

$$\left(1+\frac{1}{n}\right)^n \leq y_n \leq e,$$

and by taking limits and using the Sandwich Theorem, we get that $y_n \to e$ *as* $n \to +\infty$. ❑

332 Lemma Let n, m be strictly positive integers and let $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=y_n$. Then $y_{m+n}-y_n<\frac{1}{n\cdot n!}$.

Proof: *We have*

$$\begin{aligned} y_{m+n}-y_n &= \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots+\frac{1}{(n+m)!}\\ &< \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}\cdots+\frac{1}{(n+2)^{m-1}}\right)\\ &< \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)^2}+\cdots+\cdots\right)\\ &= \frac{1}{(n+1)!}\left(\frac{1}{1-\frac{1}{n+2}}\right)\\ &= \frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}.\end{aligned}$$

Here the second inequality follows by using the estimating trick deduced from Theorem 324. Observe that this bound is independent of m. ❑

333 Lemma Let $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}=y_n$. Then $0<e-y_n<\frac{1}{n!n}$.

Proof: *From Lemma 332,*

$$0<y_{m+n}-y_n<\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}.$$

Taking the limit as $m \to +\infty$ *we deduce*

$$0<e-y_n\leq\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}.$$

(The first inequality is strict by Theorem 331.) We only need to shew that for integer $n \geq 1$

$$\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}<\frac{1}{n!n}.$$

But working backwards (which we are allowed to do, as all quantities are strictly positive),

$$\begin{aligned}\frac{1}{(n+1)!}\cdot\frac{n+2}{n+1}<\frac{1}{n!n} &\Leftarrow n!n(n+2)<(n+1)!(n+1)\\ &\Leftarrow n(n+2)<(n+1)(n+1)\\ &\Leftarrow n^2+2n<n^2+2n+1\\ &\Leftarrow 0<1,\end{aligned}$$

and the theorem is proved. ❑

334 THEOREM e is irrational.

Proof: *Assume e is rational, with $e=\frac{p}{q}$, where p and q are positive integers and the fraction is in lowest terms. Since $qe=p$, an integer, $q!e$ must also be an integer. Also $q!y_q$ must be an integer, since*

$$q!y_q=q!\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{q!}\right).$$

But by Lemma 333,

$$0<e-y_q<\frac{1}{q!q}\implies 0<q!(e-y_q)<\frac{1}{q}\le 1.$$

That is, the integer $q!(e-y_q)$ is strictly between 0 and 1, a contradiction. ❑

335 THEOREM The sequence $\left\{n^{1/n}\right\}_{n=1}^{+\infty}$ is decreasing for $n\ge 3$. Also, $n^{1/n}\to 1$ as $n\to+\infty$.

Proof: *Consider the ratio*

$$\frac{(n+1)^n}{n^{n+1}}=\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{n}<\frac{e}{n}.$$

Thus for $n\ge 3$,

$$\frac{(n+1)^n}{n^{n+1}}<1\implies (n+1)^{1/(n+1)}<n^{1/n}.$$

Hence we have

$$3^{1/3}>4^{1/4}>5^{1/5}>\cdots.$$

Clearly, if $n>1$ then $n^{1/n}>1^{1/n}=1$. Also, by the Binomial Theorem, again, if $n>1$,

$$\left(1+\sqrt{\frac{2}{n}}\right)^n=1^n+\binom{n}{1}\left(\sqrt{\frac{2}{n}}\right)^1+\binom{n}{2}\left(\sqrt{\frac{2}{n}}\right)^2+\cdots>1+\binom{n}{2}\left(\sqrt{\frac{2}{n}}\right)^2=1+\frac{n(n-1)}{2}\left(\frac{2}{n}\right)=n.$$

We then conclude that

$$1<n^{1/n}<1+\sqrt{\frac{2}{n}},$$

and that $n^{1/n}\to 1$ follows from the Sandwich Theorem. ❑

☞ $2^{1/2}=4^{1/4}$.

Homework

336 Problem What's wrong with the following? Since the product of the limits is the limit of the product,

$$e = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = \underbrace{\left(\lim_{n\to+\infty} 1+\frac{1}{n}\right)\cdot\left(\lim_{n\to+\infty} 1+\frac{1}{n}\right)\cdots\left(\lim_{n\to+\infty} 1+\frac{1}{n}\right)}_{n \text{ times}} = \underbrace{1\cdot 1\cdots 1}_{n \text{ times}} = 1.$$

337 Problem Demonstrate that for all strictly positive integers n:

$$\cos\frac{\pi}{2^{n+1}} = \frac{1}{2}\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n \text{ radicands}},$$

$$\sin\frac{\pi}{2^{n+1}} = \frac{1}{2}\underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n \text{ radicands}}.$$

Hence deduce *Viète's Formula for π*:

$$\pi = \lim_{n\to+\infty} 2^n \underbrace{\sqrt{2-\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n \text{ radicands}}.$$

338 Problem Prove that the sequence $\left\{\sum_{k=n}^{2n}\frac{1}{k}\right\}_{n=1}^{+\infty}$ converges to $\log 2$.

339 Problem Prove that the sequence $\left\{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n}\right\}_{n=1}^{+\infty}$ converges to $\log 2$.

340 Problem Let n be a strictly positive integer and let x_n denote the unique real solution of the equation x^n+x+1. Prove that $x_n\to 1$ as $n\to+\infty$.

341 Problem Let

$$a_n = \sqrt{n+\sqrt{(n-1)+\sqrt{(n-2)+\cdots+\sqrt{2+\sqrt{1}}}}},$$

for $n\geq 1$. Prove that $a_n-\sqrt{n}\to\frac{1}{2}$.

342 Problem Prove that e is not a quadratic irrational.

343 Problem Find $\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)$.

344 Problem A *quadratic integer* is any number x that satisfies an equation

$$x^2+mx+n=0,\quad (m,n)\in\mathbb{Z}^2.$$

Prove that the real quadratic integers are dense in the reals.

3.4 Averages of Limits

Why bother? In this section we will examine some classical results that allow us to compute more complicated limits. Had we the language of matrices, most results here could be deduced from a classical result of Toeplitz. Since we don't, we will develop ad hoc methods which are interesting by themselves.

We start with the following discrete analogues of L'Hôpital's Rule.

345 THEOREM Let $\{x_n\}_{n=1}^{+\infty}$, $\{y_n\}_{n=1}^{+\infty}$, be two sequences of real numbers such that $x_n\to 0$, $y_n\to 0$. Suppose, moreover, that the x_n are eventually strictly decreasing. Then

$$\lim_{n\to+\infty}\frac{x_n-x_{n-1}}{y_n-y_{n-1}} = \lim_{n\to+\infty}\frac{x_n}{y_n},$$

provided the sinistral limit exists (be it finite or $+\infty$).

Proof: *Assume first that* $\frac{x_{n-1}-x_n}{y_{n-1}-y_n} = \frac{x_n-x_{n-1}}{y_n-y_{n-1}}\to L$, *a finite real number. Then, given* $\varepsilon>0$ *we can find* $N>0$ *such that for* $n>N$,

$$L-\varepsilon<\frac{x_{n-1}-x_n}{y_{n-1}-y_n}<L+\varepsilon,\qquad y_n<y_{n-1}.$$

Thus $(L-\varepsilon)(y_{n-1}-y_n)<x_{n-1}-x_n<(L+\varepsilon)(y_{n-1}-y_n)$, *and repeating this inequality for* $n+1,n+2,\ldots,n+m$,

$$\begin{array}{ccccc}(L-\varepsilon)(y_n-y_{n+1}) & < & x_n-x_{n+1} & < & (L+\varepsilon)(y_n-y_{n+1}),\\ (L-\varepsilon)(y_{n+1}-y_{n+2}) & < & x_{n+1}-x_{n+2} & < & (L+\varepsilon)(y_{n+1}-y_{n+2}),\\ & & \vdots & & \\ (L-\varepsilon)(y_{m+n-1}-y_{m+n}) & < & x_{m+n-1}-x_{m+n} & < & (L+\varepsilon)(y_{m+n-1}-y_{m+n}).\end{array}$$

Adding columnwise,

$$(L-\varepsilon)(y_n-y_{m+n})<x_n-x_{m+n}<(L+\varepsilon)(y_n-y_{m+n}).$$

Letting $m\to+\infty$, *and since the* y_n *are strictly positive,*

$$(L-\varepsilon)y_n<x_n<(L+\varepsilon)y_n \implies L-\varepsilon<\frac{x_n}{y_n}<L+\varepsilon \implies \frac{x_n}{y_n}\to L$$

as $n \to +\infty$.

If $\dfrac{x_{n-1}-x_n}{y_{n-1}-y_n}$ *diverges to* $+\infty$ *then for all* $M>0$ *we can find* $N'>0$ *such that for all* $n \geq N'$,

$$\frac{x_{n-1}-x_n}{y_{n-1}-y_n} > M \implies x_{n-1}-x_n > M(y_{n-1}-y_n).$$

Reasoning as above, for positive integers $m \geq 0$,

$$x_n - x_{m+n} > M(y_n - y_{m+n}).$$

Taking the limit as $m \to +\infty$,

$$x_n \geq My_n \implies \frac{x_n}{y_n} \geq M \implies \frac{x_n}{y_n} \to +\infty.$$

❑

346 THEOREM (Stolz's Theorem) Let $\{a_n\}_{n=1}^{+\infty}$, $\{b_n\}_{n=1}^{+\infty}$, be two sequences of real numbers. Suppose that $\{b_n\}_{n=1}^{+\infty}$ is strictly increasing for sufficiently large n and that $b_n \to +\infty$ as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = \lim_{n\to+\infty} \frac{a_n}{b_n},$$

provided the sinistral side exists (be it finite or infinite).

Proof: *Assume first that* $\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} \to L$, *finite. Then for every* $\varepsilon > 0$ *there is* $N > 0$ *such that* $(\forall) n \geq N$,

$$L - \varepsilon < \frac{a_{n+1}-a_n}{b_{n+1}-b_n} < L + \varepsilon, \qquad b_{n+1} > b_n.$$

This means that

$$(L-\varepsilon)(b_{n+1}-b_n) < a_{n+1}-a_n < (L+\varepsilon)(b_{n+1}-b_n)$$

By iterating the above relation for $N+1, N+2, \ldots, m+N$ *we obtain*

$$\begin{array}{ccccc}
(L-\varepsilon)(b_{N+1}-b_N) & < & a_{N+1}-a_N & < & (L+\varepsilon)(b_{N+1}-b_N), \\
(L-\varepsilon)(b_{N+2}-b_{N+1}) & < & a_{N+2}-a_{N+1} & < & (L+\varepsilon)(b_{N+2}-b_{N+1}), \\
 & & \vdots & & \\
(L-\varepsilon)(b_{m+N}-b_{m+N-1}) & < & a_{m+N}-a_{m+N-1} & < & (L+\varepsilon)(b_{m+N}-b_{m+N-1}).
\end{array}$$

Adding columnwise,

$$(L-\varepsilon)(b_{m+N}-b_N) < a_{m+N}-a_N < (L+\varepsilon)(b_{m+N}-b_N) \implies \left|\frac{a_{m+N}-a_N}{b_{m+N}-b_N} - L\right| < \varepsilon.$$

Now,

$$\frac{a_{m+N}}{b_{m+N}} - L = \frac{a_N - Lb_N}{b_{m+N}} + \left(1 - \frac{b_N}{b_{m+N}}\right)\left(\frac{a_{m+N}-a_N}{b_{m+N}-b_N} - L\right),$$

so by the Triangle Inequality

$$\left|\frac{a_{m+N}}{b_{m+N}} - L\right| \leq \left|\frac{a_N - Lb_N}{b_{m+N}}\right| + \left|1 - \frac{b_N}{b_{m+N}}\right|\left|\frac{a_{m+N}-a_N}{b_{m+N}-b_N} - L\right|.$$

Since N *is fixed,* $\dfrac{a_N - Lb_N}{b_{m+N}} \to 0$ *and* $\dfrac{b_N}{b_{m+N}} \to 0$ *as* $m \to +\infty$ *Thus the dextral side is arbitrarily small, proving that* $\dfrac{a_m}{b_m} \to L$ *as* $m \to +\infty$.

Assume now that $\dfrac{a_n - a_{n-1}}{b_n - b_{n-1}} \to +\infty$. *For sufficiently large* n *thus* $a_n - a_{n-1} > b_n - b_{n-1}$. *Thus the* a_n *are eventually increasing and* $a_n \to +\infty$ *as* $n \to +\infty$. *Applying the results above to the* $\dfrac{b_n}{a_n}$ *we obtain*

$$\lim_{n\to+\infty} \frac{b_n}{a_n} = \lim_{n\to+\infty} \frac{b_n - b_{n-1}}{a_n - a_{n-1}} = 0$$

and so $\lim_{n\to+\infty} \dfrac{a_n}{b_n} = +\infty$ *too.* ❑

347 THEOREM (Cèsaro) If a sequence of real numbers converges to a number, then its sequence of arithmetic means converges to the same number, that is, if $x_n \to a$ then $\dfrac{x_1 + x_2 + \cdots + x_n}{n} \to a$.

First Proof: *Let* $a_n = x_1 + x_2 + \cdots + x_n$ *and* $b_n = n$ *in Stolz's Theorem.* ❑

Second Proof: *It is instructive to give an ad hoc proof of this result. Given* $\varepsilon > 0$ *there exists* $N > 0$ *such that if* $n \geq N$ *then* $|x_n - a|$. *Then*

$$\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - a\right| = \left|\frac{(x_1 - a) + (x_2 - a) + \cdots + (x_n - a)}{n}\right| \leq \frac{|(x_1 - a)| + |(x_2 - a)| + \cdots + |(x_n - a)|}{n}.$$

Now we run into a slight problem. We can control the differences $|x_k - a|$ *after a certain point, but the early differences need to be taken care of. To this end we consider the differences* $x_k - a$ *with* $k \leq \lfloor\sqrt{n}\rfloor$ *or* $k > \lfloor n \rfloor$ *where* n *is so large that* $\lfloor\sqrt{n}\rfloor \geq N$. *We have*

$$\begin{aligned}\frac{|(x_1 - a)| + |(x_2 - a)| + \cdots + |(x_n - a)|}{n} &= \frac{|(x_1 - a)| + |(x_2 - a)| + \cdots + \left|(x_{\lfloor\sqrt{n}\rfloor} - a)\right|}{n} \\ &\quad + \frac{\left|(x_{\lfloor\sqrt{n}\rfloor+1} - a)\right| + |(x_2 - a)| + \cdots + |(x_n - a)|}{n} \\ &< \frac{\lfloor\sqrt{n}\rfloor \max_{1\leq k\leq\lfloor\sqrt{n}\rfloor} |x_k - a|}{n} + \frac{(n - \lfloor\sqrt{n}\rfloor)\varepsilon}{n}.\end{aligned}$$

These two last quantities tend to 0 *as* $n \to +\infty$, *from where the result follows.* ❑

348 Example Since $n^{1/n} \to 1$, $\dfrac{1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}}{n} \to 1$.

349 Example Since $\dfrac{1}{n} \to 0$, $\dfrac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}}{n} \to 0$.

350 Example Since $\left(1 + \dfrac{1}{n}\right)^n \to e$, $\dfrac{\left(1 + \frac{1}{1}\right)^1 + \left(1 + \frac{1}{2}\right)^2 + \left(1 + \frac{1}{3}\right)^3 + \cdots + \left(1 + \frac{1}{n}\right)^n}{n} \to e$.

351 Example The converse of Cèsaro's Theorem is false. For, the sequence $a_n = (-1)^n$ oscillates and does not converge. But its sequence of averages is $b_n = \dfrac{1 - 1 + 1 - 1 + \cdots + (-1)^n}{n} \to 0$ as $n \to +\infty$ since the numerator is either 0 or -1.

352 THEOREM If a sequence of positive real numbers converges to a number, then its sequence of geometric means converges to the same number, that is, if $\forall n > 0, \quad x_n \geq 0$ and $x_n \to a$ then $(x_1 x_2 \cdots x_n)^{1/n} \to a$.

Proof: *The proof mimics Cèsaro's Theorem 347. Since* $x_n \to a$, *for all* $\varepsilon > 0$ *there is* $N > 0$ *such that for all* $n \geq N$,

$$|x_n - a| < \varepsilon \implies a - \varepsilon < x_n < a + \varepsilon.$$

Then

$$\left(\min_{1\le k\le \lfloor\sqrt{n}\rfloor} x_k\right)^{\lfloor\sqrt{n}\rfloor/n} \left(x_{\lfloor\sqrt{n}\rfloor+1}\cdots x_n\right)^{1/n} \le (x_1x_2\cdots x_{\lfloor\sqrt{n}\rfloor}x_{\lfloor\sqrt{n}\rfloor+1}\cdots x_n)^{1/n} \le \left(\max_{1\le k\le \lfloor\sqrt{n}\rfloor} x_k\right)^{\lfloor\sqrt{n}\rfloor/n} \left(x_{\lfloor\sqrt{n}\rfloor+1}\cdots x_n\right)^{1/n}.$$

This gives, for $\lfloor\sqrt{n}\rfloor \ge N$,

$$\left(\min_{1\le k\le \lfloor\sqrt{n}\rfloor} x_k\right)^{\lfloor\sqrt{n}\rfloor/n} (a-\varepsilon)^{(n-\lfloor\sqrt{n}\rfloor)/n} \le (x_1x_2\cdots x_{\lfloor\sqrt{n}\rfloor}x_{\lfloor\sqrt{n}\rfloor+1}\cdots x_n)^{1/n} \le \left(\max_{1\le k\le \lfloor\sqrt{n}\rfloor} x_k\right)^{\lfloor\sqrt{n}\rfloor/n} (a+\varepsilon)^{(n-\lfloor\sqrt{n}\rfloor)/n}.$$

Now, both $\left(\min_{1\le k\le \lfloor\sqrt{n}\rfloor} x_k\right)^{\lfloor\sqrt{n}\rfloor/n}$ *and* $\left(\max_{1\le k\le \lfloor\sqrt{n}\rfloor} x_k\right)^{\lfloor\sqrt{n}\rfloor/n}$ *converge to* 1 *as* $n\to+\infty$ *by virtue of Theorem 325, and again by the same theorem,*

$$(a-\varepsilon)^{(n-\lfloor\sqrt{n}\rfloor)/n} = (a-\varepsilon)\,(a-\varepsilon))^{\lfloor\sqrt{n}\rfloor/n} \to a-\varepsilon, \qquad (a+\varepsilon)^{(n-\lfloor\sqrt{n}\rfloor)/n} = (a+\varepsilon)\,(a+\varepsilon))^{\lfloor\sqrt{n}\rfloor/n} \to a+\varepsilon$$

as $n\to+\infty$*. This gives the result.* ❑

353 Example Since $e_n = \left(\dfrac{n+1}{n}\right)^n \to e$, then by the Theorem 352

$$(e_1e_2\cdots e_n)^{1/n} = \left(\left(\frac{2}{1}\right)^1\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right)^{1/n} = \left(\frac{(n+1)^n}{n!}\right)^{1/n} \to e.$$

This gives $\dfrac{n}{(n!)^{1/n}} = \dfrac{n}{n+1}\cdot\dfrac{n+1}{(n!)^{1/n}} \to 1\cdot e = e.$

Homework

354 Problem If $\{a_n\}_{n=1}^{+\infty}$ is a sequence of strictly positive real numbers such that $\dfrac{a_n}{a_{n-1}} \to a > 0$. Prove that

$$\lim_{n\to+\infty}\frac{a_n}{a_{n-1}} = \lim_{n\to+\infty}\sqrt[n]{a_n}.$$

355 Problem Let $x_n \to a$ and $y_n \to b$. Prove that $\dfrac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} \to ab$.

356 Problem Prove that $\lim_{n\to+\infty}\left(\binom{2n}{n}\right)^{1/n} = 4$.

357 Problem Prove that $\lim_{n\to+\infty}\dfrac{1}{n}(n(n+1)\cdots(n+n))^{1/n} = 4e$.

358 Problem Prove that $\lim_{n\to+\infty}\dfrac{1}{n}(1\cdot3\cdot5\cdots(2n-1))^{1/n} = \dfrac{2}{e}$.

359 Problem Prove that $\lim_{n\to+\infty}\dfrac{1}{n^2}\left(\dfrac{(3n)!}{n!}\right)^{1/n} = \dfrac{2}{e}$.

3.5 Orders of Infinity

Why bother? It is clear that the sequences $\{n\}_{n=1}^{+\infty}$ and $\{n^2\}_{n=1}^{+\infty}$ both tend to $+\infty$ as $n\to+\infty$. We would like now to refine this statement and compare one with the other. In other words, we will examine their speed towards $+\infty$.

360 Definition We write $a_n = O(b_n)$ if $\exists C > 0$, $\exists N > 0$ such that $\forall n \ge N$ we have $|a_n| \le C|b_n|$. We then say that a_n is *Big Oh of* b_n, or that a_n *is of order at most* b_n as $n\to+\infty$. Observe that this means that $\left|\dfrac{a_n}{b_n}\right|$ is bounded for sufficiently large n. The notation $a_n << b_n$, due to Vinogradov, is often used as a synonym of $a_n = O(b_n)$.

A sequence $\{a_n\}_{n=1}^{+\infty}$ *is bounded if and only if* $a_n << 1$.

An easy criterion to identify Big Oh relations is the following.

361 Theorem If $\lim_{n\to+\infty}\dfrac{a_n}{b_n} = c \in \mathbb{R}$, then $a_n << b_n$.

Proof: *By Theorem 283, a convergent sequence is bounded, hence the sequence* $\left\{\frac{a_n}{b_n}\right\}_{n=+1}^{+\infty}$ *is bounded: so for sufficiently large* n, $\left|\frac{a_n}{b_n}\right| < C$ *for some constant* $C > 0$. *This proves the theorem.* ❑

The = in the relation $a_n = O(b_n)$ *is not a true equal sign. For example* $n^2 = O\left(n^3\right)$ *since* $\lim_{n\to+\infty} \frac{n^2}{n^3} = 0$ *and so* $n^2 << n^3$ *by Theorem 361. On the other hand,* $\lim_{n\to+\infty} \frac{n^3}{n^2} = +\infty$ *so that for sufficiently large* n*, and for all* $M > 0$, $n^3 > Mn^2$*, meaning that* $n^3 \neq O\left(n^2\right)$*. Thus the Big Oh relation is not symmetric.*[3]

362 THEOREM (Lexicographic Order of Powers) Let $(\alpha, \beta) \in \mathbb{R}$ and consider the sequences $\{n^\alpha\}_{n=1}^{+\infty}$ and $\left\{n^\beta\right\}_{n=1}^{+\infty}$. Then $n^\alpha << n^\beta \iff \alpha \leq \beta$.

Proof: *If* $\alpha \leq \beta$ *then* $\lim_{n\to+\infty} \frac{n^\alpha}{n^\beta}$ *is either* 1 *(when* $\alpha = \beta$*) or* 0 *(when* $\alpha < \beta$*), hence* $n^\alpha << n^\beta$ *by Theorem 361.*

If $n^\alpha << n^\beta$ *then for sufficiently large* n, $n^\alpha \leq Cn^\beta$ *for some constant* $C > 0$. *If* $\alpha > \beta$ *then this would mean that for all large* n *we would have* $n^{\alpha-\beta} \leq C$*, which is absurd, since for a strictly positive exponent* $\alpha - \beta$, $n^{\alpha-\beta} \to +\infty$ *as* $n \to +\infty$.❑

363 Example As $n \to +\infty$,

$$n^{1/10} << n^{1/3} << n << n^{10/9} << n^2,$$

for example.

364 THEOREM If $c \in \mathbb{R} \setminus \{0\}$ then $O(ca_n) = O(a_n)$, that is, the set of sequences of order at most ca_n is the same set at those of order at most a_n.

Proof: *We prove that* $b_n = O(ca_n) \iff b_n = O(a_n)$. *If* $b_n = O(ca_n)$ *the there are constants* $C > 0$ *and* $N > 0$ *such that* $|b_n| \leq C|ca_n|$ *whenever* $n \geq N$. *Therefore,* $|b_n| \leq C'|a_n|$ *whenever* $n \geq N$*, where* $C' = C|c|$*, meaning that* $b_n = O(a_n)$. *Similarly, if* $b_n = O(a_n)$ *the there are constants* $C_1 > 0$ *and* $N_1 > 0$ *such that* $|b_n| \leq C_1|a_n|$ *whenever* $n \geq N_1$. *Since* $c \neq 0$ *this is equivalent to* $|b_n| \leq \frac{C_1}{c}(c|a_n|) = C''(c|a_n|)$ *whenever* $n \geq N_1$*, where* $C'' = \frac{C_1}{c}$*, meaning that* $b_n = O(ca_n)$. *Therefore,* $O(a_n) = O(ca_n)$. ❑

365 Example As $n \to +\infty$,

$$O\left(n^3\right) = O\left(\frac{n^3}{3}\right) = O\left(4n^3\right).$$

366 THEOREM (Sum Rule) Let $a_n = O(x_n)$ and $b_n = O\left(y_n\right)$. Then $a_n + b_n = O(\max(|x_n|, |y_n|))$.

Proof: *There exist strictly positive constants* C_1, N_1, C_2, N_2 *such that*

$$n \geq N_1, \implies |a_n| \leq C_1|x_n| \qquad \text{and} \qquad n \geq N_2, \implies |b_n| \leq C_2\left|y_n\right|.$$

Let $N' = \max(N_1, N_2)$. *Then for* $n \geq N$*, by the Triangle inequality*

$$|a_n + b_n| \leq |a_n| + |b_n| \leq C_1|x_n| + C_2\left|y_n\right|.$$

Let $C' = \max(C_1, C_2)$. *Then*

$$|a_n + b_n| \leq C'(|x_n| + \left|y_n\right|) \leq 2C'\max(|x_n|, \left|y_n\right|),$$

whence the theorem follows. ❑

367 COROLLARY Let $a_n = k_0 n^m + k_1 n^{m-1} + k_2 n^{m-2} + \cdots + k_{m-1} n + k_n$ be a polynomial of degree m in n with real number coefficients. The $a_n = O\left(n^m\right)$, that is, a_n is of order at most its leading term.

[3]One should more properly write $a_n \in O(b_n)$, as $O(b_n)$ is the set of sequences growing to infinity no faster than b_n, but one keeps the = sign for historical reasons.

Proof: *By the Sum Rule Theorem 366 the leading term dominates.*❑

368 THEOREM (Transitivity Rule) If $a_n = O(b_n)$ and $b_n = O(c_n)$, then $a_n = O(c_n)$.

Proof: *There are strictly positive constants C_1, C_2, N_1, N_2 such that*

$$n \geq N_1, \implies |a_n| \leq C_1 |b_n| \quad \text{and} \quad n \geq N_2, \implies |b_n| \leq C_2 |c_n|.$$

If $n \geq \max(N_1, N_2)$, then $|a_n| \leq C_1 |b_n| \leq C_1 C_2 |c_n| = C |c_n|$, with $C = C_1 C_2$. This gives $a_n = O(c_n)$. ❑

369 Example By Corollary 367, $5n^4 - 2n^2 + 100n - 8 = O(5n^4)$. By Theorem 364, $O(5n^4) = O(n^4)$. Hence

$$5n^4 - 2n^2 + 100n - 8 = O(n^4).$$

370 THEOREM (Multiplication Rule) If $a_n = O(x_n)$ and $b_n = O(y_n)$, then $a_n b_n = O(x_n y_n)$.

Proof: *There are strictly positive constants C_1, C_2, N_1, N_2 such that*

$$n \geq N_1, \implies |a_n| \leq C_1 |x_n| \quad \text{and} \quad n \geq N_2, \implies |b_n| \leq C_2 |y_n|.$$

If $n \geq \max(N_1, N_2)$, then $|a_n b_n| \leq C_1 C_2 |x_n y_n| = C |x_n y_n|$, with $C = C_1 C_2$. This gives $a_n b_n = O(x_n y_n)$. ❑

371 THEOREM (Lexicographic Order of Exponentials) Let $(a, b) \in \mathbb{R}$, $a > 1$, $b > 1$, and consider the sequences $\{a^n\}_{n=1}^{+\infty}$ and $\{b^n\}_{n=1}^{+\infty}$. Then $a^n << b^n \iff a \leq b$.

Proof: *Put $r = \dfrac{a}{b}$, and use Theorems 323 and 361.* ❑

372 Example $\dfrac{1}{2^n} << 1 << 2^n << e^n << 3^n$.

373 LEMMA Let $a \in \mathbb{R}$, $a > 1$, $k \in \mathbb{N} \setminus \{0\}$. Then $n^k << a^n$.

Proof: *By Theorem 326, $\lim_{n \to +\infty} \dfrac{n^k}{a^n} = 0$. Now apply Theorem 361.* ❑

374 THEOREM ("Exponentials are faster than powers") Let $a \in \mathbb{R}$, $a > 1$, $\alpha \in \mathbb{R}$. Then $n^\alpha << a^n$.

Proof: *Put $k = \max(1, \lfloor \alpha \rfloor + 1)$. Then by Theorem 362, $n^\alpha << n^k$. By Lemma 373, $n^k << a^n$, and by the Transitivity of Big Oh (Theorem 368), $n^\alpha << n^k << a^n$.* ❑

375 Example

$$n^{100} << e^n.$$

376 THEOREM ("Logarithms are slower than powers") Let $(\alpha, \beta) \in \mathbb{R}^2$, $\alpha > 0$. Then $(\log n)^\beta << n^\alpha$.

Proof: *If $\beta \leq 0$, then $(\log n)^\beta << 1$ and the assertion is evident, so assume $\beta > 0$. For $x > 0$, then $\log x < x$. Putting $x = n^{\alpha/\beta}$, we get*

$$\log n^{\alpha/\beta} < n^{\alpha/\beta} \implies \log n < \frac{\beta n^{\alpha/\beta}}{\alpha} \implies (\log n)^\beta < \frac{\beta^\beta n^\alpha}{\alpha^\beta},$$

whence $(\log n)^\beta << n^\alpha$. ❑

By the Multiplication Rule (Theorem 370) and Theorems 362, 374, 376, in order to compare two expressions of the type $a^n n^b (\log)^c$ and $u^n n^v (\log)^w$ we simply look at the lexicographic order of the exponents, keeping in mind that logarithms are slower than powers, which are slower than exponentials.

377 Example In increasing order of growth we have

$$\frac{1}{e^n} << \frac{1}{2^n} << \frac{1}{n^2} = \frac{1}{\log n} << 1 << (\log\log n)^{10} << \sqrt{\log n} << \frac{n}{\log n} << n << n\log n << e^n.$$

378 Example Decide which one grows faster as $n \to +\infty$: $n^{\log n}$ or $(\log n)^n$.

Solution: *Since* $n^{\log n} = e^{(\log n)^2}$ *and* $(\log n)^n = e^{n\log\log n}$, *and since* $(\log n)^2 << n\log\log n$, *we conclude that* $n^{\log n} << (\log n)^n$.

We now define two more fairly common symbols in asymptotic analysis.

379 Definition We write $a_n = o(b_n)$ if $\frac{a_n}{b_n} \to 0$ as $n \to +\infty$, and say that a_n is *small oh* of b_n, or that a_n *grows slower* than b_n as $n \to +\infty$.

380 Definition A sequence $\{a_n\}_{n=1}^{+\infty}$ is said to be *infinitesimal* if $a_n = o(1)$, that is, if $a_n \to 0$ as $n \to +\infty$.

☞ *We know from above that for* $a > 1$ $\lim_{n\to+\infty} \frac{n^\alpha}{a^n} = 0$, *and so* $n^\alpha = o(a^n)$. *Also, for* $\gamma > 0$, $\lim_{n\to+\infty} \frac{(\log n)^\beta}{n^\gamma} = 0$, *and so* $(\log n)^\beta = o(n^\gamma)$.

381 Definition We write $a_n \sim b_n$ if $\frac{a_n}{b_n} \to 1$ as $n \to +\infty$, and say that a_n is *asymptotic* to b_n.

Asymptotic sequences are thus those that grow at the same rate as the index increases.

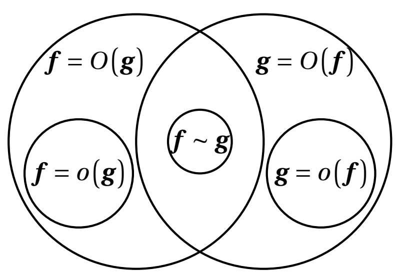


Figure 3.2: Diagram of O relations.

382 Example The sequences $\{n^2 - n\sin n\}_{n=1}^{+\infty}$, $\{n^2 + n - 1\}_{n=1}^{+\infty}$ are asymptotic since

$$\frac{n^2 - n\sin n}{n^2 + n - 1} = \frac{1 - \frac{\sin n}{n}}{1 + \frac{1}{n} - \frac{1}{n^2}} \to 1,$$

as $n \to +\infty$.

383 Theorem Let $\{a_n\}_{n=1}^{+\infty}$ and $\{b_n\}_{n=1}^{+\infty}$ be two properly diverging sequences. Then $a_n \sim b_n \iff a_n = b_n(1 + o(1))$.

Proof: *Since the limit is $1>0$, either both diverge to $+\infty$ or both to $-\infty$. Assume the former, and so, eventually, b_n will be strictly positive. Now,*

$$\begin{aligned}\lim_{n\to+\infty}\frac{a_n}{b_n}=1 &\iff \forall\varepsilon>0,\exists N>0, 1-\varepsilon<\frac{a_n}{b_n}<1+\varepsilon\\ &\iff b_n-b_n\varepsilon<a_n<b_n+b_n\varepsilon\\ &\iff |a_n-b_n|<b_n\varepsilon\\ &\iff a_n-b_n=o(b_n).\end{aligned}$$

❑

The relationship between the three symbols is displayed in figure 3.2.

Homework

384 Problem Prove that $e^n << n!$.

385 Problem Prove that $O(O(a_n)) = O(a_n)$.

386 Problem Let $k\in\mathbb{R}$ be a constant. Prove that $k+O(a_n)=O(k+a_n)=O(a_n)$.

387 Problem Let $k\in\mathbb{R}$, $k>0$, be a constant. Prove that $(a_n+b_k)^k << a_n^k+b_n^k$.

388 Problem For a sequence of real numbers $\{a_n\}_{n=1}^{+\infty}$ it is known that $a_n=O\left(n^2\right)$ and $a_n=o\left(n^2\right)$. Which of the two statements conveys more information?

389 Problem True or false: $a_n=O(n) \implies a_n=o(n)$.

390 Problem True or false: $a_n=o(n) \implies a_n=O(n)$.

391 Problem True or false: $a_n=o\left(n^2\right) \implies a_n=O(n)$.

392 Problem True or false: $a_n=o(n) \implies a_n=O\left(n^2\right)$.

3.6 Cauchy Sequences

393 Definition A sequence of real numbers $\{a_n\}_{n=1}^{+\infty}$ is called a *Cauchy Sequence* if

$$\forall\varepsilon>0,\quad \exists N>0,\quad \text{such that}\quad \forall n,m\ge N\quad |a_n-a_m|<\varepsilon.$$

394 THEOREM Cauchy sequences are bounded.

Proof: *Let $\{a_n\}_{n=1}^{+\infty}$ be Cauchy. Take $N>0$ such that for all $n\ge N$, $|a_n-a_N|<1$. Then a_n is bounded by*

$$\max(|a_1|,|a_2|,\ldots,|a_N|)+1.$$

❑

395 LEMMA If a Cauchy sequence of real numbers has a convergent subsequence, then the parent sequence converges, and it does so to the same limit as the subsequence.

Proof: *Let $\{a_n\}_{n=1}^{+\infty}$ be a Cauchy sequence of real numbers, and suppose that its subsequence $\{a_{n_k}\}_{k=1}^{+\infty}$ converges to the real number a. Given $\varepsilon>0$, take $N>0$ sufficiently large such that*

$$\forall m,n,n_k\ge N,\quad |a_n-a_m|<\varepsilon,\qquad \text{and}\quad \left|a_{n_k}-a\right|<\varepsilon.$$

By the Triangle Inequality,

$$|a_n-a|\le\left|a_n-a_{n_k}\right|+\left|a_{n_k}-a\right|<\varepsilon+\varepsilon=2\varepsilon,$$

whence $a_n\to a$.❑

396 THEOREM (General Principle of Convergence) A sequence of real numbers converges if and only if it is Cauchy.

Proof:

($\Rightarrow$) *If $a_n \to a$, given $\varepsilon > 0$, choose $N > 0$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. Then if $m, n \geq N$,*

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Since $2\varepsilon > 0$ can be made arbitrarily small, a_n is Cauchy.

($\Leftarrow$) *Suppose a_n is Cauchy. By virtue of Theorem 394 it is bounded, say that for all $n > 0$, $a_n \in \left[\alpha\,;\beta\right]$. Put*

$$\mathscr{S} = \{s : a_n \geq s \text{ for infinitely many } n\}.$$

As $\alpha \in \mathscr{S}$, $\mathscr{S} \neq \varnothing$. $\mathscr{S}$ is bounded above by β. By the Completeness Axiom, $\mathscr{S}$ has a supremum, $a = \sup \mathscr{S}$. Given $\varepsilon > 0$, $a - \varepsilon < a$ and so there is $s \in \mathscr{S}$ such that $a - \varepsilon < s$. By definition of $\mathscr{S}$, there are infinitely many n with $a_n \geq s > a - \varepsilon$. $a + \varepsilon > a$, so that $a + \varepsilon \notin \mathscr{S}$ and so there are only finitely many n for which $a_n \geq a + \varepsilon$. Thus there are infinitely many n with $a_n \in (a - \varepsilon, a + \varepsilon)$.

Choose $N > 0$ such that $|a_n - a_m| < \varepsilon$ for all $m, n \geq N$. We can find $m \geq N$ with $a_m \in (a - \varepsilon, a + \varepsilon)$ ie $|a_m - a| < \varepsilon$. Then if $n \geq N$,

$$|a_n - a| \leq |a_n - a_m| + |a_m - a| < \varepsilon + \varepsilon = 2\varepsilon$$

As 2ε can be made arbitrarily small this shews $a_n \to a$.

❑

Homework

3.7 Topology of sequences

397 THEOREM A set $X \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ is and only if for every $x \in \mathbb{R}$ there is a sequence $\{x_n\}_{n=1}^{+\infty}$ of elements of X that converge to x.

398 THEOREM Let $X \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is an accumulation point of X if and only if there exists a sequence of elements of $X \setminus \{x\}$ converging to x.

Homework

399 Problem Identify the set of accumulation points of the set $\{\sqrt{a} - \sqrt{b} : (a, b) \in \mathbb{N}^2\}$.

400 Problem Consider the following enumeration of the proper fractions

$$\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \ldots.$$

Clearly, the fraction $\frac{a}{b}$ in this enumeration occupies the $a + \frac{b(b+1)}{2}$-th place. For each integer $k \geq 1$, cover the k-th fraction $\frac{a}{b}$ by an interval of length 2^{-k} centred at $\frac{a}{b}$. Shew that the point $\frac{\sqrt{2}}{2}$ does not belong to any interval in the cover.

Chapter 4

Series

4.1 Convergence and Divergence of Series

401 Definition A *series* is the sum of a sequence. We write

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

Here s_n is the n-th *partial sum*. Observe in particular that

$$a_n = s_n - s_{n-1}.$$

402 Definition If the sequence $\{s_n\}_{n=1}^{+\infty}$ has a limit, we say that the series converges and write

$$\sum_{k=1}^{+\infty} a_k = \lim_{n\to+\infty} s_n.$$

403 THEOREM (n-th Term Test for Divergence) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to +\infty$.

Proof: *Put* $s_n = \sum_{k=1}^{n} a_k$. *Then*

$$\lim_{n\to+\infty} s_n = s \implies a_n = s_n - s_{n-1} \to s - s = 0.$$

❑

404 Example The series $\sum_{n=1}^{+\infty}\left(1+\frac{2}{n}\right)^n$ diverges, since its n-th term $\left(1+\frac{2}{n}\right)^n \to e^2$.

405 Example Even though the harmonic sequence $\frac{1}{n} \to 0$, the harmonic series $\sum_{n=1}^{+\infty}\frac{1}{n}$ diverges, by comparison with the integral $\int_1^{+\infty}\frac{dx}{x}$.

4.1.1 Geometric Series

A *geometric series* with common ratio r and first term a is one of the form

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{+\infty} ar^n.$$

If $|r| < 1$ then the series converges and we have

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{+\infty} ar^n = \frac{a}{1-r}.$$

406 Example We have

$$\sum_{n=3}^{\infty} \frac{2^n}{e^{n+1}} = \frac{2^3}{e^4} + \frac{2^4}{e^5} + \cdots = \frac{\frac{2^3}{e^4}}{1 - \frac{2}{e}} = \frac{8}{e^4 - 2e^3}.$$

4.1.2 Telescoping Series

A *telescoping series* is one where adjacent terms cancel out.

407 Example To find the sum of the series $\sum_{n=2}^{+\infty} \frac{1}{4n^2-1}$, we find through partial fractions that

$$\frac{1}{4n^2-1} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}.$$

Hence

$$\sum_{n=2}^{+\infty} \frac{1}{4n^2-1} = \left(\frac{1}{2(1)} - \frac{1}{2(3)}\right) + \left(\frac{1}{2(3)} - \frac{1}{2(5)}\right) + \left(\frac{1}{2(5)} - \frac{1}{2(7)}\right) + \cdots = \frac{1}{2(1)} = \frac{1}{2}.$$

4.2 Convergence and Divergence of Series of Positive Terms

We have several tools to establish convergence and divergence of series of positive terms. We first mention

408 Theorem (Cauchy Condensation Test)

409 Theorem (Test) If the sequence $\{a_n\}_{n=1}^{+\infty}$ is such that eventually (for $n \geq a$, say) there is a positive decreasing continuous function f such that $x_n = a_n$, then $\sum_{n=a}^{+\infty} a_n$ and $\int_a^{+\infty} f(x)\mathrm{d}x$ converge or diverge together.

410 Corollary (*p*-series test) If $p > 1$ then $\zeta(p) = \sum_{n=1}^{+\infty} \frac{1}{n^p}$ converges, but diverges when $p \leq 1$.

411 Corollary (De Morgan's Logarithmic Scale) If $p > 1$ then all of

$$\sum_{n=1}^{+\infty} \frac{1}{n^p}; \quad \sum_{n \geq e}^{+\infty} \frac{1}{n(\log n)^p}; \quad \sum_{n \geq e^e}^{+\infty} \frac{1}{n(\log n)(\log\log n)^p}; \quad \ldots$$

converge, but diverge when $p \leq 1$.

4.2.1 Comparison Tests

These tests mimic those for improper integrals. Call a divergent series of positive terms a "giant" and a converging series of positive terms a "midget." The comparison tests say that if a series is bigger than a giant it must be a giant, and if a series is smaller than a midget, it must be a midget. In symbols

$$a_n << b_n \text{ and } \sum_{n=1}^{+\infty} b_n < +\infty \implies \sum_{n=1}^{+\infty} a_n < +\infty, \quad \text{(convergence)}$$

$$a_n >> b_n \text{ and } \sum_{n=1}^{+\infty} b_n = +\infty \implies \sum_{n=1}^{+\infty} a_n = +\infty \quad \text{(divergence)}.$$

412 Example $\sum_{n=1}^{+\infty} \frac{1}{n^n}$ converges. For $n \geq 2$ we have $\frac{1}{n^n} \leq \frac{1}{n^2}$ and the series converges by direct comparison with $\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

413 Example Determine whether $\sum_{n=4}^{+\infty} \frac{(\log n)^{100}}{n^{3/2} \log\log n}$ converges.

Solution: *Since* $(\log n)^{100} << n^{1/4}$, *eventually* $\frac{(\log n)^{100}}{n^{1/4}} << 1$. *We have* $\frac{(\log n)^{100}}{n^{3/2} \log\log n} << \frac{(\log n)^{100}}{n^{1/4}} \cdot \frac{1}{n^{5/4} \log\log n}$ *and since* $\sum_{n=4}^{+\infty} \frac{1}{n^{5/4} \log\log n} < +\infty$, *we have* $\sum_{n=4}^{+\infty} \frac{(\log n)^{100}}{n^{3/2} \log\log n} < +\infty$, *that is, the series converges.*

4.2.2 Ratio and Root Test

The following two test arise by comparing to a geometric series.

414 Theorem (Ratio Test) If $\sum_{n=1}^{+\infty} a_n$ is a series with positive terms then it converges if $\frac{a_{n+1}}{a_n} \to r < 1$ and diverges if $\frac{a_{n+1}}{a_n} \to r > 1$. The test is inconclusive if $\frac{a_{n+1}}{a_n} \to r = 1$.

415 Theorem (Root Test) If $\sum_{n=1}^{+\infty} a_n$ is a series with positive terms then it converges if $(a_n)^{1/n} \to r < 1$ and diverges if $(a_n)^{1/n} \to r > 1$. The test is inconclusive if $(a_n)^{1/n} \to r = 1$.

416 Example Since

$$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \to \frac{1}{e} < 1$$

the series $\sum_{n=1}^{+\infty} \frac{n!}{n^n}$ converges.

417 Example Since

$$\left(\frac{(n!)^n}{n^{n^2}}\right)^{1/n} = \frac{n!}{n^n} \to 0$$

the series $\sum_{n=1}^{+\infty} \frac{(n!)^n}{n^{n^2}}$ converges.

4.3 Summation by Parts

4.4 Alternating Series

A series of the form $\sum_{n=1}^{+\infty} (-1)^n a_n$ where the all the $a_n \geq 0$ is called an *alternating series*.

418 Theorem (Leibniz's Alternating Series Test) The alternating series $\sum_{n=1}^{+\infty} (-1)^n a_n$ converges if all the following conditions are met

- the a_n eventually decrease, that is, $a_{n+1} \leq a_n$ for all $n \geq N$.
- $a_n \to 0$

419 Example The series $\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{1}{n}$ converges by Leibniz's Test. In fact, one can prove that it equals $\log 2$.

4.5 Absolute Convergence

If $\sum_{n=1}^{+\infty} |a_n|$ converges then $\sum_{n=1}^{+\infty} a_n$ converges. The converse is not true.

420 Example Since $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$, $\sum_{n=1}^{+\infty} \left|\frac{\sin n}{n^2}\right|$ converges by the comparison test. Thus $\sum_{n=1}^{+\infty} \frac{\sin n}{n^2}$ converges absolutely and so it converges.

421 Example Determine whether the following two infinite series converge: (I) $\sum_{n=2}^{\infty} \frac{(-1)^n \sin(3n)}{n^2}$, (II) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$.

Solution: *We have*

$$\left|(-1)^n \frac{\sin 3n}{n^2}\right| \leq \frac{1}{n^2},$$

so (I) converges absolutely. As for number (II), $f(x) = \frac{x}{x^2+2}$ *is decreasing (take the first derivative)* $\frac{n}{n^2+2} \to 0,$ *so it converges by Leibniz's Test.*

Chapter 5

Real Functions of One Real Variable

5.1 Limits of Functions

422 DEFINITION-PROPOSITION (Cauchy-Heine, Sinistral Limit) Let $f : \left]a\,;b\right[\to \mathbb{R}$ and let $x_0 \in \left]a\,;b\right[$. The following are equivalent.

1. $\forall \varepsilon > 0, \exists \delta > 0$ such that
$$x_0 - \delta < x < x_0 \implies \left|f(x) - L\right| < \varepsilon.$$

2. For each sequence $\{x_n\}_{n=1}^{+\infty}$ of points in the interval $\left]a\,;b\right[$ with $x_n < x_0$, $x_n \to x_0 \implies f(x_n) \to L$.

If either condition is fulfilled we say that f *has a sinistral limit* $f(x_0-)$ *as* x *increases towards* x_0 and we write

$$f(x_0-) = \lim_{x\to x_0-} f(x) = \lim_{x\nearrow x_0} f(x).$$

Proof:

1 $\Longrightarrow$ 2 *Suppose that* $\forall \varepsilon > 0, \exists \delta > 0$ *such that*

$$x_0 - \delta < x < x_0 \implies \left|f(x) - L\right| < \varepsilon.$$

Let $x_n < x_0$, $x_n \to x_0$. *Then*
$$|x_n - x_0| < \delta \implies x_0 - \delta < x_n < x_0 + \delta$$
for sufficiently large n. *But we are assuming that* $x_n < x_0$, *so in fact we have* $x_0 - \delta < x_n < x_0$. *By our assumption then* $\left|f(x_n) - L\right| < \varepsilon$, *and so 1 $\Longrightarrow$ 2.*

2 $\Longrightarrow$ 1 *Suppose that for each sequence* $\{x_n\}_{n=1}^{+\infty}$ *of points in the interval* $\left]a\,;b\right[$ *with* $x_n < x_0$, $x_n \to x_0 \implies f(x_n) \to L$. *If it were not true that* $f(x) \to L$ *as* $x \to x_0$, *then there exists some* $\varepsilon_0 > 0$ *such that for all* $\delta > 0$ *we can find* x *such that*
$$0 < |x - x_0| < \delta \implies \left|f(x) - L\right| \geq \varepsilon_0.$$
In particular, for each strictly positive integer n *we can find* x_n *satisfying*

$$0 < |x_n - x_0| < \frac{1}{n} \implies \left|f(x_n) - L\right| \geq \varepsilon_0,$$

a contradiction to the fact that $f(x_n) \to L$.

❑

In an analogous manner, we have the following.

423 DEFINITION-PROPOSITION (Cauchy-Heine, Dextral Limit) Let $f : \left]a\,;b\right[\to \mathbb{R}$ and let $x_0 \in \left]a\,;b\right[$. The following are equivalent.

1. For each sequence $\{x_n\}_{n=1}^{+\infty}$ of points in the interval $\left]a\,;b\right[$ with $x_n > x_0$, $x_n \to x_0 \implies f(x_n) \to L$.
2. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$$x_0 < x < x_0 + \delta \implies \left|f(x) - L\right| < \varepsilon.$$

If either condition is fulfilled we say that f *has a dextral limit* $f(x_0+)$ *as* x *decreases towards* x_0 and we write

$$f(x_0+) = \lim_{x \to x_0+} f(x) = \lim_{x \searrow x_0} f(x).$$

Upon combining Propositions 422 and 423 we obtain the following.

424 DEFINITION-PROPOSITION (Cauchy-Heine) Let $f : \left]a\,;b\right[\to \mathbb{R}$ and let $x_0 \in \left]a\,;b\right[$. The following are equivalent.

1. $f(x_0-) = f(x_0+)$
2. For each sequence $\{x_n\}_{n=1}^{+\infty}$ of points in the interval $\left]a\,;b\right[$ different from x_0, $x_n \to x_0 \implies f(x_n) \to L$.
3. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$$0 < |x - x_0| < \delta \implies \left|f(x) - L\right| < \varepsilon.$$

If either condition is fulfilled we say that f *has a (two-sided) limit* L *as* x *decreases towards* x_0 and we write

$$L = \lim_{x \to x_0} f(x).$$

We now prove analogues of the theorems that the proved for limits of sequences.

425 THEOREM (Uniqueness of Limits) Let $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, and $f : X \to \mathbb{R}$. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = L'$ then $L = L'$.

Proof: *If* $L \neq L'$ *then take* $2\varepsilon = \left|L - L'\right|$ *in the definition of limit. There is* $\delta > 0$ *such that*

$$0 < |x - a| < \delta \implies \left|f(x) - L\right| < \frac{\left|L - L'\right|}{2}, \quad \left|f(x) - L'\right| < \frac{\left|L - L'\right|}{2}.$$

By the Triangle Inequality

$$\left|L - L'\right| \leq \left|L - f(x)\right| + \left|f(x) - L'\right| < \frac{\left|L - L'\right|}{2} + \frac{\left|L - L'\right|}{2} = \left|L - L'\right|,$$

but $\left|L - L'\right| < \left|L - L'\right|$ *is a contradiction.* ❑

426 THEOREM (Local Boundedness) Let $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, and $f : X \to \mathbb{R}$. If $\lim_{x \to a} f(x) = L$ exists and is finite, then f is bounded in a neighbourhood of a.

Proof: *Take* $\varepsilon = 1$ *in the definition of limit. Then there is a* $\delta > 0$ *such that*

$$0 < |x - a| < \delta \implies \left|f(x) - L\right| < 1 \implies \left|f(x)\right| < 1 + |L|,$$

and so f *is bounded on this neighbourhood.* ❑

427 THEOREM (Order Properties of Limits) Let $X \subseteq \mathbb{R}$, $a \in \mathbb{R}$, and $f : X \to \mathbb{R}$. Let $\lim_{x \to a} f(x) = L$ exist and be finite. Then

1. If $s < L$ then there exists a neighbourhood $\mathcal{N}_a$ of a contained in X such that $\forall x \in \mathcal{N}_a$, $s < f(x)$.
2. If $L < t$ then there exists a neighbourhood $\mathcal{N}_a$ of a contained in X such that $\forall x \in \mathcal{N}_a$, $f(x) < t$.
3. If $s < L < t$ then there exists a neighbourhood $\mathcal{N}_a$ of a contained in X such that $\forall x \in \mathcal{N}_a$, $s < f(x) < t$.
4. If there exists a neighbourhood $\mathcal{N}_a \subseteq X$ such that $\forall x \in \mathcal{N}_a$, $s \leq f(x)$, then $s \leq L$.

5. If there exists a neighbourhood $\mathscr{N}_a \subseteqq X$ such that $\forall x \in \mathscr{N}_a$, $f(x) \leq t$, then $L \leq t$.

6. If there exists a neighbourhood $\mathscr{N}_a \subseteqq X$ such that $\forall x \in \mathscr{N}_a$, $s \leq f(x) \leq t$, then $s \leq L \leq t$.

Proof: *We have*

1. *Take* $\varepsilon = L - s > 0$ *in the definition of limit. There is* $\delta > 0$ *such that*

$$0 < |x - a| < \delta \implies \left|f(x) - L\right| < L - s \implies s - L + L < f(x) < 2L - s \implies s < f(x),$$

as claimed.

2. *Take* $\varepsilon = t - L > 0$ *in the definition of limit. There is* $\delta > 0$ *such that*

$$0 < |x - a| < \delta \implies \left|f(x) - L\right| < t - L \implies L - t + L < f(x) < t - L + L \implies f(x) < t,$$

as claimed.

3. *This follows by (1) and (2).*
4. *If on the said neighbourhood* $\mathscr{N}_a$ *we had, on the contrary,* $L > s$ *then (1) asserts that there is a neighbourhood of* $\mathscr{N}_a' \subseteqq \mathscr{N}_a$ *such that* $f(x) > s$, *a contradiction to the assumption that* $\forall x \in \mathscr{N}_a$, $s \leq f(x)$.
5. *If on the said neighbourhood* $\mathscr{N}_a$ *we had, on the contrary,* $L < t$ *then (2) asserts that there is a neighbourhood of* $\mathscr{N}_a' \subseteqq \mathscr{N}_a$ *such that* $f(x) < t$, *a contradiction to the assumption that* $\forall x \in \mathscr{N}_a$, $t \geq f(x)$.
6. *This follows by (4) and (5).*

❑

Analogous to the Sandwich Theorem for sequences we have

428 Theorem (Sandwich Theorem) Assume that a, b, c are functions defined on a neighbourhood $\mathscr{N}_{x_0}$ of a point x_0 except possibly at x_0 itself. Assume moreover that in $\mathscr{N}_{x_0}$ they satisfy the inequalities $a(x) \leq b(x) \leq c(x)$. Then

$$\lim_{x \to x_0} a(x) = L = \lim_{x \to x_0} c(x) \implies \lim_{x \to x_0} b(x) = L.$$

Proof: *For all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that*

$$0 < |x - x_0| < \delta \implies |a(x) - L| < \varepsilon \quad \text{and} \quad |c(x) - L| < \varepsilon \implies L - \varepsilon < a(x) < L + \varepsilon \quad \text{and} \quad L - \varepsilon < c(x) < L + \varepsilon.$$

If we now consider $x \in \mathscr{N}_{x_0} \cap \{x : 0 < |x - x_0| < \delta\}$ *then*

$$L - \varepsilon < a(x) \leq b(x) \leq c(x) < L + \varepsilon \implies L - \varepsilon < b(x) < L + \varepsilon \implies |b(x) - L| < \varepsilon,$$

whence $\lim_{x \to x_0} b(x) = L$. ❑

429 Theorem Let $X \subseteqq \mathbb{R}$, $a \in \mathbb{R}$, and $f, g : X \to \mathbb{R}$. Let $(L, L', \lambda) \in \mathbb{R}^3$. Then

1. $\lim_{x \to a} f(x) = L \implies \lim_{x \to a} \left|f(x)\right| = |L|$.
2. $\lim_{x \to a} f(x) = 0 \iff \lim_{x \to a} \left|f(x)\right| = 0$.
3. $\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = L' \implies \lim_{x \to a} (f(x) + \lambda g(x)) = L + \lambda L'$.
4. $\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = L' \implies \lim_{x \to a} (f(x)g(x)) = LL'$.
5. If $\lim_{x \to a} f(x) = 0$ and if g is bounded on a neighbourhood $\mathscr{N}_a$ of a, then $\lim_{x \to a} f(x)g(x) = 0$.
6. $\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = L' \neq 0 \implies \lim_{x \to a} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{L}{L'}$.

Proof:

1. *This follows from the inequality* $\left||f(x)|-|L|\right| \leq |f(x)-L|$.
2. *This follows from the inequalities* $-|f(x)| \leq f(x) \leq |f(x)|$ *and* $\min(-f(x),f(x)) \leq |f(x)| \leq \max(-f(x),f(x))$ *and the Sandwich Theorem.*
3. *For all* $\varepsilon>0$ *there are* $\delta_1>0$ *and* $\delta_2>0$ *such that*

$$0<|x-a|<\delta_1 \implies |f(x)-L|<\varepsilon, \quad \text{and} \quad 0<|x-a|<\delta_2 \implies |g(x)-L'|<\varepsilon.$$

Take $\delta=\min(\delta_1,\delta_2)$. *Then*

$$0<|x-a|<\delta \implies |f(x)+\lambda g(x)-(L+\lambda L')| \leq |f(x)-L|+|\lambda|\,|g(x)-L'| < (1+|\lambda|)\varepsilon.$$

Since the dextral side can be made arbitrarily small, the assertion follows.

4. *For all* $\varepsilon>0$ *there are* $\delta_1>0$ *and* $\delta_2>0$ *such that*

$$0<|x-a|<\delta_1 \implies |f(x)-L|<\varepsilon, \quad \text{and} \quad 0<|x-a|<\delta_2 \implies |g(x)-L'|<\varepsilon.$$

Also, by Theorem 426, g *is locally bounded and so there exists* $B>0$, *and* $\delta_3>0$ *such that*

$$0<|x-a|<\delta_3 \implies |g(x)|<B.$$

Take $\delta=\min(\delta_1,\delta_2,\delta_3)$. *Then*

$$|f(x)g(x)-LL'| = |(f(x)-L)g(x)+L(g(x)-L')| \leq |f(x)-L|\,|g(x)|+|L|\,|g(x)-L'| < (B+|L'|)\varepsilon.$$

As the dextral side can be made arbitrarily small, the result follows.

5. *For all* $\varepsilon>0$ *there are* $\delta_1>0$, $B>0$, *and* $\delta_2>0$ *such that*

$$0<|x-a|<\delta_1 \implies |f(x)|<\varepsilon, \quad \text{and} \quad 0<|x-a|<\delta_2 \implies |g(x)|<B.$$

Take $\delta=\min(\delta_1,\delta_2)$. *Then*

$$|f(x)g(x)| \leq |B|\,|f(x)| < B\varepsilon.$$

As the dextral side can be made arbitrarily small, the result follows.

6. *First* $|g(x)| \to |L'|$ *as* $x\to a$ *by part (1). Hence, for* $\varepsilon=\left|\dfrac{L'}{2}\right|>0$ *there is a sufficiently small* $\delta'>0$ *such that*

$$\left||g(x)|-|L'|\right| < \frac{|L'|}{2} \implies |L'|-\frac{|L'|}{2} < |g(x)| < |L'|+\frac{|L'|}{2} \implies \frac{|L'|}{2} < |g(x)| < \frac{3|L'|}{2},$$

that is, $|g(x)|$ *is bounded away from* 0 x *sufficiently close to* a. *Now, for all* $\varepsilon>0$ *there are* $\delta_1>0$ *and* $\delta_2>0$ *such that*

$$0<|x-a|<\delta_1 \implies |f(x)-L|<\varepsilon, \quad \text{and} \quad 0<|x-a|<\delta_2 \implies |g(x)-L'|<\varepsilon.$$

For $\delta=\min(\delta_1,\delta_2,\delta')$,

$$0<|x-a|<\delta \implies L-\varepsilon<f(x)<L+\varepsilon, \quad \frac{|L'|}{2}<|g(x)|<\frac{3|L'|}{2}, \quad \text{and} \quad L'-\varepsilon<g(x)<L'+\varepsilon.$$

Hence

$$\left|\frac{f(x)}{g(x)}-\frac{L}{L'}\right| = \left|\frac{L'f(x)-Lg(x)}{g(x)L'}\right| = \left|\frac{L'(f(x)-L)-L(g(x)-L')}{g(x)L'}\right| \leq \frac{|L'|\,|f(x)-L|+|L|\,|g(x)-L'|}{|g(x)|\,|L'|} < \frac{2(|L'|+|L|)\varepsilon}{|L'|\,|L'|},$$

which gives the result.

❑

In the manner of proof of Proposition 422, we may prove the following two propositions.

430 Definition-Proposition (Cauchy-Heine, Limit at $+\infty$) Let $f : \left]a\,;+\infty\right[\to \mathbb{R}$ The following are equivalent.

1. For each sequence $\{x_n\}_{n=1}^{+\infty}$ of points in the interval $\left]a\,;+\infty\right[$,
$$x_n \to +\infty \implies f(x_n) \to L.$$

2. $\forall \varepsilon > 0$, $\exists M$, $M > \max(0, a)$, such that
$$x \geq M \implies |f(x) - L| < \varepsilon.$$

If either condition is fulfilled we say that f *has a limit* L *as* x *tends towards* $+\infty$ and we write
$$L = \lim_{x\to+\infty} f(x).$$

431 Definition-Proposition (Cauchy-Heine, Limit at $-\infty$) Let $f : \left]-\infty\,;a\right[\to \mathbb{R}$ The following are equivalent.

1. For each sequence $\{x_n\}_{n=1}^{+\infty}$ of points in the interval $\left]-\infty\,;a\right[$,
$$x_n \to -\infty \implies f(x_n) \to L.$$

2. $\forall \varepsilon > 0$, $\exists M$, $M < \min(0, a)$, such that
$$x \leq M \implies |f(x) - L| < \varepsilon.$$

If either condition is fulfilled we say that f *has a limit* L *as* x *tends towards* $-\infty$ and we write
$$L = \lim_{x\to-\infty} f(x).$$

432 Definition We write $\lim_{x\to a+} f(x) = +\infty$ or $\lim_{x\searrow a} f(x) = +\infty$ if $\forall M > 0$, $\exists\delta > 0$ such that
$$x \in \left]a\,;a+\delta\right[\implies f(x) > M.$$
Similarly, we write $\lim_{x\to a-} f(x) = +\infty$ or $\lim_{x\nearrow a} f(x) = +\infty$ if $\forall M > 0$, $\exists\delta > 0$ such that
$$x \in \left]a-\delta\,;a\right[\implies f(x) > M.$$
Finally, we write $\lim_{x\to a} f(x) = +\infty$ if $\forall M > 0$, $\exists\delta > 0$ such that
$$x \in \left]a-\delta\,;a+\delta\right[\implies f(x) > M.$$

433 Definition We write $\lim_{x\to a+} f(x) = -\infty$ or $\lim_{x\searrow a} f(x) = -\infty$ if $\forall M < 0$, $\exists\delta > 0$ such that
$$x \in \left]a\,;a+\delta\right[\implies f(x) < M.$$
Similarly, we write $\lim_{x\to a-} f(x) = -\infty$ or $\lim_{x\nearrow a} f(x) = -\infty$ if $\forall M < 0$, $\exists\delta > 0$ such that
$$x \in \left]a-\delta\,;a\right[\implies f(x) < M.$$
Finally, we write $\lim_{x\to a} f(x) = -\infty$ if $\forall M < 0$, $\exists\delta > 0$ such that
$$x \in \left]a-\delta\,;a+\delta\right[\implies f(x) < M.$$

434 Theorem Let X, Y be subsets of $\mathbb{R}$, $a \in X$ and $b \in Y$, $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ such that $f(X) \subsetneqq Y$, and let $L \in \mathbb{R}$. Then
$$\lim_{x\to a} f(x) = a \quad\text{and}\quad \lim_{x\to b} g(x) = L \quad\implies\quad \lim_{x\to a} (g\circ f)(x) = L.$$

Proof:

❑

Homework

435 Problem Prove that $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist.

436 Problem Let m, n be strictly positive integers. Prove that $\lim_{x\to 1} \frac{x^n - 1}{x^m - 1} = \frac{n}{m}$.

437 Problem Let $X \subseteqq \mathbb{R}$, $a \in \mathbb{R}$, and $f, g : X \to \mathbb{R}$. If $f(x) \to +\infty$ and there exists a neighbourhood $\mathscr{N}_a \subseteqq X$ of a where $f(x) \leq g(x)$, prove that $g(x) \to +\infty$.

438 Problem Let $X \subseteqq \mathbb{R}$, $a \in \mathbb{R}$, and $f, g : X \to \mathbb{R}$. Suppose that $\lim_{x\to a} f(x) = +\infty$. Demonstrate that

1. If $\lim_{x\to a} g(x) = +\infty$, then $\lim_{x\to a} (f(x) + g(x)) = +\infty$.
2. If $\lim_{x\to a} g(x) = L \in \mathbb{R}$, then $\lim_{x\to a} (f(x) + g(x)) = +\infty$.
3. If $\lim_{x\to a} g(x) = +\infty$, then $\lim_{x\to a} (f(x)g(x)) = +\infty$.
4. If $\lim_{x\to a} g(x) = L > 0$, then $\lim_{x\to a} (f(x)g(x)) = +\infty$.

439 Problem (Cauchy Criterion for Functional Limits) Let $X \subseteqq \mathbb{R}$, $a \in X$, and $f : X \to \mathbb{R}$. Prove that f has a limit at a (finite or infinite) if and only if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $\left|x' - x''\right| < \delta$ implies $\left|f(x') - f(x'')\right| < \varepsilon$.

5.2 Continuity

440 Definition A function $f : \left]a\,;b\right[\to \mathbb{R}$ is said to be *continuous at the point* $x_0 \in \left]a\,;b\right[$, if we can exchange limiting operations, as in

$$\lim_{x\to x_0} f(x) = f\left(\lim_{x\to x_0} x\right) \qquad (= f(x_0)).$$

In other words, a function is continuous at the point x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0, \quad \text{such that} \quad |x - x_0| < \delta \implies \left|f(x) - f(x_0)\right| < \varepsilon.$$

441 Definition A function $f : \left[a\,;b\right] \to \mathbb{R}$ is said to be *right continuous at* a, if

$$f(a) = f(a+).$$

It is said to be *left continuous at* b, if

$$f(b) = f(b-).$$

In view of the above definitions and Proposition 424, we have the following

442 Theorem The following are equivalent.

1. The function $f : \left]a\,;b\right[\to \mathbb{R}$ is continuous at the point $x_0 \in \left]a\,;b\right[$.
2. $f(x_0-) = f(x_0) = f(x_0+)$.
3. If $\{x_n\}_{n=1}^{+\infty}$, and for all n, $x_n \in \left]a\,;b\right[$, then $x_n \to x_0 \implies f(x_n) \to f(x_0)$.

443 Example What are the points of discontinuity of the function

$$f : \begin{array}{ccc} \left[0\,;+\infty\right[& \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \dfrac{1}{p+q} & \text{if } x \in \mathbb{Q} \cap \left[0\,;+\infty\right[, x = \dfrac{p}{q}, \text{ in lowest terms} \\ 0 & \text{if } x \in \left[0\,;+\infty\right[\setminus \mathbb{Q} \end{cases} \end{array} \quad ?$$

Solution: *Let $a \in \mathbb{Q}$. Since $\left[0\,;+\infty\right[\setminus \mathbb{Q}$ is dense in $\left[0\,;+\infty\right[$, there exists a sequence $\{a_n\}_{n=1}^{+\infty}$ of points in $\left[0\,;+\infty\right[\setminus \mathbb{Q}$ such that $a_n \to a$ as $n \to +\infty$. Observe that $f(a_n) = 0$ but $f(a) \neq 0$. Hence $a_n \to a$ does not imply $f(a_n) \to f(a)$ and f is not continuous at a. On the other hand, let $n \in \left[0\,;+\infty\right[\setminus \mathbb{Q}$. Then $f(b) = 0$. Let $\{b_n\}_{n=1}^{+\infty}$ be a sequence in $\left[0\,;+\infty\right[\cap \mathbb{Q}$ converging to b, $b_n = \dfrac{p_n}{q_n}$ in lowest terms. By Dirichlet's Approximation Theorem we must have $p_n \to +\infty$ and $q_n \to +\infty$. Hence $\dfrac{1}{p_n + q_n} \to 0$. So f is continuous at b. In conclusion, f is continuous at every irrational in $\left[0\,;+\infty\right[$ and discontinuous at every rational in $\left[0\,;+\infty\right[$.*

444 DEFINITION-PROPOSITION (Oscillation of a function at a point) Let f be bounded. The function $\omega : \mathbf{Dom}(f) \to [0;+\infty[$, called the *oscillation of f at x* and given by

$$\omega(f,x) = \lim_{\delta\to 0+} \sup\{\left|f(a)-f(b)\right| : |a-x|<\delta, |b-x|<\delta\}$$

is well-defined. Moreover, f is continuous at x if and only if $\omega(f,x) = 0$.

Proof: *Observe that in fact*

$$\omega(f,x) = \lim_{\delta\to 0+} \sup\{\left|f(a)-f(b)\right| : |a-x|<\delta, |b-x|<\delta\} = \inf_{\delta>0}\sup\{\left|f(a)-f(b)\right| : |a-x|<\delta, |b-x|<\delta\} \leq \left|f(a)-f(b)\right| \leq 2\left|f\right| < +\infty$$

This says that $\omega(f,x)$ is well-defined.

❑

445 Definition We say that a function f is continuous on the closed interval $\left[a\,;b\right]$ if it is continuous everywhere on $\left]a\,;b\right[$, continuous on the right at a and continuous on the left at b. If $X \subseteqq \mathbb{R}$, then $f : X \to \mathbb{R}$ is said to be *continuous on X* (or *continuous*) if it is continuous at every element of X.

446 THEOREM Let $X \subseteqq \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous if and only if the the inverse image of an open set is open in X.

Proof:

$\Longrightarrow$ *Let $A \subseteqq \mathbb{R}$ be an open set. We must shew that $f^{-1}(A)$ is open in X. Let $a \in f^{-1}(A)$. Since $f(a) \in A$ and A is open in $\mathbb{R}$, there exists an $r > 0$ such that $\left]f(a)-r\,;f(a)+r\right[\subseteqq A$. Since f is continuous at a, there exists a $\delta > 0$ such that*

$$|x-a|<\delta \implies \left|f(x)-f(a)\right|<r, \quad \text{that is,} \quad x \in \left]a+\delta\,;a-\delta\right[\implies f(x) \in \left]f(a)-r\,;f(a)+r\right[,$$

$$\text{that is,} \quad x \in \left]a+\delta\,;a-\delta\right[\implies x \in f^{-1}\left(\left]f(a)-r\,;f(a)+r\right[\right),$$

$$\text{that is,} \quad \left]a+\delta\,;a-\delta\right[\subseteqq f^{-1}\left(\left]f(a)-r\,;f(a)+r\right[\right)$$

Since $f^{-1}\left(\left]f(a)-r\,;f(a)+r\right[\right) \subseteqq f^{-1}(A)$, we have shewn that $\left]a+\delta\,;a-\delta\right[\subseteqq f^{-1}(A)$, which means that for any a, a neighbourhood of a lies entirely in $f^{-1}(A)$, that is, $f^{-1}(A)$ is open.

$\Leftarrow$ *Given $\varepsilon > 0$, we must find a $\delta > 0$ such that for all $a \in X$,*

$$|x-a|<\delta \implies \left|f(x)-f(a)\right|<\varepsilon.$$

Now

$$\left|f(x)-f(a)\right|<\varepsilon \implies f(x) \in \left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\implies x \in f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right).$$

Now, $\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\subseteqq \mathbb{R}$ is open in $\mathbb{R}$, and so, by assumption, so is $f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right)$. This means that if $t \in f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right)$ then there is a $r > 0$ such that

$$\left]t-r\,;t+r\right[\subseteqq f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right).$$

But clearly $a \in f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right)$, and hence there is a $\delta > 0$ such that

$$\left]a-\delta\,;a+\delta\right[\subseteqq f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right).$$

Thus

$$x \in \left]a-\delta\,;a+\delta\right[\implies x \in f^{-1}\left(\left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[\right),$$

or equivalently,

$$|x-a|<\delta \implies f(x)\in \left]f(a)-\varepsilon\,;f(a)+\varepsilon\right[,$$

that is,

$$|x-a|<\delta \implies \left|f(x)-f(a)\right|<\varepsilon,$$

as we needed to shew.

❑

447 THEOREM Let $X \subseteq \mathbb{R}$. A function $f:X\to\mathbb{R}$ is continuous if and only if the the inverse image of a closed set is closed in X.

Proof: *Let $F\subseteq\mathbb{R}$ be a closed set. Then $\mathbb{R}\setminus F$ is open. By Theorem 446 $f^{-1}(\mathbb{R}\setminus F)$ is open in X, and so $X\setminus f^{-1}(\mathbb{R}\setminus F)$ is closed in X. But $X\setminus f^{-1}(\mathbb{R}\setminus F)=f^{-1}(F)$, proving the theorem.* ❑

448 THEOREM If two continuous functions agree on a dense set of the reals, then they are identical. That is, if $X\subseteq\mathbb{R}$ is dense in $\mathbb{R}$ and if $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ satisfy $f(x)=g(x)$ for all $x\in X$, then $f(x)=g(x)$ for all $x\in\mathbb{R}$.

Proof: *Let $a\in\mathbb{R}\setminus X$. Since X is dense in $\mathbb{R}$, there is a sequence $\{x_n\}_{n=1}^{+\infty}\subseteq X$ such that $x_n\to a$ as $n\to+\infty$. Notice that since $x_n\in X$, we have $f(x_n)=g(x_n)$. By continuity*

$$f(a)=f\left(\lim_{n\to+\infty}x_n\right)=\lim_{n\to+\infty}f(x_n)=\lim_{n\to+\infty}g(x_n)=g\left(\lim_{n\to+\infty}x_n\right)=g(a),$$

proving the theorem. ❑

449 THEOREM (Cauchy's Functional Equation) Let f be a continuous function defined over the real numbers that satisfies the *Cauchy functional equation*:

$$\forall(x,y)\in\mathbb{R}^2,\quad f(x+y)=f(x)+f(y).$$

Then f is linear, that is, there is a constant c such that $f(x)=cx$.

Proof: *Our method of proof is as follows. We first prove the assertion for positive integers n using induction. We then extend our result to negative integers. Thence we extend the result to reciprocals of integers and after that to rational numbers. Finally we extend the result to all real numbers by means of Theorem 448.*

We prove by induction that for integer $n\geq 0$, $f(nx)=nf(x)$. Using the functional equation,

$$f(0\cdot x)=f(0\cdot x+0\cdot x)=f(0\cdot x)+f(0\cdot x)\implies f(0\cdot x)=0f(x),$$

and the assertion follows for $n=0$. Assume $n\geq 1$ is an integer and that $f((n-1)x)=(n-1)f(x)$. Then

$$f(nx)=f((n-1)x+x)=f((n-1)x)+f(x)=(n-1)f(x)+f(x)=nf(x),$$

proving the assertion for all strictly positive integers.

Let $m<0$ be an integer. Then $-m>0$ is a strictly positive integer, for which the result proved in the above paragraph holds, and thus and by the above paragraph, $f(-mx)=-mf(x)$. Now,

$$0=f(0)\implies 0=f(mx+(-mx))=f(mx)+f(-mx)\implies f(mx)=-f(-mx)=-(-mf(x))=mf(x),$$

and the assertion follows for negative integers. We have thus proved the theorem for all integers.

Assume now that $x=\dfrac{a}{b}$, with $a\in\mathbb{Z}$ and $b\in\mathbb{Z}\setminus\{0\}$. Then $f(a)=f(a\cdot 1)=af(1)$ and $f(a)=f\left(b\dfrac{a}{b}\right)=bf\left(\dfrac{a}{b}\right)$ by the result we proved for integers and hence

$$af(1)=bf\left(\frac{a}{b}\right)\implies=f\left(\frac{a}{b}\right)=f(1)\left(\frac{a}{b}\right).$$

We have established that for all rational numbers $x\in\mathbb{Q}$, $f(x)=xf(1)$.

We have not used the fact that the function is continuous so far. Since the rationals are dense in the reals the extension of the result now follows from Theorem 448.❑

Homework

450 Problem Find all functions $f:\mathbb{R}\to\mathbb{R}$, continuous at $x=0$ such that $\forall x\in\mathbb{R}$, $f(x)=f(3x)$.

451 Problem Find all functions $f:\mathbb{R}\to\mathbb{R}$, continuous at $x=0$ such that $\forall x\in\mathbb{R}$, $f(x)=f\left(\frac{x}{1+x^2}\right)$.

452 Problem Determine the set of points of discontinuity of the function $f:\mathbb{R}\to\mathbb{R}$, $f:x\mapsto \lfloor x\rfloor+\sqrt{x-\lfloor x\rfloor}$.

453 Problem What are the points of discontinuity of the function $f:\begin{matrix}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & \begin{cases} x & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\end{matrix}$?

454 Problem What are the points of discontinuity of the function $f:\begin{matrix}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & \begin{cases} 0 & \text{if } x\in\mathbb{Q}\\ x & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\end{matrix}$?

455 Problem What are the points of discontinuity of the function $f:\begin{matrix}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & \begin{cases} 0 & \text{if } x\in\mathbb{Q}\\ 1 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\end{matrix}$?

456 Problem What are the points of discontinuity of the function $f:\begin{matrix}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & \begin{cases} \cos x & \text{if } x\in\mathbb{Q}\\ \sin x & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\end{matrix}$?

457 Problem Find all functions $f:\mathbb{R}\to\mathbb{R}$, continuous at $x=1$ such that $\forall x\in\mathbb{R}$, $f(x)=-f(x^2)$.

458 Problem Let $a\in\mathbb{R}$ be fixed. Find all functions $f:\mathbb{R}\to\mathbb{R}$, continuous everywhere such that $\forall(x,y)\in\mathbb{R}^2$, $f(x-y)=f(x)-f(y)+axy$.

459 Problem Let $f:\left[0\,;+\infty\right[\to\left[0\,;+\infty\right[$, $x\mapsto\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}$. Is f right-continuous at 0?

5.3 Algebraic Operations with Continuous Functions

460 THEOREM (Algebra of Continuous Functions) Let $f,g:\left]a\,;b\right[\to\mathbb{R}$ be continuous a the point $x_0\in\left]a\,;b\right[$. Then

1. $f+g$ is continuous at x_0.
2. fg is continuous at x_0.
3. if $g(x_0)\neq 0$, $\frac{f}{g}$ is continuous at x_0.

Proof: *This follows directly from Theorem 429.* ❑

461 THEOREM Let X,Y be subsets of $\mathbb{R}$, $a\in X$ and $b\in Y$, $f:X\to\mathbb{R}$, $g:Y\to\mathbb{R}$ such that $f(X)\subsetneqq Y$. If f is continuous at a and g is continuous at $f(a)$, then $g\circ f$ is continuous at a.

Proof: *This follows at once from Theorem 434.* ❑

462 THEOREM Let $f:I\to\mathbb{R}$ be a monotone function, where $I\subsetneqq\mathbb{R}$ is a non-empty interval. Then the set of points of discontinuity of f is either finite or countable.

With Theorems 460 and 461 we can now demonstrate the

5.4 Monotonicity and Inverse Image

463 Definition Let X and Y be subsets of $\mathbb{R}$. Let $f:X\to Y$, and assume that X has at least two elements. Then f is said to be

- *increasing* if $\forall(a,b)\in X^2$, $a<b\implies f(a)\leq f(b)$. Equivalently, if the ratio $\frac{f(b)-f(a)}{b-a}\geq 0$.
- *strictly increasing* if $\forall(a,b)\in X^2$, $a<b\implies f(a)<f(b)$. Equivalently, if the ratio $\frac{f(b)-f(a)}{b-a}>0$.
- *decreasing* if $\forall(a,b)\in X^2$, $a<b\implies f(a)\geq f(b)$. Equivalently, if the ratio $\frac{f(b)-f(a)}{b-a}\leq 0$.
- *strictly decreasing* if $\forall(a,b)\in X^2$, $a<b\implies f(a)>f(b)$. Equivalently, if the ratio $\frac{f(b)-f(a)}{b-a}<0$.

f is said to be *monotonic* if it is either increasing or decreasing, and *strictly monotonic* if it is either strictly increasing or strictly decreasing.

☞ *Observe that if f is increasing, then $-f$ is decreasing, and conversely. Similarly for strictly monotonic functions.*

464 THEOREM Let $X \subseteqq \mathbb{R}$ and let $f : X \to \mathbb{R}$ be strictly monotone. Then f is injective.

Proof: *Recall that f is injective if $x \neq y \implies f(x) \neq f(y)$. If f is strictly increasing then $x < y \implies f(x) < f(y)$ and if f is strictly decreasing then $x < y \implies f(x) > f(y)$. In either case, the condition for injectivity is fulfilled.* ❑

465 THEOREM Let $I \subseteqq \mathbb{R}$ be an interval and let $f : I \to f(I)$ be strictly monotone. Then f^{-1} is strictly monotone in the same sense as f.

Proof: *Assume first that f is strictly increasing and put $x = f^{-1}(a)$, $y = f^{-1}(b)$ and that $a < b$. If $x \geq y$, then, since f is strictly increasing, $f(x) \geq f(y)$. But then, $f(f^{-1}(a)) \geq f(f^{-1}(b)) \implies a \geq b$, a contradiction.*

A similar argument finishes the theorem for f strictly decreasing.

❑

The following theorem is remarkable, since it does not allude to any possible continuity of the function in question.

466 THEOREM Let $I \subseteqq \mathbb{R}$ be an interval and let $f : I \to f(I)$ be strictly monotone. Then f^{-1} is continuous.

Proof: *Let $b \in f(I)$, $b = f(a)$, and $\varepsilon > 0$. We must shew that there is $\delta > 0$ such that*

$$|y - b| < \delta \implies |f^{-1}(y) - b| < \varepsilon.$$

If a is not an endpoint of I, there is an $\alpha > 0$ such that $\left]a - \alpha\,; a + \alpha\right[\subseteqq I$. Put $\varepsilon' = \min(\varepsilon, \alpha)$. Since both f and f^{-1} are both strictly monotone

$$|f^{-1}(y) - b| < \varepsilon' \implies b - \varepsilon' < f^{-1}(y) < b + \varepsilon' \implies f(b - \varepsilon') < f(f^{-1}(y)) < f(b + \varepsilon') \implies f(b - \varepsilon') < y < f(b + \varepsilon').$$

Since f is strictly increasing and $a - \varepsilon' < a$, $f(a - \varepsilon') < f(a) = b$. Thus there must be an $\eta > 0$ such that $f(a - \varepsilon') = b - \eta < b$. Similarly, there is an η' such that $b < b + \eta' = f(a + \varepsilon')$. Putting $\eta'' = \min(\eta, \eta')$, we have that for all $y \in f(I)$,

$$\begin{aligned} |y - b| < \eta'' &\implies b - \eta'' < y < b + \eta'' \\ &\implies b - \eta < y < b + \eta' \\ &\implies a - \varepsilon' < f^{-1}(y) < a + \varepsilon' \\ &\implies |f^{-1}(y) - f^{-1}(b)| < \varepsilon', \end{aligned}$$

finishing the proof for when a is not an endpoint. If a were an endpoint, the above proof carries by suppressing one of η or η'. ❑

467 THEOREM A continuous function $f : \left[a\,; b\right] \to f\left(\left[a\,; b\right]\right)$ is invertible if and only if it is strictly monotone.

Proof:

$\implies$ *Assume f is continuous and invertible. Since f is injective, $f(a) \neq f(b)$. Assume that $f(a) < f(b)$, if $f(a) > f(b)$ the argument is similar. We would like to shew that if $a' < b' \implies f(a') < f(b')$. Consider the continuous function $g : \left[0\,; 1\right] \to \mathbb{R}$,*

$$g(t) = f((1-t)a + ta') - f((1-t)b + tb').$$

We have

$$g(0) = f(a) - f(b) < 0 \qquad \text{and} \qquad g(1) = f(a') - f(b').$$

If $g(1) = 0$*, then we must have* $a' = b'$*, contradicting* $a' < b'$*. If* $g(1) > 0$*, then by the Intermediate Value Theorem there must be an* $s \in \left]0\,;1\right[$ *such that* $g(s) = 0$*. This entails*

$$(1-s)a + sa' = (1-s)b + sb' \implies 0 > (1-s)(a-b) = s(b'-a') > 0,$$

absurd. This entails that $g(1) < 0 \implies f(a') < f(b')$*, as wanted.*

$\Leftarrow$ *Trivially,* f *is surjective. If* f *is strictly monotone, then* f *is injective by Theorem 464, and thus* f *is invertible, by Theorem 41.*

❑

5.5 Convex Functions

468 Definition Let $A \times B \subseteq \mathbb{R}^2$. A function $f : A \to B$ is *convex* in A if $\forall (a, b, \lambda) \in A^2 \times [0;1]$,

$$f(\lambda a + (1-\lambda)b) \leq f(a)\lambda + (1-\lambda)f(b).$$

It is *strictly convex* if the inequality above is strict. Similarly, a function $g : A \to B$ is *concave* in A if $\forall (a, b, \lambda) \in A^2 \times [0;1]$,

$$g(\lambda a + (1-\lambda)b) \geq g(a)\lambda + (1-\lambda)g(b).$$

It is *strictly concave* if the inequality above is strict.

5.5.1 Graphs of Functions

469 Definition Given a function f, its *graph* is the set on the plane

$$\Gamma_f = \{(x, y) \in \mathbb{R}^2 : y = f(x)\}.$$

470 Example Figures **??** through **??** shew the graphs of a few standard functions, with which we presume the reader to be familiar.

5.6 Classical Functions

5.6.1 Affine Functions

471 Definition An *affine function* is one with assignment rule of the form $x \mapsto ax + b$, where a, b are real constants.

472 Theorem The graph of an affine function is a line on the plane. Conversely, any non-vertical straight line on the plane is the graph on an affine function.

5.6.2 Quadratic Functions

5.6.3 Polynomial Functions

5.6.4 Exponential Functions

473 Definition-Proposition Let $x \in \mathbb{R}$ be fixed. The sequence $\left\{\left(1 + \frac{x}{n}\right)^n\right\}_{n > -x}^{+\infty}$ is bounded and strictly increasing. Thus it converges and we define *the natural exponential function* by

$$\exp : \mathbb{R} \to \mathbb{R}, \qquad \exp(x) := \lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n.$$

Proof: *Observe that* $1+\frac{x}{n}>0$ *for* $n>-x$. *Using the AM-GM Inequality with* $x_1=1, x_2=\cdots=x_{n+1}=1+\frac{x}{n}$

$$\left(1+\frac{x}{n}\right)^{n/(n+1)}<\frac{1+n\left(1+\frac{x}{n}\right)}{n+1}=1+\frac{x}{n+1}\implies\left(1+\frac{x}{n}\right)^{n}<\left(1+\frac{x}{n+1}\right)^{n+1},$$

whence the sequence is increasing.

For $0<x\leq 1$ *then* $\left(1+\frac{x}{n}\right)^n\leq\left(1+\frac{1}{n}\right)^n<e$, *by Theorem 328.*

If $x>1$ *then by the already proved monotonicity,*

$$\left(1+\frac{x}{n}\right)^n\leq\left(1+\frac{\lfloor x\rfloor+1}{n}\right)^n<\left(1+\frac{\lfloor x\rfloor+1}{n(\lfloor x\rfloor+1)}\right)^{n(\lfloor x\rfloor+1)}<e^{\lfloor x\rfloor+1}.$$

If $x\leq 0$ *then* $1+\frac{x}{n}\leq 1$ *and so* $\left(1+\frac{x}{n}\right)^n\leq 1$. ❑

By Theorem 328, $\exp(1)=e$. *We will later prove, in ????, that for all* $x\in\mathbb{R}$, $\exp(x)=e^x$.

5.6.5 Logarithmic Functions

5.6.6 Trigonometric Functions

474 THEOREM Let $x\in\left]0;\frac{\pi}{2}\right[$. Then $\sin x<x<\tan x$.

Proof:

❑

Homework

475 Problem How many solutions does the equation

$$\sin x=\frac{x}{100}$$

have?

476 Problem Prove that

$$\frac{2}{\pi}x\leq\sin(x)\leq x, \forall\, x\in\left[0;\frac{\pi}{2}\right].$$

477 Problem How many solutions does the equation

$$\sin x=\log x$$

have?

478 Problem How many solutions does the equation

$$\sin(\sin(\sin(\sin(\sin(x)))))=\frac{x}{3}$$

have?

479 Problem (Chebyshev Polynomials)

480 Problem (Cardano's Formula)

5.6.7 Inverse Trigonometric Functions

5.7 Continuity of Some Standard Functions.

5.7.1 Continuity Polynomial Functions

481 LEMMA Let $K\in\mathbb{R}$ be a constant. The constant function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=K$ is everywhere continuous.

Proof: *Given* $a\in\mathbb{R}$ *and* $\varepsilon>0$, *take* $\delta=\varepsilon$. *Then clearly*

$$|x-a|<\delta\implies\left|f(x)-f(a)\right|<\varepsilon,$$

since $f(x)=f(a)=K$ *and the quantity after the implication is* $0<\varepsilon$ *and we obtain a tautology.* ❑

482 Lemma The identity function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x$ is everywhere continuous.

Proof: *Given $a\in\mathbb{R}$ and $\varepsilon>0$, take $\delta=\varepsilon$. Then clearly*

$$|x-a|<\delta \implies \left|f(x)-f(a)\right|<\varepsilon,$$

since the quantity after the implication is $|x-a|<\delta$ and we obtain a tautology. ❑

483 Lemma Given a strictly positive integer n, the power function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^n$ is everywhere continuous.

Proof: *By Lemma 482, the function $x\mapsto x$ is continuous. Applying this Lemma and the product rule from Theorem 460 n times, we obtain the result.* ❑

484 Theorem (Continuity of Polynomial Functions) Let n be a fixed positive integer. Let $a_k\in\mathbb{R}$, $0\le k\le n$ be constants. Then the polynomial function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n$ is everywhere continuous.

Proof: *This follows from Lemma 483 and the sum rule from Theorem 460 applied $n+1$ times.* ❑

5.7.2 Continuity of the Exponential and Logarithmic Functions

485 Lemma Let $a>1$. The exponential function $\mathbb{R}\to\mathbb{R}$, $x\mapsto a^x$ is continuous at $x=0$.

Proof: *For integral $n>0$ we know that $\lim\limits_{n\to+\infty} a^{1/n}=1$ by virtue of Theorem 325. We wish to shew that $a^x\to 1$ as $x\to 0$. Observe first that $\lim\limits_{n\to+\infty} a^{-1/n}=\lim\limits_{n\to+\infty}\dfrac{1}{a^{1/n}}=1$ also. Thus given $\varepsilon>0$, and since $a>1$, there is $N>0$ such that*

$$1-\varepsilon<a^{-1/N}<a^{1/N}<1+\varepsilon.$$

If $x\in\left]-\dfrac{1}{N};\dfrac{1}{N}\right[$ then,

$$a^{-1/N}<a^x<a^{1/N}.$$

By the above, this implies that

$$1-\varepsilon<a^x<1+\varepsilon \implies \left|a^x-1\right|<\varepsilon \implies \left|a^x-a^0\right|<\varepsilon,$$

finishing the proof. ❑

486 Theorem (Continuity of the Exponential Function) Let $a>0$, $a\neq 1$. The exponential function $f:\mathbb{R}\to\left]0;+\infty\right[$, $x\mapsto a^x$ is everywhere continuous.

Proof: *Assume first that $a>1$. Let us shew that it is continuous at an arbitrary $u\in\mathbb{R}$. If $x\to u$ then $x-u\to 0$. Thus*

$$\lim_{x\to u} a^x = a^u\lim_{x\to u} a^{x-u} = a^u\lim_{x-u\to 0} a^{x-u} = a^u\lim_{t\to 0} a^t = a^u\cdot 1 = a^u,$$

by Lemma 485, and so the continuity is established for $a>1$.

If $0<a<1$ then $\dfrac{1}{a}>1$ and by what we have proved, $x\mapsto\dfrac{1}{a^x}$ is continuous. Then

$$\lim_{x\to u} a^x = \lim_{x\to u}\frac{1}{\frac{1}{a^x}} = \frac{1}{\frac{1}{a^u}} = a^u,$$

proving continuity in the case $0<a<1$. ❑

487 Lemma Let $a>0$, $a\neq 1$. Then $\left]0;+\infty\right[\to\mathbb{R}$, $x\mapsto\log_a x$ is everywhere continuous.

Proof: *Its inverse function* $\mathbb{R} \to \left]0\,;+\infty\right[$, $x \mapsto a^x$, *is everywhere continuous and strictly monotone. The result then follows from Theorem 466.* ❑

5.7.3 Continuity of the Power Functions

488 THEOREM Let $p \in \mathbb{R}$. Then $\left]0\,;+\infty\right[\to \left]0\,;+\infty\right[$, $x \mapsto x^p$ is everywhere continuous.

Proof: *This follows by the continuity of compositions:* $x^p = e^{p \log x}$. ❑

Homework

489 Problem Prove the continuity of the function $\mathbb{R} \to \left[-1\,;1\right]$, $x \mapsto \sin x$.

490 Problem Prove the continuity of the function $\left[-1\,;1\right] \to \left[-\frac{\pi}{2}\,;\frac{\pi}{2}\right]$, $x \mapsto \arcsin x$.

491 Problem Prove the continuity of the function $\mathbb{R} \to \left[-1\,;1\right]$, $x \mapsto \cos x$.

492 Problem Prove the continuity of the function $\left[-1\,;1\right] \to \left[0\,;\pi\right]$, $x \mapsto \arccos x$.

493 Problem Prove the continuity of the function $\mathbb{R} \setminus (2\mathbb{Z}+1)\frac{\pi}{2} \to \mathbb{R}$, $x \mapsto \tan x$.

494 Problem Prove the continuity of the function $\mathbb{R} \to \left]-\frac{\pi}{2}\,;\frac{\pi}{2}\right[$, $x \mapsto \arctan x$.

5.8 Inequalities Obtained by Continuity Arguments

The technique used Theorem 449, of proving results in a dense set of the real numbers and extending the result by continuity can be exploited in a variety of situations. We now use it to give a generalisation of Bernoulli's Inequality.

495 THEOREM (Generalisation of Bernoulli's Inequality) Let $(\alpha, x) \in \mathbb{R}^2$ with $x \geq -1$. If $0 < \alpha < 1$ then

$$(1+x)^{\alpha} \leq 1 + \alpha x.$$

If $\alpha \in \left]-\infty\,;0\right[\cup \left]1\,;+\infty\right[$ then

$$(1+x)^{\alpha} \geq 1 + \alpha x.$$

Equality holds in either case if and only if $x = 0$.

Proof: *Let* $\alpha \in \mathbb{Q}$, $0 < \alpha < 1$. *Then* $\alpha = \dfrac{m}{n}$ *for integers* m, n *with* $1 \leq m < n$. *Since* $x + 1 \geq 0$, *we may use the AM-GM Inequality to obtain*

$$\begin{aligned}
(1+x)^{\alpha} &= (1+x)^{m/n} \\
&= \left((1+x)^m \cdot 1^{n-m}\right)^{1/n} \\
&\leq \frac{m(1+x)+(n-m)\cdot 1}{n} \\
&= \frac{n+mx}{n} \\
&= 1 + \frac{m}{n}x \\
&= 1 + \alpha x.
\end{aligned}$$

Equality holds when are the factors are the same, that is, when $1 + x = 1 \implies x = 0$.

Assume now that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ *with* $0 < \alpha < 1$. *We can find a sequence of rational numbers* $\{a_n\}_{n=1}^{+\infty} \subseteq \mathbb{Q}$ *such that* $a_n \to \alpha$ *as* $n \to +\infty$. *Then*

$$(1+x)^{a_n} \leq 1 + a_n x,$$

whence by the continuity of the power functions (Theorem 488),

$$(1+x)^{\alpha} = \lim_{n\to+\infty}(1+x)^{a_n} \le \lim_{n\to+\infty}(1+a_n x) = 1+\alpha x,$$

giving the result for all real numbers α *with* $0<\alpha<1$, *except that we need to prove that equality holds only for* $x=0$. *Take a rational number* r *with* $0<\alpha<r<1$, *and recall that we are assuming that* α *is irrational. Then*

$$(1+x)^{\alpha} = (1+x)^{\alpha/r})^{r} \le \left(1+\frac{\alpha}{r}x\right)^{r}.$$

Since the exponent on the right is rational, by what we have proved above $\left(1+\frac{\alpha}{r}x\right)^{r} \le 1+x$ *with equality if and only if* $x=0$. *Hence the full result has been proved for the case* $\alpha\in\mathbb{R}$ *with* $0<\alpha<1$.

Let $\alpha>1$. *If* $1+\alpha x<0$, *then obviously* $(1+x)^{\alpha}>0>1+\alpha x$, *and there is nothing to prove. Hence we will assume that* $\alpha x\ge -1$. *By the first part of the theorem, since* $0<\frac{1}{\alpha}<1$,

$$(1+\alpha x)^{1/\alpha} \le 1+\frac{1}{\alpha}\cdot\alpha x = 1+x \implies 1+\alpha x \le (1+x)^{\alpha},$$

with equality only if $x=0$. *The theorem has been proved for* $\alpha>1$.

Finally, let $\alpha<0$. *Again, if* $1+\alpha x<0$, *then obviously* $(1+x)^{\alpha}>0>1+\alpha x$, *and there is nothing to prove. Assume thus* $\alpha x\ge -1$. *Choose a strictly positive integer* n *satisfying* $0<-\alpha<n$. *Now,*

$$1 \ge 1-\frac{\alpha^2}{n^2}x^2 = \left(1-\frac{\alpha}{n}x\right)\left(1+\frac{\alpha}{n}x\right) \implies \frac{1}{1-\frac{\alpha}{n}x} \ge 1+\frac{\alpha}{n}x,$$

and so by the first pat of the theorem

$$\begin{aligned}(1+x)^{-\alpha/n} \le 1-\frac{\alpha}{n}x \quad &\implies \quad (1+x)^{\alpha/n} \ge \frac{1}{1-\frac{\alpha}{n}x}\\ &\implies \quad (1+x)^{\alpha/n} \ge 1+\frac{\alpha}{n}x\\ &\implies \quad (1+x)^{\alpha} \ge \left(1+\frac{\alpha}{n}x\right)^{n},\end{aligned}$$

and since n *is a positive integer,* $\left(1+\frac{\alpha}{n}x\right)^{n} \ge 1+n\cdot\frac{\alpha}{n}x = 1+\alpha x$ *and so* $(1+x)^{\alpha}\ge 1+\alpha x$ *also when* $\alpha<0$. *This finishes the proof of the theorem.* ❑

496 THEOREM (Monotonicity of Power Means) Let $a_1, a_2, \ldots, a_n$ be strictly positive real numbers and let $(\alpha,\beta)\in\mathbb{R}^2$ be such that $\alpha\cdot\beta\ne 0$ and $\alpha<\beta$. Then

$$\left(\frac{a_1^{\alpha}+a_2^{\alpha}+\cdots+a_n^{\alpha}}{n}\right)^{1/\alpha} \le \left(\frac{a_1^{\beta}+a_2^{\beta}+\cdots+a_n^{\beta}}{n}\right)^{1/\beta},$$

with equality if and only is $a_1=a_2=\cdots=a_n$.

Proof: *Assume first that* $0<\alpha<\beta$. *Put* $c_{\alpha} = \left(\frac{a_1^{\alpha}+a_2^{\alpha}+\cdots+a_n^{\alpha}}{n}\right)^{1/\alpha}$ *and* $d_k = \left(\frac{a_k}{c_{\alpha}}\right)^{\alpha}$. *Observe that*

$$\frac{c_{\beta}}{c_{\alpha}} = \left(\frac{\left(\frac{a_1}{c_{\alpha}}\right)^{\beta}+\left(\frac{a_2}{c_{\alpha}}\right)^{\beta}+\cdots+\left(\frac{a_n}{c_{\alpha}}\right)^{\beta}}{n}\right)^{1/\beta} = \left(\frac{d_1^{\beta/\alpha}+d_2^{\beta/\alpha}+\cdots+d_n^{\beta/\alpha}}{n}\right)^{1/\beta},$$

and that

$$\left(\frac{d_1+d_2+\cdots+d_n}{n}\right)^{1/\alpha} = \frac{1}{c_\alpha}\left(\frac{a_1^\alpha+a_2^\alpha+\cdots+a_n^\alpha}{n}\right)^{1/\alpha} = 1 \implies d_1+d_2+\cdots+d_n = n.$$

Put $d_k = 1 + x_k$. *Then* $x_1 + x_2 + \cdots + x_n = 0$. *By Theorem 495,*

$$d_k^{\beta/\alpha} = (1+x_k)^{\beta/\alpha} \geq 1 + \frac{\beta}{\alpha}x_k. \tag{5.1}$$

Letting k *run from* 1 *through* n *and adding,*

$$d_1^{\beta/\alpha} + d_2^{\beta/\alpha} + \cdots + d_n^{\beta/\alpha} \geq n + \frac{\beta}{\alpha}(x_1+x_2+\cdots+x_n) = n.$$

Hence

$$\frac{d_1^{\beta/\alpha} + d_2^{\beta/\alpha} + \cdots + d_n^{\beta/\alpha}}{n} \geq 1 \implies \frac{c_\beta}{c_\alpha} \geq 1,$$

proving the theorem when $0 < \alpha < \beta$.

If $\alpha < \beta < 0$, *then* $0 < \dfrac{\beta}{\alpha} < 1$. *The inequality in (5.1) is reversed, giving* $\dfrac{d_1^{\beta/\alpha} + d_2^{\beta/\alpha} + \cdots + d_n^{\beta/\alpha}}{n} \leq 1$, *and since* $\beta < 0$,

$$\frac{c_\beta}{c_\alpha} = \left(\frac{d_1^{\beta/\alpha} + d_2^{\beta/\alpha} + \cdots + d_n^{\beta/\alpha}}{n}\right)^{1/\beta} \geq 1^{1/\beta} = 1,$$

proving the theorem when $\alpha < \beta < 0$.

Finally, we tackle the case $\alpha < 0 < \beta$. *By the AM-GM Inequality, putting* $G = (a_1 a_2 \cdots a_n)^{1/n}$

$$G^\alpha = (a_1^\alpha a_2^\alpha \cdots a_n^\alpha)^{1/n} \leq \frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}.$$

Raising the quantities at the extreme of the inequalities to the power $-1/\alpha$ *and remembering that* $-1/\alpha > 0$, *we gather that*

$$\left(\frac{a_1^\alpha + a_2^\alpha + \cdots + a_n^\alpha}{n}\right)^{1/\alpha} \leq G.$$

In a similar manner,

$$G^\beta = (a_1^\beta a_2^\beta \cdots a_n^\beta)^{1/n} \leq \frac{a_1^\beta + a_2^\beta + \cdots + a_n^\beta}{n},$$

and

$$G \leq \left(\frac{a_1^\beta + a_2^\beta + \cdots + a_n^\beta}{n}\right)^{1/\beta},$$

since $\beta > 0$. *This finishes the proof.* ❑

497 LEMMA Let α, a, x be real numbers with $\alpha > 1$, $a > 0$, and $x \geq 0$. Then

$$x^\alpha - ax \geq (1-\alpha)\left(\frac{a}{\alpha}\right)^{\alpha/(\alpha-1)}.$$

Proof: *By Theorem 495, since* $\alpha > 1$,

$$(1+z)^\alpha \geq 1 + \alpha z, \qquad z \geq -1,$$

with equality only is $z = 0$. *Putting* $z = 1 + y$,

$$y^\alpha \geq 1 + \alpha(y-1) \implies y^\alpha - \alpha y \geq 1 - \alpha, \qquad y \geq 0,$$

with equality only if $y = 1$. *Let* $c > 0$ *be a constant. Multiplying the above inequality by* c^α *we obtain*

$$(cy)^\alpha - \alpha c^{\alpha-1}(cy) \geq (1-\alpha)c^\alpha, \qquad \text{for} \qquad y \geq 0.$$

Putting $x = cy$ *and* $a = \alpha c^{\alpha-1}$, *we get*

$$x^{\alpha} - ax \geq (1-\alpha)\left(\frac{a}{\alpha}\right)^{\alpha/(\alpha-1)},$$

with equality if and only if $x = c = \left(\frac{a}{\alpha}\right)^{\alpha/(\alpha-1)}$.

❑

498 Theorem (Young's Inequality) Let $p > 1$ and put $\frac{1}{p} + \frac{1}{q} = 1$. Then for $(x, y) \in ([0; +\infty[)^2$ we have

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Proof: *Put* $\alpha = p$, $a = py$ *in Lemma 497, obtaining*

$$x^p - (py)x \geq (1-p)\left(\frac{py}{p}\right)^{p/(p-1)} = (1-p)y^{p/(p-1)}.$$

Now, $\frac{1}{q} = \frac{p-1}{p} \implies q = \frac{p}{p-1}$ *and* $p - 1 = \frac{p}{q}$. *Hence*

$$x^p - (py)x \geq (1-p)y^{p/(p-1)} \implies (1-p)y^{p/(p-1)} \geq -\frac{p}{q}y^q,$$

and rearranging gives the result sought. ❑

We now derive a generalisation of the Cauchy-Bunyakovsky-Schwarz Inequality.

499 Theorem (Hölder Inequality) Let x_j, y_k, $1 \leq j, k \leq n$, be real numbers. Let $p > 1$ and put $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}.$$

Proof: *If either* $\sum_{k=1}^{n} |x_k|^p = 0$ *or* $\sum_{k=1}^{n} |y_k|^q = 0$ *there is nothing to prove, so assume otherwise. From Young's Inequality we have*

$$\frac{|x_k|}{\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}} \frac{|y_k|}{\left(\sum_{k=1}^{n}|y_k|^q\right)^{1/q}} \leq \frac{|x_k|^p}{\left(\sum_{k=1}^{n}|x_k|^p\right)p} + \frac{|y_k|^q}{\left(\sum_{k=1}^{n}|y_k|^q\right)q}.$$

Adding, we deduce

$$\begin{aligned}
\sum_{k=1}^{n} \frac{|x_k|}{\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}} \frac{|y_k|}{\left(\sum_{k=1}^{n}|y_k|^q\right)^{1/q}} &\leq \frac{1}{\left(\sum_{k=1}^{n}|x_k|^p\right)p} \sum_{k=1}^{n} |x_k|^p + \frac{1}{\left(\sum_{k=1}^{n}|y_k|^q\right)q} \sum_{k=1}^{n} |y_k|^q \\
&= \frac{\sum_{k=1}^{n}|x_k|^p}{\left(\sum_{k=1}^{n}|x_k|^p\right)p} + \frac{\left(\sum_{k=1}^{n}|y_k|^q\right)q}{\left(\sum_{k=1}^{n}|y_k|^q\right)q} \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

This gives

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}.$$

The result follows by observing that

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q}.$$

❑

Finally, we derive a generalisation of Minkowski's Inequality.

500 THEOREM (Generalised Minkowski Inequality) Let $p \in]1;+\infty[$. Let x_j, y_k, $1 \leq j, k \leq n$, be real numbers. Then the following inequality holds

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}.$$

Proof: *From the triangle inequality for real numbers*

$$|x_k + y_k|^p = |x_k + y_k||x_k + y_k|^{p-1} \leq \left(|x_k| + |y_k|\right)|x_k + y_k|^{p-1}.$$

Adding

$$\sum_{k=1}^{n} |x_k + y_k|^p \leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}. \tag{5.2}$$

By the Hölder Inequality

$$\begin{aligned}\sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} &\leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |x_k + y_k|^{(p-1)q}\right)^{1/q} \\ &= \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/q}\end{aligned} \tag{5.3}$$

In the same manner we deduce

$$\sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1} \leq \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/q}. \tag{5.4}$$

Hence (5.2) gives

$$\begin{aligned}\sum_{k=1}^{n} |x_k + y_k|^p &\leq \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/q} + \left(\sum_{k=1}^{n} |y_k|^q\right)^{1/q} \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/q} \\ &= \left(\left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p}\right) \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1/q}\end{aligned}$$

,

from where we deduce the result. ❑

Homework

501 Problem Prove that if $\alpha > 0$ and $n > 0$ an integer then

$$\frac{n^{1+\alpha} - (n-1)^{1+\alpha}}{1+\alpha} < n^{\alpha} < \frac{(n+1)^{1+\alpha} - n^{1+\alpha}}{1+\alpha}.$$

Deduce that

$$\lim_{n \to +\infty} \frac{1^{\alpha} + 2^{\alpha} + \cdots + n^{\alpha}}{n^{1+\alpha}} = \frac{1}{1+\alpha}.$$

5.9 Intermediate Value Property

502 THEOREM (Intermediate Value Theorem) Let $I \subsetneqq \mathbb{R}$ and let $(a,b) \in I^2$. Let $f : I \to \mathbb{R}$ be a continuous function such that $f(a) \leq f(b)$. Then f attains every intermediate value between $f(a)$ and $f(b)$, that is,

$$\forall t \in \left[f(a)\,;f(b)\right], \exists c \in I, \qquad \text{such that} \qquad f(c) = t.$$

Proof: *Suppose on the contrary that there is a* $t \in \left[f(a)\,;f(b)\right]$ *such that for all* $c \in I$, $f(c) \neq t$. *Hence* $f(a) < t < f(b)$. *Assume, without loss of generality, that* $a < b$. *Consider the sets*

$$U = \left]-\infty\,;a\right[\cup \left\{x \in \left[a\,;b\right] : f(x) < t\right\} = \left]-\infty\,;a\right[\cup f^{-1}\left(\left]-\infty\,;t\right[\cap \left]a\,;b\right[\right),$$

and

$$V = \left]b\,;+\infty\right[\cup \left\{x \in \left[a\,;b\right] : f(x) > t\right\} = \left]b\,;+\infty\right[\cup f^{-1}\left(\left]t\,;+\infty\right[\cap \left]a\,;b\right[\right).$$

Then U, V *are open sets of* $\mathbb{R}$ *by virtue of Theorem 446. But then* $\mathbb{R} = U \cup V$ *and* $U \cap V = \varnothing$, $U \neq \varnothing$, $V \neq \varnothing$, *contradicting the fact that* $\mathbb{R}$ *is connected. Thus there must exist a* c *such that* $f(c) = t$. ❑

503 COROLLARY A continuous function defined on an interval maps that interval into an interval.

Proof: *This follows at once from the Intermediate Value Theorem and the definition of an interval.* ❑

504 THEOREM (Bolzano's Theorem) If $f : \left[u\,;v\right] \to \mathbb{R}$ is continuous and $f(u)f(v) < 0$, then there is a $w \in \left]u\,;v\right[$ such that $f(w) = 0$.

Proof: *This follows at once from the Intermediate Value Theorem by putting* $a = \min(f(u), f(v)) < 0$ *and* $b = \max(f(u), f(v)) > 0$. ❑

505 COROLLARY Every polynomial $p(x) \in \mathbb{R}[x]$ with real coefficients and odd degree has at least one real root.

Proof: *Let* $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, *with* $a_n \neq 0$ *and* n *odd. Since* p *has odd degree,* $\lim_{x\to-\infty} p(x) = (-\infty)\,\mathbf{signum}\,(a_n)$ *and* $\lim_{x\to+\infty} p(x) = (+\infty)\,\mathbf{signum}\,(a_n)$, *which are of opposite sign. The polynomial must then attain positive and negative values and between values of opposite sign, it will have a real root.* ❑

506 COROLLARY If f is continuous at the point a and $f(a) \neq 0$, then there is a neighbourhood of a where $f(x)$ has the same sign as $f(a)$.

Proof: *Take* $\varepsilon = \dfrac{|f(a)|}{2} > 0$ *in the definition of continuity. There is a* $\delta > 0$ *such that*

$$|x - a| < \delta \implies |f(x) - f(a)| < \frac{|f(a)|}{2} \implies f(a) - \frac{|f(a)|}{2} < f(x) < f(a) + \frac{|f(a)|}{2},$$

from where the result follows. ❑

507 THEOREM A continuous function defined on a compact set maps that compact set into a compact set.

Proof: *Let* $f : X \to \mathbb{R}$ *be continuous and* $X \subsetneqq \mathbb{R}$ *compact. Let* $\{y_n\}_{n=1}^{+\infty} \subsetneqq f(X)$ *be an infinite sequence of* $f(X)$. *There are* $x_n \in X$ *such that* $x_n = f(y_n)$. *Since* $\{x_n\}_{n=1}^{+\infty} \subsetneqq X$ *is an infinite sequence of* X *and* X *is compact, it has a convergent subsequence in* X, *say,* $\{x_{n_k}\}_{k=1}^{+\infty}$ *with* $x_{n_k} \to x \in X$, *by virtue of Theorem 255. Since* f *is continuous*

$$x_{n_k} \to x \implies f(x_{n_k}) \to f(x).$$

Clearly $f(x) \in f(X)$. *Thus the arbitrary sequence* $\{y_n\}_{n=1}^{+\infty} \subsetneqq f(X)$ *has the convergent subsequence* $\{y_{n_k}\}_{k=1}^{+\infty}$ *in* $f(X)$, *and one more appeal to Theorem 255 proves compactness.* ❑

508 Theorem (Weierstrass Theorem) A continuous function $f : [a ; b] \to \mathbb{R}$ attains a maximum and a minimum on $[a ; b]$.

Proof: *By Theorem 507, $f([a ; b])$ is compact, and so, by the Heine-Borel Theorem, it is closed and bounded. Thus there exists $(m, M) \in \mathbb{R}^2$ such that $m = \inf_{x \in [a ; b]} f(x)$ and $M = \sup_{x \in [a ; b]} f(x)$. We must prove that these are attained in $[a ; b]$, i.e., that there exist $\mu \in [a ; b]$ and $\mu' \in [a ; b]$ such that $f(\mu) = m$ and $f(\mu') = M$. By the Approximation Property of the Infimum and the Supremum, we may find sequences $\{m_n\}_{n=1}^{+\infty} \subseteqq [a ; b]$, and $\{M_n\}_{n=1}^{+\infty} \subseteqq [a ; b]$ such that $m \leq m_n$ and $m_n \to m$, and also, $M_n \leq M$, and $M_n \to M$ as $n \to +\infty$. By the Intermediate Value Theorem, there exist $\mu_n \in [a ; b]$ and $\mu'_n \in [a ; b]$ such that $m_n = f(\mu_n)$ and $M_n = f(\mu'_n)$. By the compactness of $[a ; b]$ the sequences $\{\mu_n\}_{n=1}^{+\infty} \subseteqq [a ; b]$ and $\{\mu'_n\}_{n=1}^{+\infty} \subseteqq [a ; b]$ have convergent subsequences $\{\mu_{n_k}\}_{k=1}^{+\infty} \subseteqq [a ; b]$ and $\{\mu'_{n_k}\}_{k=1}^{+\infty} \subseteqq [a ; b]$ such that $\mu_{n_k} \to \mu \in [a ; b]$ and $\mu'_{n_k} \to \mu' \in [a ; b]$. By continuity and the uniqueness of limits,*

$$\mu_{n_k} \to \mu \implies m_{n_k} = f(\mu_{n_k}) \to m = f(\mu), \qquad \text{and} \qquad \mu'_{n_k} \to \mu' \implies M_{n_k} = f(\mu'_{n_k}) \to M = f(\mu'),$$

and so f attains both extrema in $[a ; b]$. ❑

509 Theorem (Fixed Point Theorem) Let $f : [a ; b] \to [a ; b]$ be continuous. Then f has a fixed point, that is, there is $c \in [a ; b]$ such that $f(c) = c$.

Proof: *If either $f(a) = a$ or $f(b) = b$ we are done. Assume then that $f(a) > a$ and $f(b) < b$. Put $g(x) = f(x) - x$. Then g is continuous, $g(a) > 0$ and $g(b) < 0$. By Bolzano's Theorem, there must be a $c \in]a ; b[$ such that $g(c) = 0$, that is, $f(c) - c = 0$, finishing the proof.* ❑

Homework

510 Problem Let $p(x), q(x)$ be polynomials with real coefficients such that

$$p(x^2 + x + 1) = p(x)q(x).$$

Prove that p must have even degree.

511 Problem A function f defined over all real numbers is continuous and for all real x satisfies

$$(f(x)) \cdot ((f \circ f)(x)) = 1.$$

Given that $f(1000) = 999$, find $f(500)$.

512 Problem Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to -\infty} f(x) = 0 = \lim_{x \to +\infty} f(x)$. If f is strictly negative somewhere on $\mathbb{R}$ then f attains a finite absolute minimum on $\mathbb{R}$. If f is strictly positive somewhere on $\mathbb{R}$ then f attains a finite absolute maximum on $\mathbb{R}$.

513 Problem Let $f : [0 ; 1] \to [0 ; 1]$ be continuous. Prove that there is no $c \in [0 ; 1]$ such that $f^{-1}(\{c\})$ has exactly two elements.

514 Problem Let f, g be continuous functions from $[0 ; 1]$ to $[0 ; 1]$ such that

$$\forall x \in [0 ; 1] \qquad f(g(x)) = g(f(x)).$$

Prove that f and g have a common fixed point in $[0 ; 1]$.

515 Problem A continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\forall x \in \mathbb{R} \qquad f(x + f(x)) = f(x).$$

Prove that f is constant.

516 Problem Let I be a closed and bounded interval on the line and let f be continuous on I. Suppose that for each $x \in I$, there exists a $y \in I$ such that

$$|f(y)| \leq \frac{1}{2}|f(x)|.$$

Prove the existence of a $t \in I$ such that $f(t) = 0$.

517 Problem Find all continuous functions that satisfy the functional equation

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right),$$

for all $-1 < x, y < 1$.

518 Problem (Putnam 1947) A real valued continuous function satisfies for all real x, y the functional equation

$$f(\sqrt{x^2 + y^2}) = f(x)f(y).$$

Prove that $f(x) = (f(x))^{x^2}$.

519 Problem Suppose that $f : [0 ; 1] \to [0 ; 1]$ is continuous. Prove that there is a number c in $[0 ; 1]$ such that $f(c) = 1 - c$.

520 Problem (Universal Chord Theorem) Suppose that f is a continuous function of $[0 ; 1]$ and that $f(0) = f(1)$. Let n be a strictly positive integer. Prove that there is some number $x \in [0 ; 1]$ such that $f(x) = f(x + 1/n)$.

521 Problem Under the same conditions of problems 520 prove that there are no universal chords of length $a, 0 < a < 1, a \neq 1/n$.

5.10 Variation of a Function and Uniform Continuity

522 Definition A *partition* $\mathscr{P}$ of the interval $\left[a\,;b\right]$ is any finite set of points $x_0, x_1, \ldots, x_n$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

A partition $\mathscr{P}'$ of $\left[a\,;b\right]$ is said to be *finer* than the partition $\mathscr{P}$ if $\mathscr{P} \subseteq \mathscr{P}'$.

523 Definition The *mesh* or *norm* of $\mathscr{P}$ is

$$\left|\left|\mathscr{P}\right|\right| = \max_{1 \le k \le n} |x_k - x_{k-1}|.$$

☞ *If $\mathscr{P} \subseteq \mathscr{P}'$ then clearly $\left|\left|\mathscr{P}'\right|\right| \le \left|\left|\mathscr{P}\right|\right|$, since the finer partition has probably more points which will make the corresponding subintervals smaller.*

524 Definition Let f be a bounded function on an interval $\left[a\,;b\right]$ and let $I \subseteq \left[a\,;b\right]$ be a subinterval. The *oscillation of f on I* is defined and denoted by

$$\omega(f, I) = \sup_{x \in I} f(x) - \inf_{x \in I} f(x).$$

525 Theorem Let $f : \left[a\,;b\right] \to \mathbb{R}$ be a continuous function. Given $\varepsilon > 0$ there exists a partition of $\left[a\,;b\right]$ into a finite number of subintervals of equal length such that the oscillation of f on each of these subintervals is at most ε.

Proof: *Let P_ε mean the following: there is an $\varepsilon > 0$ such that for all partitions of $\left[a\,;b\right]$ into a finite number of intervals of equal length, the oscillation of f is $\ge \varepsilon$. By bisecting $\left[a\,;b\right]$, at least one of the halves must have property P_ε, say $\left[a_1\,;b_1\right]$. If $\left[a\,;b\right]$ we to have property P_ε, then by bisecting $\left[a_1\,;b_1\right]$, at least one of the halves must have property P_ε, say $\left[a_2\,;b_2\right]$. Continuing in this way we have constructed a sequence of imbricated intervals*

$$\left[a\,;b\right] \supseteq \left[a_1\,;b_1\right] \supseteq \left[a_2\,;b_2\right] \supseteq \cdots \supseteq \left[a_n\,;b_n\right] \supseteq \cdots$$

where the length of $\left[a_n\,;b_n\right]$ is $b_n - a_n = \dfrac{b-a}{2^n} \to 0$ as $n \to +\infty$. By the Cantor Intersection Theorem, there is a point $c \in \bigcap_{n=1}^{\infty} \left[a_n\,;b_n\right]$. Moreover, we have $\omega(f, \left[a_n\,;b_n\right]) \ge \varepsilon$. Since $c \in \left[a\,;b\right]$, f is continuous at c. Hence there is a $\delta > 0$ such that

$$x \in \left]c - \delta\,; c + \delta\right[\implies |f(x) - f(c)| < \frac{\varepsilon}{2}$$

. Taking $(x', x'') \in \left]c - \delta\,; c + \delta\right[^2$ we have

$$|f(x') - f(x'')| \le |f(x') - f(c)| + |f(c) - f(x'')| < \varepsilon,$$

whence

$$\omega(f, \left[a\,;b\right] \cap \left]c - \delta\,; c + \delta\right[) < \varepsilon.$$

Now, if there was an $\varepsilon > 0$ such that for all partitions of $\left[a\,;b\right]$ into a finite number of intervals of equal length, the oscillation of f is $\ge \varepsilon$, then by taking n large enough above we could find one of the $\left[a_n\,;b_n\right]$ completely inside one of the subintervals of the partition. By the above, the oscillation there would be $< \varepsilon$, a contradiction. ❑

526 Theorem Let $f : \left[a\,;b\right] \to \mathbb{R}$ be a continuous function. Given $\varepsilon > 0$ there exists a $\delta > 0$ such that on any subinterval $I \subseteq \left[a\,;b\right]$ having length $< \delta$ the oscillation of f on I is $< \varepsilon$.

Proof: *Let* $\delta = \dfrac{b-a}{n}$*. By Theorem 525 we may choose* n *so large that the oscillation of* f *on each of*

$$\left[a\,;a+\delta\right],\quad \left[a+\delta\,;a+2\delta\right],\quad \ldots\quad,\quad \left[a+(n-1)\delta\,;b\right], \tag{5.5}$$

is $<\dfrac{\varepsilon}{2}$*. Let* $I \subseteqq \left[a\,;b\right]$ *be any subinterval of length* $<\delta$ *and let* $x' \in I$ *be the point where* f *achieves its largest value and* $x'' \in I$ *be the point where* f *achieves its smallest value. Then* x' *and* x'' *either belong to the same interval in 5.5—in which case* $|f(x') - f(x'')| < \dfrac{\varepsilon}{2}$*—or since* I *has length smaller than* δ*, to two consecutive subintervals*

$$\left[a+(j-1)\delta\,;a+j\delta\right], \left[a+j\delta\,;a+(j+1)\delta\right].$$

In this case

$$f(x') - f(x'') = (f(x') - f(a+j\delta)) + (f(a+j\delta) - f(x'')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The theorem now follows.❑

527 Definition A function f is said to be *uniformly continuous* on $\left[a\,;b\right]$ if $\forall \varepsilon > 0$ there exists $\delta > 0$ depending only on ε such that for any $(u,v) \in \left[a\,;b\right]^2$,

$$|u-v| < \delta \implies \left|f(u) - f(v)\right| < \varepsilon.$$

528 Theorem If $f : \left[a\,;b\right] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

Proof: *This follows from Theorem 526.* ❑

529 Theorem (Heine's Theorem) If $f : X \to \mathbb{R}$ is continuous and X is compact, then f is uniformly continuous.

Proof: *This follows from Theorem 528.* ❑

530 Theorem Let f be an increasing function on an open interval $\left]a\,;b\right[$. Then, for any x satisfying $a < x < b$,

$$\sup_{t \in \left]a\,;x\right[} f(t) = f(x-) \le f(x) \le \inf_{t \in \left]x\,;b\right[} f(t) = f(x+).$$

Moreover, if $a < x < y < b$, then $f(x+) \le f(y-)$.

Proof: *The set* $\{f(t) : a < u < x\}$ *is bounded above by* $f(x)$ *and hence it has a supremum* $\sup_{t \in \left]a\,;x\right[} f(t) = A$ *and clearly* $A \le f(x)$ *as* f *is increasing. Let us shew that* $A = f(x-)$*. By the Approximation Property of the Supremum, there is* $\delta > 0$ *such that* $a < x - \delta < x$ *and* $A - \varepsilon < f(x-\delta) \le A$*. But as* f *is increasing,*

$$x - \delta < t < x \implies f(x-\delta) \le f(t) < A \implies \left|f(x) - A\right|,$$

whence $f(x-) = A$*.*

A similar reasoning gives $\inf_{t \in \left]x\,;b\right[} f(t) = f(x+)$*.*

Now, if $a < x < y < b$*, then by what has already been proved we obtain*

$$f(x+) = \inf_{x<t<b} f(t) = \inf_{x<t<y} f(t),$$

again, remembering that f *is increasing. Similarly,*

$$f(y-) = \sup_{a<t<y} f(t) = \inf_{x<t<y} f(t),$$

from where $f(x+) \le f(y-)$*.* ❑

531 THEOREM Let f be an increasing function defined on the interval $\left[a\,;b\right]$ and let

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

be $n+1$ points partitioning the interval. Then

$$\sum_{k=1}^{n-1} \left(f(x_k+) - f(x_k-)\right) \leq f(b) - f(a).$$

Proof: *Let* $y_k \in \left]x_k\,;x_{k+1}\right[$. *For* $1 \leq k \leq n-1$, *by Theorem* **??**,

$$f(x_k+) \leq f(y_k) \qquad \text{and} \quad f(y_{k-1}) \leq f(x_k-) \implies f(x_k+) - f(x_k-) \leq f(y_k) - f(y_{k-1}).$$

Adding,

$$\sum_{k=1}^{n-1} \left(f(x_k+) - f(x_k-)\right) \leq \sum_{k=1}^{n-1} \left(f(y_k) - f(y_{k-1})\right) = f(y_{n-1}) - f(y_0).$$

The proof is completed upon noticing that $f(y_{n-1}) - f(y_0) \leq f(b) - f(a)$. ❑

532 THEOREM Let $f : \left[a\,;b\right] \to \mathbb{R}$ be a monotone function, Then the set of points of discontinuity of f is either finite or countable.

Proof: *Assume* f *is increasing, for if* f *were decreasing, we may apply the same argument to* $-f$. *Let* $m > 0$ *be an integer, and let*

$$\mathscr{S}_m = \left\{x \in \left]a\,;b\right[: f(x+) - f(x-) \geq \frac{1}{m}\right\}.$$

If $x_1 < x_2 < \cdots < x_n$ *are in* $\mathscr{S}_m$ *then by Theorem 531,*

$$\frac{n}{m} \leq f(b) - f(a),$$

which implies that $\mathscr{S}_m$ *is a finite set. The set of discontinuities of* f *in* $\left[a\,;b\right]$ *is* $\bigcup_{m=1}^{\infty} \mathscr{S}_m$, *the countable union of finite sets, and hence it is countable.* ❑

533 Definition Let f be a function defined on the interval $\left[a\,;b\right]$ and let

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

be $n+1$ points partitioning the interval. If there exists $V > 0$ such that

$$\sum_{k=1}^{n} \left|f(x_k) - f(x_{k-1})\right| \leq V$$

for all partitions of $\left[a\,;b\right]$, the we say that f *is of bounded variation on* $\left[a\,;b\right]$.

534 THEOREM If f is monotonic on $\left[a\,;b\right]$, then f is bounded variation on $\left[a\,;b\right]$.

Proof: *Let*

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $\left[a\,;b\right]$. *Then*

$$\sum_{n=1}^{n} \left|f(x_k) - f(x_{k-1})\right| = \max(f(b) - f(a), f(a) - f(b)),$$

the first choice occurring when f *is increasing and the second when* f *is decreasing. Then* $V = \left|f(b) - f(a)\right|$ *satisfies the definition of bounded variation for an arbitrary partition.* ❑

535 THEOREM If f is of bounded variation on $\left[a\,;b\right]$ then f is bounded on $\left[a\,;b\right]$.

Proof: *Let* $x \in \left]a\,;b\right[$ *and consider the partition* $a < x < b$ *of* $\left[a\,;b\right]$. *Since* f *is of bounded variation there is a* $V > 0$ *such that*

$$\left|f(a) - f(x)\right| + \left|f(x) - f(b)\right| \leq V.$$

But then

$$\left|f(x)\right| \leq \left|f(x) - f(a)\right| + \left|f(a)\right| \leq V + \left|f(a)\right|.$$

and so f *is bounded by the constant quantity* $V + \left|f(a)\right|$. ❑

Homework

536 Problem Shew that $\left]0\,;+\infty\right[\to \left]0\,;+\infty\right[,\ x \mapsto x^2$, is not uniformly continuous.

5.11 Classical Limits

537 LEMMA If $0 < x \leq 1$ then

$$1 \leq \frac{\exp(x) - 1}{x} \leq 1 + x(e-2).$$

If $-\dfrac{1}{2} \leq x < 0$ then

$$1 + x \leq \frac{\exp(x) - 1}{x} \leq 1 + \frac{x}{4}.$$

Proof:
Since $\left(1 + \dfrac{x}{n}\right)^n \leq \exp(x)$ *for* $n > -x$ *by Proposition 473, we have* $1 + x \leq \exp(x)$ *for all* $x > -1$. *Now, for* $n \geq 2$ *and* $0 < x \leq 1$,

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &= 1 + \binom{n}{1}\frac{x}{n} + \binom{n}{2}\frac{x^2}{n^2} + \binom{n}{3}\frac{x^3}{n^3} + \cdots + \binom{n}{n}\frac{x^n}{n^n}\\ &= 1 + x + x^2\left(\frac{1}{2!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)x + \cdots + \frac{1}{n!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)x^{n-2}\right)\\ &\leq 1 + x + x^2\left(\frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)\\ &< 1 + x + x^2(e-2),\end{aligned}$$

upon using Theorem 331. This proves the first set of inequalities.

For $x > -2$, $1 + x + \dfrac{x^2}{4} = \left(1 + \dfrac{x}{2}\right)^2 \leq \exp(x)$ *by Proposition 473. Now we assume that* $-\dfrac{1}{2} \leq x \leq 0$. *As before,*

$$\left(1+\frac{x}{n}\right)^n = 1 + x + x^2\left(\frac{1}{2!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)x + \cdots + \frac{1}{n!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)x^{n-2}\right).$$

Since $x^k \leq 0$ *for odd* k *and* $x^k \leq \dfrac{1}{2^k}$ *for even* k *we may delete the odd terms from the dextral side and so*

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n &\leq 1 + x + x^2\left(\frac{1}{2!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right) + 0 + \cdots + \frac{1}{n!}\left(\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{2k-1}{n}\right)x^{2k} + \cdots\right)\\ &\leq 1 + x + x^2\left(\frac{1}{2} + \frac{1}{2^2} + \cdots\right)\\ &\leq 1 + x + x^2.\end{aligned}$$

On taking limits $\exp(x) \le 1 + x + x^2$ *for* $-\frac{1}{2} \le x \le 0$. *Thus we have*

$$-\frac{1}{2} \le x < 0 \implies 1 + x + \frac{x^2}{4} \le \exp(x) \le 1 + x + x^2 \implies 1 + x \le \frac{\exp(x) - 1}{x} \le 1 + \frac{x}{4},$$

since division by negative x *reverses the sense of the inequalities.* ❑

538 THEOREM $\lim_{x \to 0} \frac{\exp(x) - 1}{x} = 1.$

Proof: *We prove that* $\lim_{x \to 0+} \frac{\exp(x) - 1}{x} = 1$ *and that* $\lim_{x \to 0-} \frac{\exp(x) - 1}{x} = 1$. *Let us start with the first assertion. For* $0 < x \le 1$ *we have, by the Sandwich Theorem, and Lemma 537,*

$$1 \le \frac{\exp(x) - 1}{x} \le 1 + x(e - 2) \implies \lim_{x \to 0+} \frac{\exp(x) - 1}{x} = 1,$$

proving the first assertion.

For $-\frac{1}{2} \le x \le 0$ *we have, by the Sandwich Theorem and Lemma 537,*

$$1 + x + \frac{x^2}{4} \le \exp(x) \le 1 + x + x^2 \implies 1 + \frac{x}{4} \le \frac{\exp(x) - 1}{x} \le 1 + x \implies \lim_{x \to 0-} \frac{\exp(x) - 1}{x} = 1,$$

proving the second assertion. ❑

539 LEMMA For $0 < x \le 1$,

$$1 - \frac{x(e-2)}{1+x} \le \frac{\log(1+x)}{x} \le 1$$

and for $-\frac{1}{2} \le x \le 0$,

$$1 \le \frac{\log(1+x)}{x} \le 1 - \frac{x}{1+x}.$$

Proof: *Since* $x \mapsto \log(1+x)$ *is strictly increasing, we have by Lemma 537 for* $0 < x \le 1$,

$$1 + x \le \exp(x) \le 1 + x + x^2(e-2) \implies \log(1+x) \le x \le \log(1 + x + x^2(e-2)).$$

Notice that we have established that $\log(1+x) \le x$ *for* $0 < x \le 1$. *Now*

$$\log(1 + x + x^2(e-2)) = \log(1+x)\left(1 + \frac{x^2(e-2)}{1+x}\right) = \log(1+x) + \left(1 + \frac{x^2(e-2)}{1+x}\right).$$

Since for $x > 0$, $x \mapsto \frac{x^2}{1+x}$ *is strictly increasing,* $\frac{x^2(e-2)}{1+x} < \frac{e-2}{2} < 1$ *for* $0 < x < 1$. *Thus we may use* $\log(1+y) \le y$, $0 \le y \le 1$ *with* $y = \frac{x^2(e-2)}{1+x}$ *obtaining*

$$\log\left(1 + \frac{x^2(e-2)}{1+x}\right) \le \frac{x^2(e-2)}{1+x}.$$

Hence

$$x \le \log(1 + x + x^2(e-2)) \le \log(1+x) + \frac{x^2(e-2)}{1+x}.$$

In conclusion,

$$0 < x \le 1 \implies \log(1+x) \le x \le \log(1+x) + \frac{x^2(e-2)}{1+x} \implies 1 - \frac{x(e-2)}{1+x} \le \frac{\log(1+x)}{x} \le 1.$$

Similarly, for $-\frac{1}{2} \le x < 0$, *by Lemma 537,*

$$1 + x + \frac{x^2}{4} \le \exp(x) \le 1 + x + x^2 \implies \log\left(1 + x + \frac{x^2}{4}\right) \le x \le \log(1 + x + x^2).$$

Since $x \mapsto \log(1+x)$ *is increasing, plainly*

$$\log(1+x) \le \log\left(1 + x + \frac{x^2}{4}\right) \le x.$$

Now observe that $-\frac{1}{2} \le x < 0 \implies 0 < \frac{x^2}{1+x} \le \frac{1}{2} < 1$ *and so*

$$\log(1 + x + x^2) = \log(1+x) + \log\left(1 + \frac{x^2}{1+x}\right) \le \log(1+x) + \frac{x^2}{1+x} \implies x \le \log(1+x) + \frac{x^2}{1+x}.$$

In conclusion,

$$-\frac{1}{2} \le x < 0 \implies \log(1+x) \le x \le \log(1+x) + \frac{x^2}{1+x} \implies 1 \le \frac{\log(1+x)}{x} \le 1 - \frac{x}{1+x},$$

since division by negative x *reverses the sense of the inequalities.* ❑

540 THEOREM $\lim_{x \to 0} \frac{\log(1+x) - x}{x} = 0.$

Proof: *By Lemma 539, for* $0 < x \le 1$,

$$1 - \frac{x(e-2)}{1+x} \le \frac{\log(1+x)}{x} \le 1 \implies \lim_{x \to 0+} \frac{\log(1+x)}{x} = 1,$$

by the Sandwich Theorem. Again, by Lemma 539 and the Sandwich Theorem,

$$-\frac{1}{2} \le x \le 0 \implies 1 \le \frac{\log(1+x)}{x} \le 1 - \frac{x}{1+x} \implies \lim_{x \to 0-} \frac{\log(1+x)}{x} = 1.$$

Combining both results, the theorem follows.❑

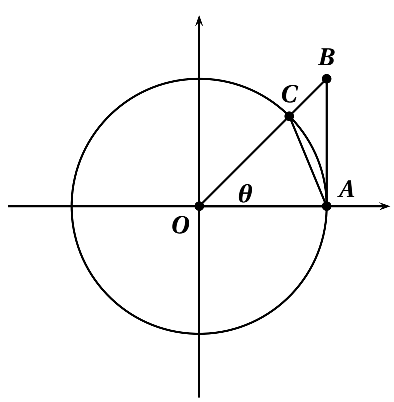


Figure 5.1: Theorem 542.

541 THEOREM If $a \in \mathbb{R}$, then $\lim_{x \to 0} \frac{(1+x)^a - 1}{x} = a.$

Proof: *This is evident for* $a = 0$. *Assume now* $a \neq 0$. *Since* $x \mapsto \exp(x)$ *is continuous and since* $a\log(1+x) \to 0$ *as* $x \to 0$, *by Theorems 538 and 540,*

$$\lim_{x\to 0} \frac{(1+x)^a - 1}{x} = a \lim_{x\to 0} \frac{\exp(a\log(1+x)) - 1}{a\log(1+x)} \cdot \lim_{x\to 0} \frac{\log(1+x)}{x} = a \cdot 1 \cdot 1 = a.$$

❑

542 THEOREM $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$

Proof: *We first prove that* $\lim_{\theta\to 0+} \frac{\sin\theta}{\theta} = 1$. *Since* $\theta \mapsto \frac{\sin\theta}{\theta}$ *is an even function it will also follow that* $\lim_{\theta\to 0-} \frac{\sin\theta}{\theta} = 1$.

Assume $0 < \theta < \frac{\theta}{2}$ *and consider* $\triangle OAB$ *right-angled at* A, *with* $OA = 1$ *and* $\angle BOA = \theta$. C *is the point where line* OB *meets the unit circle with centre at* O *and* D *is its perpendicular projection. The area of* $\triangle OAC$ *is smaller than the area of the circular sector* OAC, *which is smaller than the area of* $\triangle OAB$. *Hence*

$$\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta \implies \frac{1}{\cos\theta} < \frac{\sin\theta}{\theta} < 1 \implies \lim_{\theta\to 0+} \frac{\sin\theta}{\theta} = 1$$

by the Sandwich Theorem, proving the theorem.❑

Chapter 6

Differentiable Functions

6.1 Derivative at a Point

543 Definition Let I be an interval, $a \in \mathring{I}$, and $f : I \to \mathbb{R}$. We say that f is differentiable at a if the limit

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists and is finite. In such a case we denote this limit by $f'(a)$, $Df(a)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}(a)$ and we call this quantity *the derivative of f at a*.

544 Definition Let I be an interval, $a \in \mathring{I}$, and $f : I \to \mathbb{R}$. If

$$\lim_{x\to a+} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0+} \frac{f(a+h)-f(a)}{h}$$

exists and is finite we say that f *is differentiable at a on the right* and write $f'_+(a)$ for this limit. If

$$\lim_{x\to a-} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0-} \frac{f(a+h)-f(a)}{h}$$

exists and is finite we say that f *is differentiable at a on the left* and write $f'_-(a)$ for this limit.

545 THEOREM Let I be an interval, $a \in \mathring{I}$, and $f : I \to \mathbb{R}$. Then f is differentiable at a if and only if both $f_+(a)$ and $f_-(a)$ exist and are equal. In this case $f_+(a) = f'(a) = f_-(a)$.

Proof: *Obvious.* ❑

546 THEOREM Let I be an interval, $a \in \mathring{I}$, and $f : I \to \mathbb{R}$. If f is differentiable at a then it is continuous at a.

Proof: *We have*

$$\lim_{h\to 0} f(a+h) - f(a) = \lim_{h\to 0} \left(\frac{f(a+h)-f(a)}{h}\right) h = \left(\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}\right)\left(\lim_{h\to 0} h\right) = f'(a)\cdot 0 = 0.$$

Thus $\lim_{h\to 0} f(a+h) - f(a) = 0 \implies \lim_{h\to 0} f(a+h) = f(a)$ *and so f is continuous.* ❑

547 THEOREM Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \to \mathbb{R}$ is identically constant, then $f'(I) = 0$.

Proof: *Assume that $f(I) = K$, a constant. Let $c \in \mathring{I}$. Then* $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{K-K}{x-c} = 0.$ *If c is an endpoint of I, then the argument is modified to be either the left or right derivative.* ❑

Homework

548 Problem Let $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) \begin{cases} x+1 & \text{if } x \in \mathbb{Q} \\ 2-x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Prove that f is nowhere differentiable.

549 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$. Prove that f is not differentiable at $x = 0$ and that for $x \neq 0$, $f'(x) = \text{signum}(x)$.

550 Problem Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x|x|$. Determine whether $f'(0)$ exists.

6.2 Differentiation Rules

551 Theorem Let I be an interval, $a \in \mathring{I}$, $\lambda \in \mathbb{R}$ a constant, and $f, then, g : I \to \mathbb{R}$. If f and g are differentiable at a then

1. **(Linearity Rule)** $f + \lambda g$ is differentiable at a and $(f + \lambda g)'(a) = f'(a) + \lambda g'(a)$
2. **(Product Rule)** fg is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
3. if $g(a) \neq 0$, $\frac{1}{g}$ is differentiable at a and $\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{(g(a))^2}$
4. **(Quotient Rule)** if $g(a) \neq 0$, $\frac{f}{g}$ is differentiable at a and $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$

Proof:

1. *This follows by the linearity of limits.*
2. *We have*

$$\begin{aligned} (fg)'(a) &= \lim_{h \to 0} \frac{(fg)(a+h) - (fg)(a)}{h} \\ &= \lim_{h \to 0} \frac{g(a+h)(f(a+h) - f(a)) + f(a)(g(a+h) - g(a))}{h} \\ &= \lim_{h \to 0} g(a+h) \lim_{h \to 0} \frac{(f(a+h) - f(a))}{h} + \lim_{h \to 0} f(a) \lim_{h \to 0} \frac{(g(a+h) - g(a))}{h} \\ &= g(a)f'(a) + f(a)g'(a), \end{aligned}$$

as desired.

3. *We have*

$$\begin{aligned} \left(\frac{1}{g}\right)'(a) &= \lim_{h \to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h} \\ &= \lim_{h \to 0} \frac{\frac{g(a) - g(a+h)}{g(a+h)g(a)}}{h} \\ &= \lim_{h \to 0} \frac{g(a) - g(a+h)}{h} \lim_{h \to 0} \frac{1}{g(a+h)g(a)} \\ &= \left(-g'(a)\right)\left(\frac{1}{g(a)g(a)}\right) \\ &= -\frac{g'(a)}{g(a)^2}, \end{aligned}$$

as desired.

4. *Using (2) and (3),*

$$\begin{aligned} \left(\frac{f}{g}\right)'(a) &= f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a) \\ &= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}, \end{aligned}$$

as desired.

❑

552 THEOREM (Chain Rule) Let I, J be intervals of $\mathbb{R}$, with $a \in I$. Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be such that $f(I) \subsetneqq J$. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $(g \circ f)' = g'(f(a))f'(a)$.

Proof: *Put* $b = f(a)$, *and*

$$\varphi(y) = \begin{cases} \dfrac{g(y) - g(b)}{y - b} & \text{if } y \neq b \\ g'(b) & \text{if } y = b \end{cases}$$

Since g *is differentiable at* b, φ *is continuous at* $y = b$. *Now, for* $x \neq a$,

$$\frac{g(f(x)) - g(f(a))}{x - a} = \varphi(f(x)) \frac{f(x) - f(a)}{x - a}.$$

(If $f(x) \neq f(a)$ *this follows directly from the definition of* φ. *If* $f(x) = f(a)$, *both sides of the equality are* 0.*)*

By the continuity of f *at* a *and of* φ *at* b,

$$\lim_{x \to a} \varphi(f(x)) = \varphi(f(a)) = g'(f(a)),$$

whence

$$\begin{aligned} (g \circ f)'(a) &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \\ &= \lim_{x \to a} \varphi(f(x)) \frac{f(x) - f(a)}{x - a} \\ &= g'(f(a))f'(a), \end{aligned}$$

as desired.

❑

553 THEOREM (Inverse Function Rule) Let I be an interval of $\mathbb{R}$, with $a \in I$. Let $f : I \to \mathbb{R}$ be strictly monotonic and continuous over I. If f is differentiable at a and $f'(a) \neq 0$, then the inverse $f^{-1} : f(I) \to \mathbb{R}$ is differentiable at $f(a)$ and

$$(f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Proof: *Put* $b = f(a)$. *Observe that* $\lim_{y \to b} f^{-1}(y) = a$, *and by the composition rule for limits,*

$$\lim_{y \to b} \frac{f^{-1}(y) - f^{1}(b)}{y - a} = \lim_{y \to b} \frac{f^{-1}(y) - a}{f(f^{-1}(y)) - a} = \frac{1}{f'(a)},$$

proving the theorem. ❑

☞ *Once it is known that* $(f^{-1})'$ *exists, we may proceed as follows. Since* $f^{-1}(f(x)) = x$, *differentiating on both sides, using the Chain Rule on the sinistral side,*

$$(f^{-1})'(f(x))f'(x) = 1,$$

from where the result follows.

554 Definition Let I be an interval of $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be differentiable at every point of I. The function $f' : I \to \mathbb{R}$, $x \mapsto f'(x)$ is called the *derivative function* or *derivative* of the function f.

555 THEOREM Let $n \geq 0$ be an integer. Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^n$. Then f is everywhere differentiable and $f' : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto nx^{n-1}$.

Proof: *Assume first n is strictly positive. By Theorem 99,*

$$\begin{aligned}\lim_{x\to a}\frac{x^n-a^n}{x-a} &= \lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1})}{x-a}\\ &= \lim_{x\to a}(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1})\\ &= na^{n-1}.\end{aligned}$$

Observe that this is true for all $a\in\mathbb{R}$.

If $n=0$ then f is constant, say $f(x)=K$ for all x and so

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{K-K}{x-a}=0.$$

❑

556 Theorem Let $n>0$ be an integer and $f:\left]0\,;+\infty\right[\to\left]0\,;+\infty\right[,\ x\mapsto\frac{1}{x^n}$. Then f' exists everywhere in $\left]0\,;+\infty\right[$ and $f':\left]0\,;+\infty\right[\to\left]0\,;+\infty\right[$ is given by $f'(x)=-\frac{n}{x^{n+1}}$.

Proof: *We use the result above, part (3) of Theorem 551, and the Chain Rule, to get*

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x^n}=-\frac{nx^{n-1}}{(x^n)^2}=-\frac{n}{x^{n+1}},$$

and the theorem follows.❑

557 Lemma Let $q\in\mathbb{Z}$, $q>0$ be an integer, and $f:\left]0\,;+\infty\right[\to\left]0\,;+\infty\right[,\ x\mapsto x^{1/q}$. Then f' exists everywhere in $\left]0\,;+\infty\right[$ and $f':\left]0\,;+\infty\right[\to\left]0\,;+\infty\right[$ is given by $f'(x)=\frac{x^{1/q-1}}{q}$.

Proof: *We have $(f(x))^q=x$. Using the Chain Rule $qf'(x)(f(x))^{q-1}=1$. Since $f(x)\neq 0$,*

$$f'(x)=\frac{1}{q(f(x))^{q-1}}=\frac{1}{q(x^{1/q})^{q-1}}=\frac{1}{q}x^{1/q-1}.$$

❑

558 Theorem Let $r\in\mathbb{Q}$ and let $f:\left]0\,;+\infty\right[\to\left]0\,;+\infty\right[,\ x\mapsto x^r$. Then f' exists everywhere in $\left]0\,;+\infty\right[$ and $f':\left]0\,;+\infty\right[\to\left]0\,;+\infty\right[$ is given by $f'(x)=rx^{r-1}$.

Proof: *Let $r=\frac{a}{b}$, where a,b are integers, with $b>0$. We use the Chain Rule, Lemma 557, and Theorem 556. Then*

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{a/b}=\frac{\mathrm{d}}{\mathrm{d}x}(x^{1/b})^a=a(x^{1/b})^{a-1}\cdot\frac{1}{b}x^{1/b-1}=\frac{a}{b}x^{a/b-1}=rx^{r-1},$$

proving the theorem.

❑

559 Theorem (Derivative of the Exponential Function) Let $\exp:\mathbb{R}\to\mathbb{R},\ x\mapsto e^x$. Then $\exp$ is everywhere differentiable and $\exp':\mathbb{R}\to\mathbb{R}$ is given by $x\mapsto e^x$.

Proof: *Using Theorem 538, we have, with* $h = x - a$,

$$\begin{aligned}\lim_{x\to a}\frac{e^x - e^a}{x-a} &= e^a\lim_{x\to a}\frac{e^{x-a}-1}{x-a}\\ &= e^a\lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^a\cdot 1\\ &= e^a.\end{aligned}$$

❑

560 THEOREM (Derivative of the Logarithmic Function) Let $f: \left]0\,;+\infty\right[\to \left]-\infty\,;+\infty\right[$, $x \mapsto \log x$. Then f' exists everywhere in $\left]0\,;+\infty\right[$ and $f': \left]0\,;+\infty\right[\to \mathbb{R}\setminus\{0\}$ is given by $f'(x) = \frac{1}{x}$.

Proof: *Let* $a > 0$. *Then, with* $h = \frac{x}{a} - 1$, *and using Theorem 540,*

$$\begin{aligned}\lim_{x\to a}\frac{\log x - \log a}{x-a} &= \lim_{x\to a}\frac{\log\frac{x}{a}}{x-a}\\ &= \frac{1}{a}\cdot\lim_{x\to a}\frac{\log\left(1+\frac{x}{a}-1\right)}{\frac{x}{a}-1}\\ &= \frac{1}{a}\cdot\lim_{h\to 0}\frac{\log(1+h)}{h}\\ &= \frac{1}{a}\cdot 1\\ &= \frac{1}{a}.\end{aligned}$$

❑

561 THEOREM (Power Rule) Let $t \in \mathbb{R}$ and let $f: \left]0\,;+\infty\right[\to \left]0\,;+\infty\right[$, $x \mapsto x^t$. Then f' exists everywhere in $\left]0\,;+\infty\right[$ and $f': \left]0\,;+\infty\right[\to \left]0\,;+\infty\right[$ is given by $f'(x) = tx^{t-1}$.

Proof: *Using the Chain Rule,*

$$\frac{\mathrm{d}}{\mathrm{d}x}x^t = \frac{\mathrm{d}}{\mathrm{d}x}\left(\exp(t\log x)\right) = \frac{t}{x}\cdot\left(\exp(t\log x)\right) = \frac{t}{x}\cdot x^t = tx^{t-1}.$$

❑

562 THEOREM (Derivative of sin) . Let $\sin: \mathbb{R}\to\mathbb{R}$, $x \mapsto \sin x$. Then $\sin$ is everywhere differentiable and $\sin': \mathbb{R}\to\mathbb{R}$ is given by $x \mapsto \cos x$.

Proof: *We make a change of variables, and use Theorem 542,*

$$\begin{aligned}
\lim_{x\to a}\frac{\sin x-\sin a}{x-a} &= \lim_{x\to a}\frac{\sin(x-a+a)-\sin a}{x-a}\\
&= \lim_{x\to a}\frac{\sin(x-a)\cos a+\cos(x-a)\sin a-\sin a}{x-a}\\
&= (\cos a)\lim_{x\to a}\frac{\sin(x-a)}{x-a}+(\sin a)\lim_{x\to a}\frac{\cos(x-a)-1}{x-a}\\
&= (\cos a)\lim_{h\to 0}\frac{\sin h}{h}+(\sin a)\lim_{h\to 0}\frac{\cos h-1}{h}\\
&= (\cos a)\cdot 1+(\sin a)\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}\\
&= (\cos a)\cdot 1+(\sin a)\lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h+1)}\\
&= (\cos a)+(\sin a)\lim_{h\to 0}\frac{\sin h}{h}\cdot\lim_{h\to 0}\frac{-\sin h}{\cos h+1}\\
&= \cos a,
\end{aligned}$$

and the theorem follows. ❑

563 THEOREM (Derivatives of the Goniometric Functions)

$$\begin{array}{llll}
1. & \dfrac{d}{dx}\sin x = & \cos x & x\in\mathbb{R}\\
2. & \dfrac{d}{dx}\cos x = & \sin x & x\in\mathbb{R}\\
3. & \dfrac{d}{dx}\tan x = & \sec^2 x & x\in\mathbb{R}\setminus(2\mathbb{Z}+1)\dfrac{\pi}{2}\\
4. & \dfrac{d}{dx}\sec x = & \sec x\tan x & x\in\mathbb{R}\setminus(2\mathbb{Z}+1)\dfrac{\pi}{2}\\
5. & \dfrac{d}{dx}\csc x = & -\csc x\cot x & x\in\mathbb{R}\setminus\mathbb{Z}\pi\\
6. & \dfrac{d}{dx}\cot x = & -\csc^2 x & x\in\mathbb{R}\setminus\mathbb{Z}\pi
\end{array}$$

Proof: *(1) is Theorem 562. To prove (2), observe that*

$$\frac{d}{dx}\cos x=\frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right)=-\cos\left(\frac{\pi}{2}-x\right)=-\sin x.$$

To prove (3), we use the Quotient Rule,

$$\frac{d}{dx}\tan x=\frac{d}{dx}\frac{\sin x}{\cos x}=\frac{(\cos x)(\cos x)-(-\sin x)(\sin x)}{\cos^2 x}=\frac{1}{\cos^2 x}=\sec^2 x.$$

To prove (4), we use once again the Quotient Rule,

$$\frac{d}{dx}\sec x=\frac{d}{dx}\frac{1}{\cos x}=\frac{(0)(\cos x)-(-\sin x)(1)}{\cos^2 x}=\frac{\sin x}{\cos^2 x}=\sec x\tan x.$$

To prove (5), observe that

$$\frac{d}{dx}\csc x=\frac{d}{dx}\sec\left(\frac{\pi}{2}-x\right)=-\sec\left(\frac{\pi}{2}-x\right)\tan\left(\frac{\pi}{2}-x\right)=-\csc x\cot x.$$

To prove (6), observe that

$$\frac{d}{dx}\cot x=\frac{d}{dx}\tan\left(\frac{\pi}{2}-x\right)=-\sec^2\left(\frac{\pi}{2}-x\right)=-\csc^2 x.$$

❑

564 Definition (Higher Order Derivatives) Let I be an interval of $\mathbb{R}$ and let $f: I \to \mathbb{R}$. For $a \in I$ we define the successive derivatives of f at a, inductively. Put $f(a) = f^{(0)}(a)$. If $n \geq 1$,

$$f^{(n)}(a) = f'(f^{(n-1)}(a)),$$

provided f is differentiable at $f^{(n-1)}(a)$.

☞ *We usually write* f'' *instead of* $f^{(2)}$.

565 THEOREM (Leibniz's Rule) Let n be a positive integer.

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

Proof: *This is a generalisation of the Product Rule. The proof is by induction on* n. *For* $n = 0$ *and* $n = 1$ *the assertion is obvious. Assume that* $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$. *Observe that*

$$\begin{aligned}
(fg)^{(n+1)} &= ((fg)^{(n)})' \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right)' \\
&= \sum_{k=0}^{n} \binom{n}{k} (f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}) \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= f^{(0)} g^{(n+1)} + \sum_{k=0}^{n} \left(\binom{n}{k} + \binom{n}{k+1}\right) f^{(k)} g^{(n+1-k)} + f^{(n+1)} g^{(0)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n+1-k)},
\end{aligned}$$

proving the statement.

❑

Homework

566 Problem Prove that

$$\frac{2}{x^2 - 1} = \frac{1}{x-1} - \frac{1}{x+1}$$

and use this result to find the 100th derivative of $f(x) = \frac{2}{x^2 - 1}$.

567 Problem Find the 100-th derivative of $x \mapsto x^2 \sin x$.

568 Problem Demonstrate that the polynomial $p(x) \in \mathbb{R}[x]$ has a zero at $x = a$ of multiplicity k if and only if

$$p(a) = p'(a) = \cdots = p^{(k-1)}(a) = 0.$$

569 Problem Demonstrate that if for all $x \in \mathbb{R}$ there holds the identity

$$\sum_{k=0}^{n} a_k (x-a)^k = \sum_{k=0}^{n} b_k (x-b)^k,$$

then $a_k = \sum_{j=k}^{n} \binom{n}{j} b_j (a-b)^{j-k}$.

570 Problem Let p be a polynomial of degree r and consider the polynomial F with

$$F(x) = p(x) + p'(x) + p''(x) + \cdots + p^{(r)}(x).$$

Prove that

$$\frac{\mathrm{d}\left(F(x)\exp(-x)\right)}{\mathrm{d}x} = -\exp(-x)p(x).$$

6.3 Rolle's Theorem and the Mean Value Theorem

571 THEOREM (Rolle's Theorem) Let $(a, b) \in \mathbb{R}^2$ such that $a < b$, $f: \left[a\,;b\right] \to \mathbb{R}$ be such that f is continuous on $\left[a\,;b\right]$ and differentiable in $\left]a\,;b\right[$, and $f(a) = f(b)$. Then there exists $c \in \left]a\,;b\right[$ such that $f'(c) = 0$.

Proof: *Since f is continuous on $\left[a\,;b\right]$, by Weierstrass' Theorem 508,*

$$m = \inf_{x\in\left[a\,;b\right]} f(x), \qquad M = \sup_{x\in\left[a\,;b\right]} f(x),$$

exist. If $m = M$, then f is constant and so by Theorem 547, f' is identically 0 and there is nothing to prove. Assume that $m < M$. Since $f(a) = f(b)$, one may not simultaneously have $M = f(a)$ and $m = f(a)$. Assume thus without loss of generality that $M \neq f(a)$. Then there exists $c \in \left]a\,;b\right[$ such that $f(c) = M$. Now

$$\lim_{x\to c-} \frac{f(x)-f(c)}{x-c} \geq 0, \qquad \lim_{x\to c+} \frac{f(x)-f(c)}{x-c} \leq 0,$$

whence it follows that $f'(c) = 0$, proving the theorem. ❑

572 THEOREM (Mean Value Theorem) Let $(a,b) \in \mathbb{R}^2$ such that $a < b$, $f : \left[a\,;b\right] \to \mathbb{R}$ be such that f is continuous on $\left[a\,;b\right]$ and differentiable on $\left]a\,;b\right[$. Then there exists $c \in \left]a\,;b\right[$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof: *Put*

$$g : \left[a\,;b\right] \to \mathbb{R}, \quad g(x) = f(x) - \frac{f(b)-f(a)}{b-a}x.$$

Then g is continuous on $\left[a\,;b\right]$ and differentiable on $\left]a\,;b\right[$, and $g(a) = g(b)$. Since g satisfies the hypotheses of Rolle's Theorem, there is $c \in \left]a\,;b\right[$ such that

$$g'(c) = 0 \implies f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \implies f'(c) = \frac{f(b)-f(a)}{b-a},$$

proving the theorem.❑

573 THEOREM If $f : I \to \mathbb{R}$ is continuous on the interval I, differentiable on $\mathring{I}$, and if $\forall x \in \mathring{I}$, $f'(x) = 0$ then f is constant on I.

Proof: *Let $(a,b) \in I^2$, $a < b$. By the Mean Value Theorem, there is $c \in \left]a\,;b\right[$ such that*

$$f(b) - f(a) = f'(c)(b-a) = 0 \cdot (b-a) \implies f(b) = f(a),$$

thus any two outputs have exactly the same value and f is constant. ❑

574 THEOREM If $f : I \to \mathbb{R}$ is continuous on the interval I, and differentiable on $\mathring{I}$. Then f is increasing on I if and only if $\forall x \in \mathring{I}$, $f'(x) \geq 0$ and f is decreasing on I if and only if $\forall x \in \mathring{I}$, $f'(x) \leq 0$.

Proof:

$\implies$ *Suppose f is increasing. Let $x_0 \in \mathring{I}$. If $h \neq 0$ is so small that $x_0 + h \in \mathring{I}$, then*

$$\frac{f(x_0+h)-f(x_0)}{h} \geq 0 \implies \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \geq 0 \implies f'(x_0) \geq 0.$$

If f is decreasing we apply what has just been proved to $-f$.

$\Leftarrow$ *Suppose that for all $x \in \mathring{I}$, $f'(x) \geq 0$. Let $(a,b) \in I^2$, $a < b$. By the Mean Value Theorem, there is $c \in \left]a\,;b\right[$ such that*

$$f(b) - f(a) = (b-a)f'(c) \geq 0,$$

and so f is increasing. If for all $x \in \mathring{I}$, $f'(x) \leq 0$ we apply what we just proved to $-f$.

❑

575 Theorem If $f : I \to \mathbb{R}$ is continuous on the interval I, and differentiable on $\mathring{I}$. Then f is strictly increasing on I if and only if $\forall x \in I$, $f'(x) \geq 0$ and the set $\overset{\circ}{\overbrace{\{x \in I^\circ : f'(x) = 0\}}} = \varnothing$. Also, f is strictly decreasing on I if and only if $\forall x \in I$, $f'(x) \leq 0$ and $\overset{\circ}{\overbrace{\{x \in I^\circ : f'(x) = 0\}}} = \varnothing$.

Proof:

$\Longrightarrow$ *Suppose f is strictly increasing. From Theorem 574 we know that $\forall x \in \mathring{I}$, $f'(x) \geq 0$. Assume that $\overset{\circ}{\overbrace{\{x \in I^\circ : f'(x) = 0\}}} \neq \varnothing$. Then there is $c \in \overset{\circ}{\overbrace{\{x \in I^\circ : f'(x) = 0\}}}$ and $\varepsilon > 0$ such that $\left]c-\varepsilon\,;c+\varepsilon\right[\subseteqq I$ and $\forall x \in \left]c-\varepsilon\,;c+\varepsilon\right[, f'(x) = 0$. By Theorem 573, f must be constant on $\left]c-\varepsilon\,;c+\varepsilon\right[$ and so it is not strictly increasing, a contradiction. If f is strictly decreasing, we apply what has been proved to $-f$.*

$\Leftarrow$ *Conversely, suppose that $\forall x \in I$, $f'(x) \geq 0$. and the set $\overset{\circ}{\overbrace{\{x \in I^\circ : f'(x) = 0\}}} = \varnothing$. From Theorem 574, f is increasing on I. Suppose that there exist $(a,b) \in I^2$, $a < b$ such that $f(a) = f(b)$. Since f is increasing, we have $\forall x \in \left[a\,;b\right]$, $f(x) = f(a)$. But then $\left]a\,;b\right[\subseteqq \{x \in I^\circ : f'(x) = 0\}$, a contradiction, since this last set was assumed empty. If $f'(x) \leq 0$ we apply what has been proved to $-f$.*

❑

Homework

576 Problem Shew, by means of Rolle's Theorem, that $5x^4 - 4x + 1 = 0$ has a solution in $[0;1]$.

577 Problem Let $a_0, a_1 \ldots, a_n$ be real numbers satisfying

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0.$$

Shew that the polynomial

$$a_0 + a_1 x + \cdots + a_n x^n$$

has a root in $\left]0\,;1\right[$.

578 Problem Let a, b, c be three functions such that $a' = b$, $b' = c$, and $c' = a$. Prove that the function $a^3 + b^3 + c^3 - 3abc$ is constant.

579 Problem Suppose that $f : \left[0\,;1\right] \to \mathbb{R}$ is differentiable, $f(0) = 0$ and $f(x) > 0$ for $x \in \left]0\,;1\right[$. Is there a number $d \in \left]0\,;1\right[$ such that

$$\frac{2f'(c)}{f(c)} = \frac{f'(1-c)}{f(1-c)}?$$

580 Problem Let $n \geq 1$ be an integer and let $f : [0;1] \to \mathbb{R}$ be differentiable and such that $f(0) = 0$ and $f(1) = 1$. Prove that there exist distinct points $0 < a_0 < a_2 < \cdots < a_{n-1} < 1$ such that

$$\sum_{k=0}^{n-1} f'(a_k) = n.$$

581 Problem Let $n \geq 1$ be an integer and let $f : [0;1] \to \mathbb{R}$ be differentiable and such that $f(0) = 0$ and $f(1) = 1$. Prove that there exist distinct points $0 < a_0 < a_2 < \cdots < a_{n-1} < 1$ such that

$$\sum_{k=0}^{n-1} \frac{1}{f'(a_k)} = n.$$

582 Problem (Putnam 1946) Let $p(x)$ is a quadratic polynomial with real coefficients satisfying $\max_{x \in \left[-1\,;1\right]} \left|f(x)\right| \leq 1$. Prove that $\max_{x \in \left[-1\,;1\right]} \left|f'(x)\right| \leq 4$.

583 Problem (Generalised Mean Value Theorem) Let f, g be continuous of $\left[a\,;b\right]$ and differentiable on $\left]a\,;b\right[$. Then there is $c \in \left]a\,;b\right[$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

584 Problem (First L'Hôpital Rule) Let I be an open interval (finite or infinite) having c has an endpoint (which may be finite or infinite). Assume f, g are differentiable on I, g and g' never vanish on I and that $\lim_{x \to c} f(x) = 0 = \lim_{x \to c} g(x)$. Prove that if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ (where L is finite or infinite), then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$

585 Problem (Second L'Hôpital Rule) Let I be an open interval (finite or infinite) having c has an endpoint (which may be finite or infinite). Assume f, g are differentiable on I, g and g' never vanish on I and that $\lim_{x \to c} \left|f(x)\right| = \lim_{x \to c} \left|g(x)\right| = +\infty$. Prove that if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = L$ (where L is finite or infinite), then $\lim_{x \to c} \frac{f(x)}{g(x)} = L$

586 Problem If f' exists on an interval containing c, then

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c-h)}{2h}.$$

587 Problem If f'' exists on an interval containing c, then

$$f''(c) = \lim_{h \to 0} \frac{f(c+h) + f(c-h) - 2c}{h^2}.$$

6.4 Extrema

588 Definition Let $X \subseteqq \mathbb{R}$, $f : X \to \mathbb{R}$.

1. We say that f has a *local maximum at a* if there exists a neighbourhood of a, $\mathscr{N}_a$ such that $\forall x \in \mathscr{N}_a$, $f(x) \leq f(a)$.
2. We say that f has a *local minimum at a* if there exists a neighbourhood of a, $\mathscr{N}_a$ such that $\forall x \in \mathscr{N}_a$, $f(x) \geq f(a)$.
3. We say that f has a *strict local maximum at a* if there exists a neighbourhood of a, $\mathscr{N}_a$ such that $\forall x \in \mathscr{N}_a$, $f(x) < f(a)$.

4. We say that f has a *strict local minimum at* a if there exists a neighbourhood of a, $\mathcal{N}_a$ such that $\forall x \in \mathcal{N}_a$, $f(x) > f(a)$.
5. We say that f has a *local extremum at* a if f has either a local maximum or a local minimum at a.
6. We say that f has a *strict local extremum at* a if f has either a strict local maximum or a strict local minimum at a. The plural of extremum is *extrema*.

589 THEOREM If $f : I \to \mathbb{R}$ is continuous on the interval I, differentiable on $\mathring{I}$, and if f has a local extremum at $a \in \mathring{I}$, then $f'(a) = 0$.

Proof: *Suppose f admits a local maximum at a. Let $h \neq 0$ be so small that $a + h \in I$. Now*

$$h > 0 \implies \frac{f(a+h) - f(a)}{h} \leq 0, \qquad h < 0 \implies \frac{f(a+h) - f(a)}{h} \geq 0.$$

Upon taking limits as $h \to 0$, $f'(a) \leq 0$ and $f'(a) \geq 0$, whence $f'(a) = 0$. ❑

590 Definition Let $f : I \to \mathbb{R}$. The points $x \in I$ where $f'(x) = 0$ are called *critical points or stationary points* of f.

591 THEOREM Let $f : \left[a\,;b\right] \to \mathbb{R}$ be a twice differentiable function having a critical point at $c \in \left]a\,;b\right[$. If $f''(c) < 0$ then f has a relative maximum at $x = c$, and if $f''(c) > 0$ then f has a relative minimum at $x = c$.

Proof: *Assume that $f'(c) = 0$ and $f''(c) < 0$. Since*

$$\lim_{x \to c} \frac{f'(x)}{x - c} = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = f''(c) < 0,$$

there exists $\delta > 0$ such that $f'(x) > 0$ when $c - \delta < x < c$ and $f'(x) > 0$ when $c < x < x + \delta$. Consequently, f is strictly increasing on $\left]c - \delta\,;c\right[$ and strictly decreasing on $\left]c\,;c + \delta\right[$. Hence

$$|x - c| < \delta \implies f(x) \leq f(c),$$

and so $x = c$ is a local maximum. If $f'' > 0$ then we apply what has been proved to $-f$. ❑

592 THEOREM (Darboux's Theorem) Let f be differentiable on $\left[a\,;b\right]$ and suppose that $f'(a) < C < f'(b)$. Then there exists $c \in \left]a\,;b\right[$ such that $f'(c) = C$.

Proof: *Put $g(x) = f(x) - Cx$. Then g is differentiable on $\left[a\,;b\right]$. Now $g'(a) = f'(a) - C < 0$ so g is strictly increasing at $x = a$. Similarly, $g'(b) = f'(b) - C < 0$ so g is strictly decreasing at $x = b$. Since g is continuous, g must have a local maximum at some point $c \in \left]a\,;b\right[$, where $g'(c) = f'(c) - C = 0$, proving the theorem.* ❑

Homework

593 Problem Let f be a polynomial with real coefficients of degree n such that $\forall x \in \mathbb{R} \quad f(x) \geq 0$. Prove that

$$\forall x \in \mathbb{R} \qquad f(x) + f'(x) + f''(x) + \cdots + f^{(n)}(x) \geq 0.$$

594 Problem Put $f(0) = 1$, $f(x) = x^x$ for $x > 0$. Find the minimum value of f.

6.5 Convex Functions

595 Definition Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be *convex* if

$$\forall (a,b) \in I^2, \forall \lambda \in \left[0\,;1\right], f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b).$$

We say that f is *concave* if $-f$ is convex.

f is convex if given any two points on its graph, the straight line joining these two points lies above the graph of f. See figure 6.1.

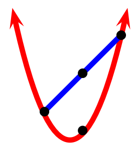


Figure 6.1: A convex curve

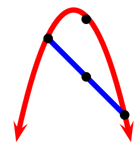


Figure 6.2: A concave curve.

596 Definition Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and let $\lambda_k \in \left[0\,;1\right]$ be such that $\sum_{k=1}^{n} \lambda_k = 1$. The sum

$$\sum_{k=1}^{n} \lambda_k x_k$$

is called a *convex combination* of the x_k.

597 THEOREM If $(x_1, x_2, \ldots, x_n) \in \left[a\,;b\right]^n$, then any convex combination of the x_k also belongs to $\left[a\,;b\right]$.

Proof: *Assume $\lambda_k \in \left[0\,;1\right]$ be such that $\sum_{k=1}^{n} \lambda_k = 1$. Since the $\lambda_k \geq 0$ we have*

$$a \leq x_k \leq b \implies \lambda_k a \leq \lambda_k x_k \leq \lambda_k b.$$

Adding, and bearing in mind that $\sum_{k=1}^{n} \lambda_k = 1$,

$$\left(\sum_{k=1}^{n} \lambda_k\right) a \leq \sum_{k=1}^{n} \lambda_k x_k \leq \left(\sum_{k=1}^{n} \lambda_k\right) b \implies a \leq \sum_{k=1}^{n} \lambda_k x_k \leq b,$$

proving the theorem. ❑

598 THEOREM (Jensen's Inequality) Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. Let $n \geq 1$ be an integer, $x_k \in I$, and $\lambda_k \in \left[0\,;1\right]$ be such that $\sum_{k=1}^{n} \lambda_k = 1$. Then

$$f\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k f(x_k).$$

Proof: *The proof is by induction on n. For $n = 2$ we must shew that given $(x_1, x_2) \in \left[a\,;b\right]^2$,*

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

As $\lambda_1 + \lambda_2 = 1$, we may put $\lambda = \lambda_2 = 1 - \lambda_1$ and so the above inequality becomes

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2),$$

retrieving the definition of convexity.

Assume now that $f\left(\sum_{k=1}^{n-1}\mu_k x_k\right) \leq \sum_{k=1}^{n-1}\mu_k f(x_k)$, *when* $\sum_{k=1}^{n-1}\mu_k = 1$, $\mu_k \in \left]0\,;1\right[$. *We must prove that* $f\left(\sum_{k=1}^{n}\lambda_k x_k\right) \leq \sum_{k=1}^{n}\lambda_k f(x_k)$, *when* $\sum_{k=1}^{n}\lambda_k = 1$, $\lambda_k \in \left]0\,;1\right[$.

If $\lambda_n = 1$ *the assertion is trivial, since then* $\lambda_1 = \cdots = \lambda_{n-1} = 0$. *So assume that* $\lambda_n \neq 1$. *Observe that* $\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n} = \frac{\left(\sum_{k=1}^{n}\lambda_k\right)-\lambda_n}{1-\lambda_n} = \frac{1-\lambda_n}{1-\lambda_n} = 1$ *so that* $\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}x_k$ *is a convex combination of the* x_k *and hence also belongs to* $\left[a\,;b\right]$, *by Theorem 597. Since* f *is convex,*

$$\begin{aligned} f\left(\sum_{k=1}^{n}\lambda_k x_k\right) &= f\left(\sum_{k=1}^{n-1}\lambda_k x_k + \lambda_n x_n\right) \\ &= f\left((1-\lambda_n)\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}x_k + \lambda_n x_n\right) \\ &\leq (1-\lambda_n) f\left(\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}x_k\right) + \lambda_n f(x_n) \end{aligned}$$

By the inductive hypothesis, with $\mu_k = \frac{\lambda_k}{1-\lambda_n} = 1$,

$$f\left(\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}x_k\right) \leq \sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}f(x_k).$$

Finally, we gather,

$$\begin{aligned} f\left(\sum_{k=1}^{n}\lambda_k x_k\right) &\leq (1-\lambda_n) f\left(\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}x_k\right) + \lambda_n f(x_n) \\ &\leq (1-\lambda_n)\sum_{k=1}^{n-1}\frac{\lambda_k}{1-\lambda_n}f(x_k) + \lambda_n f(x_n) \\ &= \sum_{k=1}^{n-1}\lambda_k f(x_k) + \lambda_n f(x_n) \\ &= \sum_{k=1}^{n}\lambda_k f(x_k), \end{aligned}$$

proving the theorem. ❑

599 THEOREM Let $I \subseteqq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$. For $a \in I$ we put

$$T_a : \begin{array}{ccc} I \setminus \{a\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{f(x)-f(a)}{x-a} \end{array}.$$

Then f is convex if and only if $\forall a \in I$, T_a is increasing over $I \setminus \{a\}$.

Proof: *Let* $a < b < c$ *as in figure 6.3. Consider the points* $A(a, f(a))$, $B(b, f(b))$, *and* $C(c, f(c))$. *The slopes*

$$m_{AB} = \frac{f(b)-f(a)}{b-a}, \qquad m_{BC} = \frac{f(c)-f(b)}{c-b}, \qquad m_{CA} = \frac{f(c)-f(a)}{c-a},$$

satisfy

$$m_{AB} \leq m_{AC}, \qquad m_{AC} \leq m_{BC}, \qquad m_{AB} \leq m_{BC},$$

and the theorem follows. An analytic proof may be obtained by observing that from Theorem 597, any $\lambda a + (1-\lambda)c$ *lies in the interval* $\left[a\,;c\right]$ *for* $\lambda \in \left[0\,;1\right]$*. Conversely, given* $b \in \left[a\,;c\right]$*, we may solve for* λ *the equation*

$$b = \lambda a + (1-\lambda)c \implies \lambda = \frac{c-b}{c-a} \in \left[0\,;1\right].$$

Hence

$$f(\lambda a + (1-\lambda)c) \leq \lambda f(a) + (1-\lambda) f(c) \iff f(b) \leq \frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c) \iff \frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(b)}{c-b}. \tag{6.1}$$

This gives

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(b)}{c-b} \tag{6.2}$$

from where the theorem follows.

❑

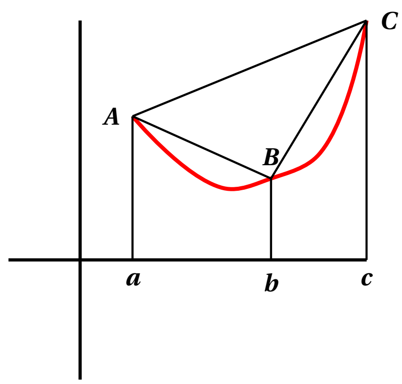


Figure 6.3: Theorem 599.

600 THEOREM Let $I \subsetneqq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a convex function. Then f is left and right differentiable on every point of $\mathring{I}$ and for $(a,b,c) \in I^3$ with $a < b < c$,

$$\frac{f(b)-f(a)}{b-a} \leq f_-(b) \leq f_+(b) \leq \frac{f(c)-f(b)}{c-b}.$$

Proof: *Since* f *is convex,* $\forall b \in \mathring{I}$, $T_b : \begin{array}{ccc} I\setminus\{b\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{f(x)-f(b)}{x-b} \end{array}$ *is increasing, by virtue of Theorem 599. Thus* $\forall u \in \left]a\,;b\right[, \forall v \in \left]b\,;c\right[$

$$T_b(a) \leq T_b(u) \leq T_b(v) \leq T_b(c).$$

This means that T_b *is increasing on* $\left]b\,;c\right[$ *and bounded below by* $T_b(u)$*. It follows by Theorem 530 that* $T_b(b+)$ *exists, and so* f *is right-differentiable at* b*. Moreover,*

$$T_b(a) \leq T_b(u) \leq f'_+(b) \leq T_b(c).$$

Similarly, T_b *is increasing and bounded above by* $f'_+(b)$*. Appealing again to Theorem 530,* f *is left-differentiable at* b *and*

$$T_b(a) \leq f'_-(b) \leq f'_+(b) \leq T_b(c).$$

❑

601 COROLLARY If f is convex on an interval I, then f is continuous on $\mathring{I}$.

Proof: *Given $b \in \mathring{I}$, we know that f is both left and right differentiable at b (though we may have $f'_-(b) < f'_+(b)$). Regardless, this makes f left and right continuous at b: hence both $f(b-) = f(b)$ and $f(b+) = f(b)$. But then $f(b-) = f(b+)$ and so f is continuous at b.* ❑

602 THEOREM Let $I \subsetneqq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be differentiable on I. Then f is convex if and only if f' is increasing on I.

Proof:

$\Longrightarrow$ *Assume f is convex. Let $a < x < c$. By (6.2),*

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(c)-f(a)}{c-a} \leq \frac{f(c)-f(x)}{c-x}.$$

Taking limits as $x \to a+$,

$$f'_+(a) \leq \frac{f(c)-f(a)}{c-a}.$$

Taking limits as $x \to c-$,

$$\frac{f(c)-f(a)}{c-a} \leq f'_-(c).$$

Thus $f'_+(a) \leq f'_-(c)$. Since f is differentiable, $f'_+(a) = f'(a)$ and $f'_-(c) = f'(c)$, and so $f'(a) \leq f'(c)$ proving that f' is increasing.

$\Leftarrow$ *Assume f' is increasing and that $a < x < b$. By the Mean Value Theorem, there exists $\alpha \in \left]a\,; x\right[$ and $\alpha' \in \left]x\,; b\right[$ such that*

$$\frac{f(x)-f(a)}{x-a} = f(\alpha), \qquad \frac{f(b)-f(x)}{b-x} = f(\alpha').$$

Since $f'(\alpha) \leq f(\alpha')$ we must have

$$\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x},$$

and so f is convex in view of (6.1).

❑

603 COROLLARY Let $I \subsetneqq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be twice differentiable on I. Then f is convex if and only if $f'' \geq 0$.

Proof: *This follows from Theorems 574 and 602.* ❑

604 Definition An *inflexion point* is a point on the graph of a function where the graph changes from convex to concave or viceversa.

Homework

605 Problem (Putnam 1991) A there any polynomials $p(x)$ with real coefficients of degree $n \geq 2$ all whose n roots are distinct real numbers and all whose $n-1$ zeroes of $p'(x)$ are the midpoints between consecutive roots of $p(x)$?

606 Problem Prove that the inflexion points of $x \mapsto \dfrac{x}{\tan x}$ are aligned.

607 Problem By considering $f : \begin{array}{ccc} [0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & x^k - k(x-1) \end{array}$ for $0 < k < 1$ and using first and second derivative arguments, obtain a new proof of Young's Inequality 498.

6.6 Inequalities Obtained Through Differentiation

608 THEOREM Let $x > 0$. Then $\dfrac{x^2}{2} < \exp(x)$.

Proof: *Let* $f(x) = \exp(x) - \frac{x^2}{2}$. *Then* $f'(x) = \exp(x) - x$ *and* $f''(x) = \exp(x) - 1$. *Since* $x > 0$, $f''(x) > 0$ *and so* f' *is strictly increasing. Thus* $f'(x) > f'(0) = 1 > 0$ *and so* f *is increasing. Thus*

$$f(x) > f(0) \implies \exp(x) - \frac{x^2}{2} > 0,$$

proving the theorem.❑

609 Theorem $\lim_{x\to+\infty} \frac{x}{\exp(x)} = 0.$

Proof: *From Theorem 608, for* $x > 0$,

$$0 < \frac{x}{\exp(x)} < \frac{2}{x} \implies 0 \le \lim_{x\to+\infty} \frac{x}{\exp(x)} \le \lim_{x\to+\infty} \frac{2}{x} = 0,$$

and the theorem follows from the Sandwich Theorem. ❑

610 Theorem Let $\alpha \in \mathbb{R}$. Then $\lim_{x\to+\infty} \frac{x^\alpha}{\exp(x)} = 0.$

Proof: *If* $\alpha < 1$ *then*

$$\frac{x^\alpha}{\exp(x)} = \frac{x}{\exp(x)} \cdot x^{\alpha-1} \to 0 \cdot 0,$$

by Lemma 609. If $\alpha \ge 1$ *then*

$$\frac{x^\alpha}{\exp(x)} = \alpha^{-\alpha}\left(\frac{\alpha x}{\exp(\alpha x)}\right)^\alpha \to \alpha^{-\alpha} \cdot 0^\alpha = 0,$$

by continuity and by Lemma 609. ❑

611 Theorem Let $x > 0$. Then $\log x < x$.

Proof: *Put* $f(x) = x - \log x$. *Then* $f'(x) = 1 - \frac{1}{x}$. *For* $x < 1$, $f'(x) < 0$, *for* $x = 1$, $f'(x) = 0$, *and for* $x > 1$, $f'(x) > 0$, *which means that* f *has a minimum at* $x = 1$. *Thus*

$$f(x) > f(1) \implies x - \log x > 1.$$

Since $x - \log x > 1$ *then* a fortiori *we must have* $x - \log x > 0$ *and the theorem follows.*❑

612 Lemma $\lim_{x\to+\infty} \frac{\log x}{x} = 0.$

Proof: *From Theorem 611,* $\log x^2 < x^2$. *For* $x > 1$, $\log x > 0$ *and hence,*

$$x > 1 \implies 0 < \frac{\log x}{x} < \frac{1}{2x},$$

whence $\lim_{x\to+\infty} \frac{\log x}{x} = 0$ *by the Sandwich Theorem.* ❑

613 Theorem Let $\alpha \in \left]0\,;+\infty\right[$. Then $\lim_{x\to+\infty} \frac{\log x}{x^\alpha} = 0.$

Proof: *If* $\alpha > 1$ *then*

$$\frac{\log x}{x^\alpha} = \frac{\log x}{x} \cdot x^{1-\alpha} \to 0 \cdot 0,$$

by Lemma 612. If $0 < \alpha \le 1$ *then*

$$\frac{\log x}{x^\alpha} = \frac{\log x^\alpha}{\alpha x^\alpha} \to \frac{1}{\alpha} \cdot 0 = 0,$$

by continuity and by Lemma 612. ❑

614 THEOREM For $x \in \left]0\,;\frac{\pi}{2}\right[$, $\sin x < x < \tan x$.

Proof: *Observe that we gave a geometrical argument for this inequality in Theorem 542. First, let f(x)=* $\sin x - x$*. Then* $f'(x) = \cos x - 1 < 0$*, since for* $x \in \left]0\,;\frac{\pi}{2}\right[$*, the cosine is strictly positive. This means that* f *is strictly decreasing. Thus for all* $x \in \left]0\,;\frac{\pi}{2}\right[$*,*

$$f(0) > f(x) \implies 0 > \sin x - x \implies \sin x < x,$$

giving the first half of the inequality.

For the second half, put $g(x) = \tan x - x$*. Then* $g'(x) = \sec^2 x - 1$*. Now, since* $|\cos x| < 1$ *for* $x \in \left]0\,;\frac{\pi}{2}\right[$, $\sec^2 x > 1$*. Hence* $g'(x) > 0$*, and so* g *is strictly increasing. This gives*

$$g(0) < g(x) \implies 0 < \tan x - x \implies x < \tan x,$$

obtaining the second inequality.❑

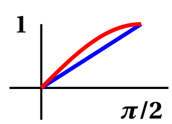


Figure 6.4: Jordan's Inequality

615 THEOREM (Jordan's Inequality) For $x \in \left]0\,;\frac{\pi}{2}\right[$, $\quad \frac{2}{\pi}x < \sin x < x.$

Proof: *This inequality says that the straight line joining* $(0,0)$ *to* $\left(\frac{\pi}{2}, 1\right)$ *lies below the curve* $y = \sin x$ *for* $x \in \left]0\,;\frac{\pi}{2}\right[$*. See figure 6.4. Put* $f(x) = \frac{\sin x}{x}$ *for* $x \neq 0$ *and* $f(0) = 1$*. Then* $f'(x) = (\cos x)\left(\frac{x - \tan x}{x^2}\right) < 0$ *since* $\cos x > 0$ *and* $x - \tan x < 0$ *for* $x \in \left]0\,;\frac{\pi}{2}\right[$*. Thus* f *is strictly decreasing for* $x \in \left]0\,;\frac{\pi}{2}\right[$ *and so*

$$f(x) > f\left(\frac{\pi}{2}\right) \implies \frac{\sin x}{x} > \frac{2}{\pi},$$

proving the theorem.❑

616 Definition If $w_1, w_2, \ldots, w_n$ are positive real numbers such that $w_1 + w_2 + \cdots + w_n = 1$, we define the *r-th weighted power mean* of the x_i as:

$$M_w^r(x_1, x_2, \ldots, x_n) = \left(w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r\right)^{1/r}.$$

When all the $w_i = \dfrac{1}{n}$ we get the standard power mean. The weighted power mean is a continuous function of r, and taking limit when $r \to 0$ gives us

$$M_w^0 = x_1^{w_1} x_2^{w_2} \cdots w_n^{w_n}.$$

617 THEOREM (Generalisation of the AM-GM Inequality) If $r < s$ then

$$M_w^r(x_1, x_2, \ldots, x_n) \le M_w^s(x_1, x_2, \ldots, x_n).$$

Proof: *Suppose first that* $0 < r < s$ *are real numbers, and let* $w_1, w_2, \ldots, w_n$ *be positive real numbers such that* $w_1 + w_2 + \cdots + w_n = 1$.

Put $t = \dfrac{s}{r} > 1$ *and* $y_i = x_i^r$ *for* $1 \le i \le n$. *This implies that* $y_i^t = x_i^s$. *The function* $f :]0; +\infty[\to]0; +\infty[$, $f(x) = x^t$ *is strictly convex, since its second derivative is* $f''(x) = \dfrac{1}{t(t-1)} x^{t-2} > 0$ *for all* $x \in]0; +\infty[$. *By Jensen's inequality,*

$$\begin{aligned} (w_1 y_1 + w_2 y_2 + \cdots + w_n y_n)^t &= f(w_1 y_1 + w_2 y_2 + \cdots + w_n y_n) \\ &\le w_1 f(y_1) + w_2 f(y_2) + \cdots + w_n f(y_n) \\ &= w_1 y_1^t + w_2 y_2^t + \cdots + w_n y_n^t. \end{aligned}$$

with equality if and only if $y_1 = y_2 = \cdots = y_n$. *By substituting* $t = \dfrac{s}{r}$ *and* $y_i = x_i^r$ *back into this inequality, we get*

$$(w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{s/r} \le w_1 x_1^s + w_2 x_2^s + \cdots + w_n x_n^s$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$. *Since* s *is positive, the function* $x \mapsto x^{1/s}$ *is strictly increasing, so raising both sides to the power* $1/s$ *preserves the inequality:*

$$(w_1 x_1^r + w_2 x_2^r + \cdots + w_n x_n^r)^{1/r} \le (w_1 x_1^s + w_2 x_2^s + \cdots + w_n x_n^s)^{1/s},$$

which is the inequality we had to prove. Equality holds if and only if all the x_i *are equal.*

The cases $r < 0 < s$ *and* $r < s < 0$ *can be reduced to the case* $0 < r < s$.❑

Homework

618 Problem Complete the following steps (due to George Pólya) in order to prove the AM-GM Inequality (Theorem 146).

1. Prove that $\forall x \in \mathbb{R}$, $x \le e^{x-1}$.
2. Put
$$A_k = \frac{n a_k}{a_1 + a_2 + \cdots + a_n},$$
and $G_n = a_1 a_2 \cdots a_n$. Prove that
$$A_1 A_2 \cdots A_n = \frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n},$$
and that
$$A_1 + A_2 + \cdots + A_n = n.$$
3. Deduce that
$$G_n \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n.$$
4. Prove the AM-GM inequality by assembling the results above.

6.7 Asymptotic Preponderance

619 Definition Let $I \subseteq \overline{\overline{\mathbb{R}}}$ be an interval, and let $a \in I$. A function $\alpha : I \to \mathbb{R}$ is said to be *infinitesimal* as $x \to a$ if $\lim_{x \to a} \alpha(x) = 0$. We say that α is *negligible* in relation to β as $x \to a$ or that β is *preponderant* in relation to α as $x \to a$, if $\forall \varepsilon > 0, \exists \delta > 0$ such that

$$x \in \left]a - \delta ; a + \delta\right[\implies |\alpha(x)| \le \varepsilon \left|\beta(x)\right|.$$

We express the condition above with the notation $\alpha(x) = o_{x \to a}\left(\beta(x)\right)$ (read "α of x is small oh of β of x as x tends to a").

Finally, we say that α *is Big Oh of* β *around* $x = a$—written $\alpha(x) = O_{x \to a}\left(\beta(x)\right)$, or $\alpha(x) <<_{x \to a} \left(\beta(x)\right)$—if $\exists C > 0$ and $\exists \delta > 0$ such that $\forall x \in \left]a - \delta ; a + \delta\right[$, $|\alpha(x)| \le C\left|\beta(x)\right|$.

☞ *Notice that a above may be finite or $\pm\infty$. If a is understood, we prefer to write $\alpha(x) = o\big(\beta(x)\big)$ rather $\alpha(x) = o_{x\to a}\big(\beta(x)\big)$. Also*

$$\alpha = o_{x\to a}(\beta) \iff \lim_{x\to a}\frac{\alpha(x)}{\beta(x)} = 0 \quad\text{and}\quad \beta(a) = 0 \implies \alpha(a) = 0.$$

620 Example $\sin:\mathbb{R}\to[-1;1]$ is infinitesimal as $x\to 0$, since $\lim_{x\to 0}\sin x = 0$.

621 Example $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}, x\mapsto\dfrac{1}{x}$ is infinitesimal as $x\to+\infty$, since $\lim_{x\to+\infty}\dfrac{1}{x} = 0$.

622 Example We have $x^2 = o(x)$ as $x\to 0$ since

$$\lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0}x = 0.$$

623 Example We have $x = o\big(x^2\big)$ as $x\to+\infty$ since

$$\lim_{x\to+\infty}\frac{x}{x^2} = \lim_{x\to+\infty}\frac{1}{x} = 0.$$

624 Definition We write $\alpha(x) = \gamma(x) + o\big(\beta(x)\big)$ as $x\to a$ if $\alpha(x) - \gamma(x) = o\big(\beta(x)\big)$ as $x\to a$. Similarly, $\alpha(x) = \gamma(x) + O\big(\beta(x)\big)$ as $x\to a$ means that $\alpha(x) - \gamma(x) = O\big(\beta(x)\big)$ as $x\to a$.

625 Example We have $\sin x = x + o(x)$ as $x\to 0$ since

$$\lim_{x\to 0}\frac{\sin x - x}{x} = \lim_{x\to 0}\frac{\sin x}{x} - \lim_{x\to 0}1 = 1 - 1 = 0.$$

626 THEOREM Let $f, g, \alpha, \beta, u, v$ be real-valued functions defined on an interval containing $a\in\overline{\overline{\mathbb{R}}}$. Let $\lambda\in\mathbb{R}$ be a constant. Let h be a real valued function defined on an interval containing $b\in\overline{\overline{\mathbb{R}}}$. Then

1. $f = o\big(g\big) \implies f = O\big(g\big)$.
2. $f = o(\alpha) \implies \lambda f = o(\alpha)$.
3. $f = o(\alpha), g = o(\alpha) \implies f + g = o(\alpha)$.
4. $f = o(\alpha), g = o\big(\beta\big) \implies fg = o\big(\alpha\beta\big)$.
5. $f = O(\alpha) \implies \lambda f = O(\alpha)$.
6. $f = O(\alpha), g = O(\alpha) \implies f + g = O(\alpha)$.
7. $f = O(\alpha), g = O\big(\beta\big) \implies fg = O\big(\alpha\beta\big)$.
8. $f = O(\alpha), g = o\big(\beta\big) \implies fg = o\big(\alpha\beta\big)$.
9. $f = O(\alpha), \alpha = O\big(\beta\big) \implies f = O\big(\beta\big)$.
10. $f = o(\alpha), \alpha = O\big(\beta\big) \implies f = o\big(\beta\big)$.
11. $f = O(\alpha), \alpha = o\big(\beta\big) \implies f = o\big(\beta\big)$.
12. $f = o(\alpha), \lim_{x\to b}h(x) = a \implies f\circ h = o_{x\to b}(\alpha\circ h)$.
13. $f = O(\alpha), \lim_{x\to b}h(x) = a \implies f\circ h = O_{x\to b}(\alpha\circ h)$.

Proof: *These statements follow directly from the definitions.*

1. *If* $f = o(g)$ *then* $\forall \varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in \left]a-\delta\,;a+\delta\right[\implies \left|\frac{f(x)}{g(x)} - 0\right| < \varepsilon \implies |f(x)| < \varepsilon |g(x)| \implies f = O(g),$$

using $C = \varepsilon$ *in the definition of Big Oh.*

2. *This follows by Theorem 429.*
3. *This follows by Theorem 429.*
4. *Both* $\lim_{x\to a} \frac{f(x)}{\alpha(x)} = 0$ *and* $\lim_{x\to a} \frac{g(x)}{\beta(x)} = 0$. *Hence* $\lim_{x\to a} \frac{f(x)g(x)}{\alpha(x)\beta(x)} = \lim_{x\to a} \frac{f(x)}{\alpha(x)} \cdot \lim_{x\to a} \frac{g(x)}{\beta(x)} = 0 \implies fg = o(\alpha\beta)$.
5. *If* $f = O(\alpha)$ *then there is* $\delta > 0$ *and* $C > 0$ *such that*

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)| \le C|g(x)| \implies |\lambda f(x)| \le C|\lambda| \cdot |g(x)| \implies \lambda f = O(\alpha)$$

6. *There exists* $\delta_1 > 0, \delta_2 > 0$ *and* $C_1 > 0$, $C_2 > 0$ *such that*

$$x \in \left]a-\delta_1\,;a+\delta_1\right[\implies |f(x)| \le C_1 |\alpha(x)| \quad \text{and} \quad x \in \left]a-\delta_2\,;a+\delta_2\right[\implies |g(x)| \le C_2 |\alpha(x)|.$$

Thus if $\delta = \min(\delta_1, \delta_2)$,

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)+g(x)| \le |f(x)| + |g(x)| \le C_1\alpha(x) + C_2\alpha(x) = (C_1+C_2)\alpha(x) \implies f+g = O(\alpha).$$

7. *There exists* $\delta_1 > 0, \delta_2 > 0$ *and* $C_1 > 0$, $C_2 > 0$ *such that*

$$x \in \left]a-\delta_1\,;a+\delta_1\right[\implies |f(x)| \le C_1 |\alpha(x)| \quad \text{and} \quad x \in \left]a-\delta_2\,;a+\delta_2\right[\implies |g(x)| \le C_2 |\beta(x)|.$$

Thus if $\delta = \min(\delta_1, \delta_2)$,

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)g(x)| = |f(x)||g(x)| \le C_1 |\alpha(x)| \cdot C_2 |\beta(x)| = (C_1C_2)|\alpha(x)\beta(x)| \implies fg = O(\alpha\beta).$$

8. *There exists* $\delta_1 > 0, \delta_2 > 0$ *and* $C_1 > 0$, *such that* $\forall \varepsilon > 0$

$$x \in \left]a-\delta_1\,;a+\delta_1\right[\implies |f(x)| \le C_1 |\alpha(x)| \quad \text{and} \quad x \in \left]a-\delta_2\,;a+\delta_2\right[\implies |g(x)| \le \varepsilon |\beta(x)|.$$

Thus if $\delta = \min(\delta_1, \delta_2)$,

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)g(x)| = |f(x)||g(x)| \le C_1 |\alpha(x)| \cdot \varepsilon |\beta(x)| = \varepsilon(C_1)|\alpha(x)\beta(x)| \implies fg = o(\alpha\beta).$$

9. *There exists* $\delta_1 > 0, \delta_2 > 0$ *and* $C_1 > 0$, $C_2 > 0$ *such that*

$$x \in \left]a-\delta_1\,;a+\delta_1\right[\implies |f(x)| \le C_1 |\alpha(x)| \quad \text{and} \quad x \in \left]a-\delta_2\,;a+\delta_2\right[\implies |\alpha(x)| \le C_2 |\beta(x)|.$$

Thus if $\delta = \min(\delta_1, \delta_2)$,

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)| \le C_1 |\alpha(x)| \le C_1C_2 |\beta(x)| \implies f = O(\beta).$$

10. *There exists* $\delta_1 > 0, \delta_2 > 0$ *and* $C > 0$, *such that* $\forall \varepsilon > 0$

$$x \in \left]a-\delta_1\,;a+\delta_1\right[\implies |f(x)| \le \varepsilon |\alpha(x)| \quad \text{and} \quad x \in \left]a-\delta_2\,;a+\delta_2\right[\implies |\alpha(x)| \le C |\beta(x)|.$$

Thus if $\delta = \min(\delta_1, \delta_2)$,

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)| \le \varepsilon |\alpha(x)| \le C\varepsilon |\beta(x)| \implies f = o(\beta).$$

11. *There exists* $\delta_1 > 0, \delta_2 > 0$ *and* $C > 0$, *such that* $\forall \varepsilon > 0$

$$x \in \left]a-\delta_1\,;a+\delta_1\right[\implies |f(x)| \le C |\alpha(x)| \quad \text{and} \quad x \in \left]a-\delta_2\,;a+\delta_2\right[\implies |\alpha(x)| \le \varepsilon |\beta(x)|.$$

Thus if $\delta = \min(\delta_1, \delta_2)$,

$$x \in \left]a-\delta\,;a+\delta\right[\implies |f(x)| \le C |\alpha(x)| \le C\varepsilon |\beta(x)| \implies f = o(\beta).$$

12. *There exists* $\delta_1 > 0, \delta_2 > 0$ *such that* $\forall \varepsilon > 0$

$$x \in \left]a-\delta_1 ; a+\delta_1\right[\implies \left|f(x)\right| \leq \varepsilon |\alpha(x)| \quad \text{and} \quad x \in \left]b-\delta_2 ; b+\delta_2\right[\implies |h(x)-a| \leq \varepsilon \implies h(x) \in \left]a-\varepsilon ; a+\varepsilon\right[.$$

Thus if $\delta = \min(\delta_1, \delta_2, \varepsilon)$,

$$x \in \left]b-\delta ; b+\delta\right[\implies \left|(f \circ h)(x)\right| \leq \varepsilon |(\alpha \circ h)(x)| \implies f \circ h = o_{x \to b}(\alpha \circ h).$$

13. *There exists* $\delta_1 > 0, \delta_2 > 0, C > 0$ *such that* $\forall \varepsilon > 0$

$$x \in \left]a-\delta_1 ; a+\delta_1\right[\implies \left|f(x)\right| \leq C |\alpha(x)| \quad \text{and} \quad x \in \left]b-\delta_2 ; b+\delta_2\right[\implies |h(x)-a| \leq \varepsilon \implies h(x) \in \left]a-\varepsilon ; a+\varepsilon\right[.$$

Thus if $\delta = \min(\delta_1, \delta_2, \varepsilon)$,

$$x \in \left]b-\delta ; b+\delta\right[\implies \left|(f \circ h)(x)\right| \leq C |(\alpha \circ h)(x)| \implies f \circ h = O_{x \to b}(\alpha \circ h).$$

❑

☞ *In the above theorem, (8), (10), and (11) essentially say that* $O(o) = o(O) = o(o) = o$ *and (9) says that* $O(O) = O$.

The following corollary is immediate.

627 Corollary Let α and β be infinitesimal functions as $x \to a$. Then the following hold.

1. The sum of two infinitesimals is an infinitesimal:
$$o\left(\beta(x)\right) + o\left(\beta(x)\right) = o\left(\beta(x)\right).$$
2. The difference of two infinitesimals is an infinitesimal:
$$o\left(\beta(x)\right) - o\left(\beta(x)\right) = o\left(\beta(x)\right).$$
3. $\forall c \in \mathbb{R} \setminus \{0\}, o\left(c\beta(x)\right) = o\left(\beta(x)\right).$
4. $\forall n \in \mathbb{N}, n \geq 2, 1 \leq k \leq n-1,\ o\left((\beta(x))^n\right) = o\left((\beta(x))^k\right).$
5. $o\left(o\left(\beta(x)\right)\right) = o\left(\beta(x)\right).$
6. $\forall n \in \mathbb{N}, n \geq 1,\ (\beta(x))^n o\left(\beta(x)\right) = o\left((\beta(x))^{n+1}\right).$
7. $\forall n \in \mathbb{N}, n \geq 2, \dfrac{o\left((\beta(x))^n\right)}{\beta(x)} = o\left((\beta(x))^{n-1}\right).$
8. $\dfrac{o\left(\beta(x)\right)}{\beta(x)} = o(1).$
9. If c_k are real numbers, then $o\left(\sum_{k=1}^{n} c_k(\beta(x))^k\right) = o\left(\beta(x)\right).$
10. $(\alpha\beta)(x) = o(\alpha(x))$ and $(\alpha\beta)(x) = o\left(\beta(x)\right).$
11. If $\alpha \sim \beta$, then $(\alpha-\beta)(x) = o(\alpha(x))$ and $(\alpha-\beta)(x) = o\left(\beta(x)\right).$

628 Theorem (Canonical small oh Relations) The following relationships hold

1. $\forall (\alpha, \beta) \in \mathbb{R}^2, x^\alpha = o_{x \to +\infty}\left(x^\beta\right) \iff \alpha < \beta.$
2. $\forall (\alpha, \beta) \in \mathbb{R}^2, x^\alpha = o_{x \to 0+}\left(x^\beta\right) \iff \alpha > \beta.$

3. $\log x = o_{x\to+\infty}(x)$.

4. $\forall (\alpha, \beta) \in \mathbb{R}^2, \beta > 0$, $(\log x)^{\alpha} = o_{x\to+\infty}\left(x^{\beta}\right)$.

5. $\forall (\alpha, \beta) \in \mathbb{R}^2, \beta < 0$, $\left|\log x\right|^{\alpha} = o_{x\to 0+}\left(x^{\beta}\right)$.

6. $\forall (\alpha, a) \in \mathbb{R}^2, a > 1$, $x^{\alpha} = o_{x\to+\infty}\left(a^{x}\right)$

7. $\forall (\alpha, a) \in \mathbb{R}^2, a > 1$, $a^{x} = o_{x\to-\infty}\left(|x|^{\alpha}\right)$

Proof:

1. *Immediate.*

2. *Immediate.*

3. *This follows from Lemma 612.*

4. *If $\alpha = 0$ then eventually $(\log x)^{\alpha} = 1$ and so the assertion is immediate. If $\alpha < 0$ the assertion is also immediate, since then $(\log x)^{\alpha} \to 0$ as $x \to +\infty$. If $\alpha > 0$, by Theorem 613,*

$$\frac{\log x}{x^{\beta/\alpha}} \to 0,$$

whence

$$\frac{(\log x)^{\alpha}}{x^{\beta}} = \left(\frac{\log x}{x^{\beta/\alpha}}\right)^{\alpha} \to 0^{\alpha} = 0.$$

5. *If $x \to 0+$ then $\dfrac{1}{x} \to +\infty$. Hence by the preceding part and by continuity, as $x \to 0+$ and for $\gamma > 0$,*

$$\frac{\left(\left|\log \frac{1}{x}\right|\right)^{\alpha}}{\left(\frac{1}{x}\right)^{\gamma}} \to 0.$$

But

$$\frac{\left(\left|\log \frac{1}{x}\right|\right)^{\alpha}}{\left(\frac{1}{x}\right)^{\gamma}} = \frac{\left(\left|-\log x\right|\right)^{\alpha}}{\left(\frac{1}{x}\right)^{\gamma}} = x^{\gamma}\left|\log x\right|^{\alpha},$$

and so $\left|\log x\right|^{\alpha} = o_{x\to 0+}\left(x^{-\gamma}\right)$, and so putting $\beta = -\gamma < 0$ we have $\left|\log x\right|^{\alpha} = o_{x\to 0+}\left(x^{\beta}\right)$.

6. *For $\alpha < 1$ we have*

$$\frac{x^{\alpha}}{a^{x}} = \frac{x \log a}{\exp(x \log a)} \cdot \frac{x^{\alpha-1}}{\log a} \to 0 \cdot 0,$$

since $\dfrac{x \log a}{\exp(x \log a)} \to 0$ by continuity and Theorem 610, and $\dfrac{x^{\alpha-1}}{\log a} \to 0$ since $\alpha - 1 < 0$. If $\alpha > 1$ then

$$\frac{x^{\alpha}}{a^{x}} = \left(\frac{x}{\left(a^{1/\alpha}\right)^{x}}\right)^{\alpha} = \frac{\alpha^{\alpha}}{(\log a)^{\alpha}} \cdot \left(\frac{x\dfrac{\log a}{\alpha}}{\exp\left(x\dfrac{\log a}{\alpha}\right)}\right)^{\alpha} \to \frac{\alpha^{\alpha}}{(\log a)^{\alpha}} \cdot 0^{\alpha} = 0,$$

by continuity and Theorem 610.

7. *If $\alpha > 0$, $a > 1$ then $|x|^{\alpha} \to +\infty$ but $a^{x} \to 0$ as $x \to -\infty$, hence there is nothing to prove. If $\alpha = 0$, again the result is obvious. Assume $\alpha < 0$. If $x \to -\infty$ then $-x \to +\infty$ and so by the preceding part*

$$\frac{|x|^{-\alpha}}{a^{-x}} \to 0$$

since the above result is valid regardless of the sign of α. *Now*

$$\frac{a^x}{|x|^\alpha} = \frac{|x|^{-\alpha}}{a^{-x}},$$

proving the result.

❑

629 Example In view of Corollary 627 and Theorem 628, we have

$$o\left(-2x^3 + 8x^2\right) = o(x),$$

as $x \to 0$.

630 Example In view of Corollary 627 and Theorem 628, we have

$$o\left(-2x^3 + 8x^2\right) = o\left(x^4\right),$$

as $x \to +\infty$.

Homework

631 Problem Which one is faster as $x \to +\infty$, $(\log\log x)^{\log x}$ or $(\log x)^{\log\log x}$?

6.8 Asymptotic Equivalence

632 Definition Let $I \subseteq \overline{\overline{\mathbb{R}}}$ be an interval, and let $a \in I$. We say that α is *asymptotic* to a function $\beta : I \to \mathbb{R}$ as $x \to a$, and we write $\alpha \sim \beta$, if $\alpha \sim \beta \iff \alpha - \beta = o_a(\beta)$.

☞ *If in a neighbourhood* $\mathcal{N}_a$ *of* a $\beta \neq 0$ *then*

$$\alpha \sim \beta \iff \begin{cases} \frac{\alpha}{\beta} \sim 1 \\ \beta(a) = 0 \implies \alpha(a) = 0 \end{cases}$$

633 Example We have $\sin x \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

634 Example We have $x^2 + x \sim x$ as $x \to 0$, since $\lim_{x\to 0} \frac{x^2 + x}{x} = 1$.

635 Example We have $x^2 + x \sim x^2$ as $x \to +\infty$, since $\lim_{x\to +\infty} \frac{x^2 + x}{x^2} = 1$.

636 Theorem

$$\alpha \sim \beta \implies \begin{cases} \alpha = O(\beta) \\ \beta = O(\alpha) \end{cases}$$

Proof: *If* $\alpha - \beta = o(\beta)$ *there is a neighbourhood* $\mathcal{N}_a$ *of* a *such that*

$$\forall \varepsilon > 0, x \in \mathcal{N}_a \implies |\alpha(x) - \beta(x)| \leq \varepsilon |\beta(x)|.$$

In particular, for $\varepsilon = \frac{1}{2}$, *we have*

$$x \in \mathcal{N}_a \implies |\alpha(x) - \beta(x)| \leq \frac{1}{2} |\beta(x)|.$$

Hence

$$x \in \mathscr{N}_a \implies |\alpha(x)| = \left|\alpha(x) - \beta(x) + \beta(x)\right| \le \left|\alpha(x) - \beta(x)\right| + \left|\beta(x)\right| \le \frac{3}{2}\left|\beta(x)\right| \implies \alpha = O\left(\beta\right),$$

and

$$x \in \mathscr{N}_a \implies \left|\beta(x)\right| = \left|\beta(x) - \alpha(x) + \alpha(x)\right| \le \left|\beta(x) - \alpha(x)\right| + |\alpha(x)| \le \frac{1}{2}\left|\beta(x)\right| + |\alpha(x)| \implies \left|\beta(x)\right| \le 2|\alpha(x)| \implies \beta = O(\alpha).$$

❑

637 THEOREM The relation of asymptotic equivalence $\sim$ is an equivalence relation on the set of functions defined on a neighbourhood of a.

Proof: *We have*

Reflexivity $\alpha - \alpha = 0 = o(\alpha)$.

Symmetry $\alpha - \beta = o\left(\beta\right) \implies \beta = O(\alpha)$ *by Theorem 636. Now by (10) of Theorem 626,*

$$\alpha - \beta = o\left(\beta\right) \quad \text{and} \beta = O(\alpha) \implies \alpha - \beta = o(\alpha) \implies \beta - \alpha = o(\alpha),$$

whence $\beta \sim \alpha$.

Transitivity *Assume* $\alpha - \beta = o\left(\beta\right)$ *and* $\beta - \gamma = o\left(\gamma\right)$. *Then by Theorem 636 we also have* $\beta = O\left(\gamma\right)$. *Hence* $\alpha - \beta = o\left(\gamma\right)$ *by (10) of Theorem 626. Finally* $\alpha - \beta = o\left(\gamma\right)$ *and* $\beta - \gamma = o\left(\gamma\right)$ *give* $\alpha - \gamma = o\left(\gamma\right)$ *by (3) of Theorem 626.*

❑

The relationship between o, O, and $\sim$ is displayed in figure 6.5.

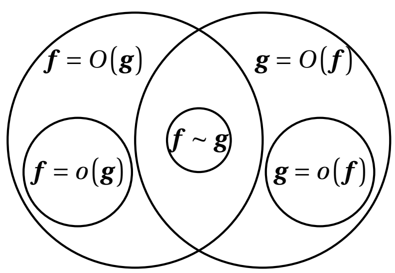


Figure 6.5: Diagram of Big Oh relations.

638 THEOREM The relation of asymptotic equivalence $\sim$ possesses the following properties.

1. $\begin{cases} \alpha \sim \beta \\ \gamma \sim \delta \end{cases} \implies \alpha\gamma \sim \beta\delta.$

2. $\begin{cases} \alpha \sim \beta \\ n \in \mathbb{N} \setminus \{0\} \end{cases} \implies \alpha^n \sim \beta^n$

3. if $\alpha \sim \beta$ and if there is a neighbourhood $\mathscr{N}_a$ of a where $\forall x \in \mathscr{N}_a \setminus \{a\}, \beta(a) \neq 0$, then $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ are defined on $\mathscr{N}_a \setminus \{a\}$ and $\dfrac{1}{\alpha} \sim_a \dfrac{1}{\beta}$.

4. $\begin{cases} \alpha = o(\beta) \\ \beta \sim \gamma \end{cases} \implies \alpha = o(\gamma).$

5. $\begin{cases} \alpha \sim \beta \\ \beta = o(\gamma) \end{cases} \implies \alpha = o(\gamma).$

6. if $\alpha \sim \beta$ and if there is a neighbourhood $\mathscr{N}_a$ of a where $\forall x \in \mathscr{N}_a \setminus \{a\}, \beta(a) > 0$, and if $r \in \mathbb{R}$ then $\alpha^r \sim_a \beta^r$.

7. **(Dextral Composition)** If $\alpha \sim_a \beta$ and if $\lim_{x\to b} \gamma(x) = a$, then $\alpha \circ \gamma \sim_a \beta \circ \gamma$.

Proof: *We prove the assertions in the given order.*

1. *Since* $\alpha - \beta = o(\beta)$ *and* $\gamma - \delta = o(\delta)$ *then* $\alpha = O(\beta)$, *and so*
$$\alpha\gamma - \beta\delta = \alpha(\gamma - \delta) - \delta(\beta - \alpha) = O(\beta)\, o(\delta) - \delta\, o(\beta) = o(\beta\delta).$$
2. *This follows upon applying the preceding product rule* $n-1$ *times, using* $\gamma = \alpha$ *and* $\delta = \beta$.
3. *Observe that*
$$\frac{1}{\alpha} - \frac{1}{\beta} = \frac{\beta - \alpha}{\alpha\beta} = \frac{o(\alpha)}{\alpha\beta} = o\left(\frac{1}{\beta}\right),$$
upon using $\beta - \alpha = o(\alpha)$ *and (8) of Corollary 627.*
4. *We have* $\alpha = o(\beta)$ *and* $\beta - \gamma = o(\gamma)$. *This last implies that* $\beta = O(\gamma)$ *by Theorem 636. Hence*
$$\alpha = o(\beta) = o(O(\gamma)) = o(\gamma).$$
5. *We have* $\alpha - \beta = o(\beta)$ *and* $\beta = o(\gamma)$. *This last implies that* $\alpha = O(\beta)$ *by Theorem 636. Hence*
$$\alpha = O(\beta) = O(o(\gamma)) = o(\gamma).$$
6. *Since* β *is eventually strictly positive, so is* α. *Hence* $\alpha \sim \beta \iff \frac{\alpha}{\beta}(x) \to 1$ *as* $x \to a$. *Since the function* $x \mapsto x^r$ *is continuous in* $]0\,;+\infty[$,
$$\frac{\alpha}{\beta}(x) \to 1 \implies \frac{\alpha^r}{\beta^r}(x) \to 1 \implies \alpha^r \sim \beta^r.$$
7. *We have* $\frac{\alpha(x) - \beta(x)}{\beta(x)} \to 0$ *as* $x \to a$. *Now if* $\gamma(x) \to a$ *as* $x \to b$ *then as* $x \to b$,
$$\frac{\alpha(\gamma(x)) - \beta(\gamma(x))}{\beta(\gamma(x))} \to 0.$$

❑

639 THEOREM (Exponential Composition) $\exp(\alpha) \sim_a \exp(\beta) \iff \alpha - \beta \sim_a 0$.

Proof: *We have*
$$\begin{aligned}
\exp(\alpha) \sim_a \exp(\beta) &\iff \exp(\alpha) - \exp(\beta) = o(\exp(\beta)) \\
&\iff (\exp(-\beta))(\exp(\alpha) - \exp(\beta)) = (\exp(-\beta))\, o(\exp(\beta)) \\
&\iff \exp(\alpha - \beta) - 1 = o(1) \\
&\iff \alpha - \beta = o(0).
\end{aligned}$$

❑

☞ *The above theorem* does not say *that* $\alpha \sim \beta \implies \exp(\alpha) \sim \exp(\beta)$. *That this last assertion is false can be seen from the following counterexample:* $x+1 \sim x$ *as* $x \to 0$, *but* $\exp(x+1) = e\exp(x)$ *is not asymptotic to* $\exp(x)$.

640 THEOREM (Logarithmic Composition) Suppose there is a neighbourhood of a $\mathscr{N}_a$ such that $\forall x \in \mathscr{N}_a \setminus \{a\}, \beta(x) > 0$. Suppose, moreover, that $\alpha \sim_a \beta$ and that $\lim_{x\to a} \beta(x) = l$ with $l \in \left[0\,;+\infty\right] \setminus \{1\}$. Then $\log\circ\alpha \sim_a \log\circ\beta$.

Proof: *Either* $l \in \left]0\,;+\infty\right[\setminus \{1\}$ *or* $l = +\infty$ *or* $l = 0$.

In the first case, $\log\alpha(x) \to \log l$ *and* $\log\beta(x) \to \log l$ *as* $x \to a$ *hence*

$$\log\alpha \sim \log l \sim \log\beta, \qquad \text{as } x \to a.$$

In the second case $\beta(x) > 1$ *eventually, and thus* $\log\beta(x) \neq 0$. *Hence*

$$\frac{\log\alpha(x)}{\log\beta(x)} - 1 = \frac{\log\alpha(x) - \log\beta(x)}{\log\beta(x)} = \frac{\log\dfrac{\alpha(x)}{\beta(x)}}{\log\beta(x)} \to \frac{\log 1}{+\infty} = 0,$$

since $\dfrac{\alpha(x)}{\beta(x)} \to 1$ *and* $\log\beta(x) \to +\infty$ *as* $x \to +\infty$.

The third case becomes the second case upon considering $\dfrac{1}{\alpha}$ *and* $\dfrac{1}{\beta}$. ❑

641 THEOREM (Addition of Positive Terms) If $\alpha \sim \beta$ and $\gamma \sim \delta$ and there exists a neighbourhood of a $\mathscr{N}_a$ such that $\forall x \in \mathscr{N}_a \setminus \{a\}, \beta(x) > 0, \delta(x) > 0$ then

$$\alpha + \gamma \sim \beta + \delta.$$

Proof: *We have* $\alpha - \beta = o\left(\beta\right)$ *and* $\gamma - \delta = o(\delta)$. *Hence*

$$\begin{aligned}(\alpha+\gamma) - (\beta+\delta) &= (\alpha-\beta) + (\gamma-\delta) \\ &= o\left(\beta\right) + o(\delta) \\ &= o\left(\beta+\delta\right),\end{aligned}$$

which means $\alpha + \gamma \sim \beta + \delta$. ❑

642 THEOREM The following asymptotic expansions hold as $x \to 0$:

1. $\exp(x) - 1 \sim x$ and thus $\exp(x) = 1 + x + o(x)$
2. $\log(1+x) \sim x$ and thus $\log(1+x) = x + o(x)$
3. $\sin x \sim x$ and thus $\sin(x) = x + o(x)$
4. $\tan x \sim x$ and thus $\tan(x) = x + o(x)$
5. $\arcsin x \sim x$ and thus $\arcsin(x) = x + o(x)$
6. $\arctan x \sim x$ and thus $\tan(x) = x + o(x)$
7. for $\alpha \in \mathbb{R}$ constant, $(1+x)^{\alpha} - 1 \sim \alpha x$ and thus $(1+x)^{\alpha} = 1 + \alpha x + o(x)$
8. $1 - \cos x \sim \dfrac{x^2}{2}$ and thus $\cos(x) = 1 - \dfrac{x^2}{2} + o\left(x^2\right)$

Proof: *Results 1—7 follow from the fact that*

$$f'(a) \neq 0, \quad \frac{f(x)-f(a)}{x-a} \to f'(a) \implies f(x)-f(a) \sim f'(a)(x-a).$$

Property 8 follows from the identity $1-\cos x = 2\sin^2\frac{x}{2}$. ❑

643 Example Since $\tan x = x + o(x)$, we have

$$\tan\frac{x^2}{2} = \frac{x^2}{2} + o\left(\frac{x^2}{2}\right) = \frac{x^2}{2} + o\left(x^2\right),$$

as $x \to 0$. Also,

$$(\tan x)^3 = (x+o(x))^3 = x^3 + 3x^2 o(x) + 3x o\left(x^2\right) + (o(x))^3 = x^3 + o\left(x^3\right).$$

644 Example Since $\cos x = 1 - \frac{x^2}{2} + o\left(x^2\right)$, we have

$$\cos 3x^2 = 1 - \frac{9x^4}{2} + o\left(x^4\right).$$

645 Example Find an asymptotic expansion of $\cot^2 x$ of type $o\left(x^{-2}\right)$ as $x \to 0$.

Solution: *Since* $\tan x \sim x$ *we have*

$$\cot^2 x \sim \frac{1}{x^2}.$$

We can write this as $\cot^2 x = \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$.

646 Example Calculate

$$\lim_{x\to 0}\frac{\sin\sin\tan\frac{x^2}{2}}{\log\cos 3x}.$$

Solution: *We use theorems 642 and 627.*

$$\begin{aligned}
\sin\sin\tan\frac{x^2}{2} &= \sin\sin\left(\frac{x^2}{2}+o\left(x^2\right)\right)\\
&= \sin\left(\frac{x^2}{2}+o\left(x^2\right)+o\left(\frac{x^2}{2}+o\left(x^2\right)\right)\right)\\
&= \sin\left(\frac{x^2}{2}+o\left(x^2\right)\right)\\
&= \frac{x^2}{2}+o\left(x^2\right),
\end{aligned}$$

and

$$\begin{aligned}
\log\cos 3x &= \log\left(1-\frac{9x^2}{2}+o\left(x^2\right)\right)\\
&= -\frac{9x^2}{2}+o\left(x^2\right)+o\left(-\frac{9x^2}{2!}+o\left(x^2\right)\right)\\
&= -\frac{9x^2}{2}+o\left(x^2\right)
\end{aligned}$$

The limit is thus equal to

$$\lim_{x\to 0}\frac{\frac{x^2}{2}+o\left(x^2\right)}{-\frac{9x^2}{2}+o\left(x^2\right)} = \lim_{x\to 0}\frac{\frac{1}{2}+o(1)}{-\frac{9}{2}+o(1)} = -\frac{1}{9}.$$

647 Example Find $\lim_{x\to 0}(\cos x)^{(\cot^2 x)}$.

Solution: *By example 645, we have* $\cot^2 x = \frac{1}{x^2} + o\left(\frac{1}{x^2}\right)$. *Also,*

$$\log\cos x = \log\left(1 - \frac{x^2}{2} + o\left(x^2\right)\right) = -\frac{x^2}{2} + o\left(x^2\right).$$

Hence

$$\begin{aligned}(\cos x)^{\cot^2 x} &= \exp\left((\cot^2 x)\log\cos x\right)\\ &= \exp\left(\left(\frac{1}{x^2} + o\left(\frac{1}{x^2}\right)\right)\left(-\frac{x^2}{2} + o\left(x^2\right)\right)\right)\\ &= \exp\left(-\frac{1}{2} + o(1)\right)\\ &\to e^{-1/2},\end{aligned}$$

as $x \to 0$.

Homework

648 Problem Prove that $\frac{\log(1+2\tan x)}{\sin x} \to 2$ as $x \to 0$.

649 Problem Prove that $\left(1+\frac{1}{x}\right)^x \to e$ as $x \to +\infty$.

650 Problem Prove that $(\tan x)^{\cot 4x} \to e^{1/2}$ as $x \to \frac{\pi}{4}$.

6.9 Asymptotic Expansions

651 Definition Let $n \in \mathbb{N}$ and let $f : \mathscr{N}_0 \to \mathbb{R}$ where $\mathscr{N}_0$ is a neighbourhood of 0. We say that f admits an *asymptotic expansion* of order n about $x = 0$ if there exists a polynomial p of degree n such that

$$\forall x \in \mathscr{N}_0, \quad f(x) = p(x) + o_0\left(x^n\right).$$

The polynomial p is called the *regular part of the asymptotic expansion about* $x = 0$ *of* f.

652 THEOREM If f admits an asymptotic expansion about 0, its regular part is unique.

Proof: *Assume* $f(x) = p(x) + o_0\left(x^n\right)$ *and* $f(x) = q(x) + o_0\left(x^n\right)$, *where* $p(x) = p_n x^n + \cdots + p_1 x + p_0$ *and* $q(x) = q_n x^n + \cdots + q_1 x + q_0$ *are polynomials of degree* n. *If* $p \neq q$ *let* k *be the largest* k *for which* $p_k \neq q_k$. *Then subtracting both equivalencies, as* $x \to 0$,

$$p(x) - q(x) = o\left(x^n\right) \implies (p_n - q_n)x^n + (p_{n-1} - q_{n-1})x^{n-1} + \cdots + (p_1 - q_1)x = o\left(x^n\right) \implies (p_k - q_k)x^k + \cdots + \cdots = o\left(x^n\right).$$

But $(p_k - q_k)x^k + \cdots + \cdots O\left(x^k\right)$ *as* $x \to 0$, *a contradiction, since* $k \leq n$. ❑

653 Definition Let $n \in \mathbb{N}$, $a \in \mathbb{R}$, and let $f : \mathscr{N}_a \to \mathbb{R}$ where $\mathscr{N}_a$ is a neighbourhood of a. We say that f admits an *asymptotic expansion* of order n about $x = a$ if there exists a polynomial p of degree n such that

$$\forall x \in \mathscr{N}_a, \quad f(x) = p(x - a) + o_a\left((x - a)^n\right).$$

The polynomial p is called the *regular part of the asymptotic expansion about* $x = a$ *of* f.

654 Definition Let $n \in \mathbb{N}$, and let $f : \mathscr{N}_{+\infty} \to \mathbb{R}$ where $\mathscr{N}_{+\infty}$ is a neighbourhood of $+\infty$. We say that f admits an *asymptotic expansion* of order n about $+\infty$ if there exists a polynomial p of degree n such that

$$\forall x \in \mathscr{N}_a \cap \left]0 ; +\infty\right[, \quad f(x) = p\left(\frac{1}{x}\right) + o_{+\infty}\left(\frac{1}{x^n}\right).$$

The polynomial p is called the *regular part of the asymptotic expansion about* $+\infty$ *of* f.

655 THEOREM Let $f : \mathscr{N}_0 \to \mathbb{R}$ be a function with an asymptotic expansion $f(x) = p(x) + o_0\left(x^n\right)$, where p is a polynomial. Then, if f is even, then p is even and if f is odd, then p is odd.

Proof: *Let $f(x) = p(x) + o\left(x^n\right)$ as $x \to 0$, where p is a polynomial of degree n. Then $f(-x) = p(-x) + o\left(x^n\right)$. If f is even then*

$$p(x) + o\left(x^n\right) = f(x) = f(-x) = p(-x) + o\left(x^n\right),$$

and so by uniqueness of the regular part of an asymptotic expansion we must have $p(x) = p(-x)$, so p is even. Similarly if f is odd then

$$-p(x) + o\left(x^n\right) = -f(x) = f(-x) = p(-x) + o\left(x^n\right),$$

and so by uniqueness of the regular part of an asymptotic expansion we must have $-p(x) = p(-x)$, so p is odd.

❑

We want to expand the function f in powers of $x - a$:

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots,$$

and that we will truncate at the n-th term, obtaining thereby a polynomial of degree n in powers of $x-a$. We must determine what the coefficients a_k are, and what the remainder

$$f(x) - a_0 - a_1(x-a) - a_2(x-a)^2 - \cdots - a_n(x-a)^n = R(x)$$

is. We hope that this remainder is $o_a\left((x-a)^n\right)$. The coefficients a_k are easily found. For $0 \le k \le n$ since f is $n+1$ times differentiable, differentiating k times,

$$f^{(k)}(x) = k!a_k + ((k+1)(k)\cdots 2)a_{k+1}(x-a) + ((k+2)(k+1)\cdots 3)a_{k+2}(x-a)^2 + \cdots + R^{(k)}(x), \implies \frac{f^{(k)}(a)}{k!} = a_k,$$

as long as $R(a) = R'(a) = R''(a) = \cdots = R^{(n)}(a) = 0$. We write our ideas formally in the following theorems.

656 THEOREM (Taylor-Lagrange Theorem) Let $I \subsetneqq \mathbb{R}$, $I \neq \varnothing$ be an interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be $n+1$ times differentiable in I. Then if $(x, a) \in I^2$,, there exist c with $\inf(x, a) < c < \sup(x, a)$ such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Proof: *If $x = a$ then there is nothing to prove. If $x < a$ then replace $x \mapsto f(x)$ with $x \mapsto f(-x)$, which then verifies the same hypotheses given in the theorem. Thus it remains to prove the theorem for $x > a$. Consider the function $\phi : \left[a\,;x\right] \to \mathbb{R}$ with*

$$\phi(t) = f(x) - \sum_{k=0}^{n} f^{(k)}(t)\,\frac{(x-t)^k}{k!} - R\frac{(x-t)^{n+1}}{(n+1)!},$$

where R is a constant. Observe that $\phi(x) = 0$. We now choose the constant R so that $\phi(a) = 0$. Observe that ϕ is differentiable and that it satisfies the hypotheses of Rolle's Theorem on $\left[a\,;x\right]$. Therefore, there exists $c \in \left]a\,;x\right[$ such that $\phi'(c) = 0$. Now

$$\phi'(t) = -\sum_{k=1}^{n}\left(f^{(k+1)}(t)\frac{(x-t)^k}{k!} - f^{(k)}(t)\frac{(x-t)^{k-1}}{(k-1)!}\right) + R\frac{(x-t)^n}{n!} = -\frac{(x-t)^n}{n!}f^{(n+1)}(t) + R\frac{(x-t)^n}{n!},$$

from where we gather, that $R = f^{(n+1)}(c)$ and the theorem follows. ❑

657 COROLLARY (Taylor-Young Theorem) Let $f : \mathscr{N}_a \to \mathbb{R}$ be $n+1$ times differentiable in $\mathscr{N}_a$. Then f admits the asymptotic expansion of order n about a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + o_a\left((x-a)^n\right).$$

Proof: *Follows at once from Theorem 656.* ❑

The following theorem follows at once from Corollary 657.

658 THEOREM Let $x \to 0$. Then

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o\left(x^{2n+2}\right)$.
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o\left(x^{2n+1}\right)$.
3. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o\left(x^5\right)$.
4. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o\left(x^n\right)$
5. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + o\left(x^n\right)$.
6. $(1+x)^\tau = 1 + \tau x + \frac{\tau(\tau-1)}{2}x^2 + \cdots + \frac{\tau(\tau-1)(\tau-2)(\tau-3)\cdots(\tau-n+1)}{n!}x^n + o\left(x^n\right)$.

659 Example Find an asymptotic development of $\log(2\cos x + \sin x)$ around $x = 0$ of order $o\left(x^4\right)$.

Solution: *By theorem 658,*

$$\begin{aligned}
2\cos x + \sin x &= 2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o\left(x^5\right)\right) + \left(x - \frac{x^3}{6} + o\left(x^4\right)\right) \\
&= 2 + x - x^2 - \frac{x^3}{6} + \frac{x^4}{12} + o\left(x^4\right) \\
&= 2\left(1 + \frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right),
\end{aligned}$$

and so,

$$\begin{aligned}
\log(2\cos x + \sin x) &= \log 2\left(1 + \frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right) \\
&= \log 2 + \log\left(1 + \frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right) \\
&= \log 2 + \left(\frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right) \\
&\quad - \frac{1}{2}\left(\frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right)^2 \\
&\quad + \frac{1}{3}\left(\frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right)^3 \\
&\quad - \frac{1}{4}\left(\frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24} + o\left(x^4\right)\right)^4 + o\left(x^4\right) \\
&= \log 2 + \left(\frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{12} + \frac{x^4}{24}\right) - \frac{1}{2}\left(\frac{x^2}{4} - \frac{x^3}{2} + \frac{x^4}{6}\right) \\
&\quad + \frac{1}{3}\left(\frac{x^3}{8} - \frac{3x^4}{8}\right) - \frac{1}{4}\cdot\frac{x^4}{16} + o\left(x^4\right) \\
&= \log 2 + \frac{x}{2} - \frac{5x^2}{8} + \frac{5x^3}{24} - \frac{35x^4}{192} + o\left(x^4\right)
\end{aligned}$$

as $x \to 0$.

Homework

660 Problem Prove that the limit

$$\lim_{n\to+\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n,$$

exists. The constant

$$\gamma=\lim_{n\to+\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)-\log n$$

is called the *Euler-Mascheroni* constant. It is not known whether γ is irrational.

Chapter 7

Integrable Functions

7.1 The Area Problem

661 Definition Let $f : \left[a\,;b\right] \to \mathbb{R}$ be bounded, say with $m \leq f(x) \leq M$ for all $x \in \left[a\,;b\right]$. Corresponding to each partition $\mathscr{P} = \{x_0, x_1, \ldots, x_n\}$ of $\left[a\,;b\right]$, we define the *upper Darboux sum*

$$U(f, \mathscr{P}) = \sum_{k=1}^{n} (\sup_{x_{k-1} \leq x \leq x_k} f(x))(x_k - x_{k-1}),$$

and the *lower Darboux sum*

$$L(f, \mathscr{P}) = \sum_{k=1}^{n} (\inf_{x_{k-1} \leq x \leq x_k} f(x))(x_k - x_{k-1}).$$

Clearly

$$L(f, \mathscr{P}) \leq U(f, \mathscr{P}).$$

Finally, we put

$$\overline{\int_a^b} f(x) \mathrm{d}x = \inf_{\mathscr{P} \text{ is a partition of } \left[a\,;b\right]} U(f, \mathscr{P}),$$

which we call the *upper Riemann integral of* f and

$$\underline{\int_a^b} f(x) \mathrm{d}x = \sup_{\mathscr{P} \text{ is a partition of } \left[a\,;b\right]} L(f, \mathscr{P}).$$

which we call the *lower Riemann integral of* f.

662 Definition Let $f : \left[a\,;b\right] \to \mathbb{R}$ be bounded. We say that f *is Riemann integrable* if $\overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x$. In this case, we denote their common value by $\int_a^b f(x)\mathrm{d}x$ and call it the *Riemann integral of* f *over* $\left[a\,;b\right]$.

663 Theorem Let f be a bounded function on $\left[a\,;b\right]$ and let $\mathscr{P} \subseteqq \mathscr{P}'$ be two partitions of $\left[a\,;b\right]$. Then

$$L(f, \mathscr{P}) \leq L(f, \mathscr{P}') \leq U(f, \mathscr{P}') \leq U(f, \mathscr{P}).$$

Proof: *Clearly is enough to prove this when* $\mathscr{P}'$ *has exactly one more point than* $\mathscr{P}$. *Let*

$$\mathscr{P} = \{x_0, x_1, \ldots, x_n\}$$

with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. *Let* $\mathscr{P}'$ *have the extra point* x_* *with* $x_i < x_* < x_{i+1}$. *Observe that we have both* $\inf_{x_i \le x \le x_{i+1}} f(x) \le \inf_{x_i \le x \le x_*} f(x)$ *and* $\inf_{x_i \le x \le x_{i+1}} f(x) \le \inf_{x_* \le x \le x_{i+1}} f(x)$ *since the larger interval may contain smaller values of* f. *Then*

$$\begin{aligned}
\inf_{x_i \le x \le x_{i+1}} f(x)(x_{i+1} - x_i) &= \inf_{x_i \le x \le x_{i+1}} f(x)(x_{i+1} - x_* + x_* - x_i) \\
&= \inf_{x_i \le x \le x_{i+1}} f(x)(x_* - x_i) + \inf_{x_i \le x \le x_{i+1}} f(x)(x_{i+1} - x_*) \\
&\le \inf_{x_i \le x \le x_*} f(x)(x_* - x_i) + \inf_{x_* \le x \le x_{i+1}} f(x)(x_{i+1} - x_*).
\end{aligned}$$

Thus

$$\begin{aligned}
L(f, \mathscr{P}) &= (\inf_{x_0 \le x \le x_1} f(x))(x_1 - x_0) + \cdots + (\inf_{x_i \le x \le x_{i+1}} f(x))(x_{i+1} - x_i) + \cdots + (\inf_{x_{n-1} \le x \le x_n} f(x))(x_n - x_{n-1}) \\
&\le (\inf_{x_0 \le x \le x_1} f(x))(x_1 - x_0) + \cdots + (\inf_{x_i \le x \le x_*} f(x))(x_* - x_i) + (\inf_{x_* \le x \le x_{i+1}} f(x))(x_{i+1} - x_*) + \cdots + (\inf_{x_{n-1} \le x \le x_n} f(x))(x_n - x_{n-1}) \\
&= L(f, \mathscr{P}').
\end{aligned}$$

A similar argument shews that $U(f, \mathscr{P}') \le U(f, \mathscr{P})$. *The we have*

$$L(f, \mathscr{P}) \le L(f, \mathscr{P}') \le U(f, \mathscr{P}') \le U(f, \mathscr{P})$$

proving te theorem.❑

664 THEOREM Let f be a bounded function on $\left[a\,;b\right]$ and let $\mathscr{P}_1$ and $\mathscr{P}_2$ be any two partitions of $\left[a\,;b\right]$. Then

$$L(f, \mathscr{P}_1) \le U(f, \mathscr{P}_2)$$

Proof: *Let* $\mathscr{P}' = \mathscr{P}_1 \cup \mathscr{P}_2$ *be a common refinement for* $\mathscr{P}_1$ *and* $\mathscr{P}_2$. *By Theorem 663,*

$$L(f, \mathscr{P}_1) \le L(f, \mathscr{P}_1 \cup \mathscr{P}_2) \le U(f, \mathscr{P}_1 \cup \mathscr{P}_2) \le U(f, \mathscr{P}_1),$$

and

$$L(f, \mathscr{P}_2) \le L(f, \mathscr{P}_1 \cup \mathscr{P}_2) \le U(f, \mathscr{P}_1 \cup \mathscr{P}_2) \le U(f, \mathscr{P}_2),$$

whence the theorem follows.❑

665 THEOREM Let f be a bounded function on $\left[a\,;b\right]$. Then $\underline{\int_a^b} f(x)\mathrm{d}x \le \overline{\int_a^b} f(x)\mathrm{d}x$.

Proof: *By Theorem 664,*

$$L(f, \mathscr{P}_1) \le U(f, \mathscr{P}_2) \implies \underline{\int_a^b} f(x)\mathrm{d}x = \sup_{\mathscr{P}_1 \text{ is a partition of } \left[a\,;b\right]} L(f, \mathscr{P}_1) \le U(f, \mathscr{P}_2),$$

and so

$$\underline{\int_a^b} f(x)\mathrm{d}x \le U(f, \mathscr{P}_2.$$

Taking now the infimum,

$$\underline{\int_a^b} f(x)\mathrm{d}x \le \inf_{\mathscr{P}_2 \text{ is a partition of } \left[a\,;b\right]} U(f, \mathscr{P}_2) = \overline{\int_a^b} f(x)\mathrm{d}x,$$

and the result is established.❑

666 THEOREM Let f be a bounded function on $\left[a\,;b\right]$. Then f is Riemann integrable if and only if $\forall \varepsilon > 0$, $\exists \mathscr{P}$ a partition of $\left[a\,;b\right]$ such that

$$U(f,\mathscr{P}) - L(f,\mathscr{P}) < \varepsilon.$$

Proof:

$\Leftarrow$ *If for all* $\varepsilon > 0$, $U(f,\mathscr{P}) - L(f,\mathscr{P}) < \varepsilon$ *then by Theorem 665,*

$$L(f,\mathscr{P}) \leq \underline{\int_a^b} f(x)\mathrm{d}x \leq \overline{\int_a^b} f(x)\mathrm{d}x \leq U(f,\mathscr{P}) \implies 0 \leq \overline{\int_a^b} f(x)\mathrm{d}x - \underline{\int_a^b} f(x)\mathrm{d}x < \varepsilon,$$

and so $\overline{\int_a^b} f(x)\mathrm{d}x = \underline{\int_a^b} f(x)\mathrm{d}x$, *which means that* f *is Riemann-integrable.*

$\Longrightarrow$ *Suppose* f *is Riemann integrable. By the Approximation property of the supremum and infimum, for all* $\varepsilon > 0$ *there exist partitions* $\mathscr{P}_1$ *and* $\mathscr{P}_2$ *such that*

$$U(f,\mathscr{P}_2) - \int_a^b f(x)\mathrm{d}x < \frac{\varepsilon}{2}, \qquad \int_a^b f(x)\mathrm{d}x - L(f,\mathscr{P}_1) < \frac{\varepsilon}{2}.$$

Hence by taking $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ *then*

$$U(f,\mathscr{P}) \leq U(f,\mathscr{P}_2) < \int_a^b f(x)\mathrm{d}x + \frac{\varepsilon}{2} < L(f,\mathscr{P}_1) + \varepsilon < L(f,\mathscr{P}) + \varepsilon,$$

from where $U(f,\mathscr{P}) - L(f,\mathscr{P}) < \varepsilon$.

❑

667 Example

- $f(x) = \begin{cases} 0 & x \text{ irrational}, \\ 1 & x \text{ rational}. \end{cases} \quad x \in [0;1]$

  Then $U(f,\mathscr{P}) = 1, L(f,\mathscr{P}) = 0$, for any partition $\mathscr{P}$, and so f is not Riemann integrable.

- $f(x) = \begin{cases} 0 & x \text{ irrational}, \\ \dfrac{1}{q} & x \text{ rational} = \dfrac{p}{q} \text{ in lowest terms}. \end{cases} \quad x \in [0;1]$

  is Riemann integrable with

$$\int_0^1 f(x)\,\mathrm{d}x = 0$$

668 Definition Let f be a bounded function on $\left[a\,;b\right]$ and let $\mathscr{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $\left[a\,;b\right]$. If t_k are selected so that $x_{k-1} \leq t_k \leq x_k$, put

$$S(f,\mathscr{P}) = \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}),$$

is the *Riemann sum of* f *associated with* $\mathscr{P}$.

669 THEOREM Let $f_1, f_2, \ldots, f_m$ be Riemann integrable over $\left[a\,;b\right]$, and let $f : \left[a\,;b\right] \to \mathbb{R}$. If for any subinterval $I \subsetneqq \left[a\,;b\right]$ there exists strictly positive numbers $a_1, a_2, \ldots, a_m$ such that

$$\omega(f,I) \leq a_1\omega(f_1,I) + a_2\omega(f_2,I) + \cdots + a_m\omega(f_m,I),$$

then f is also Riemann integrable.

Proof: *Let* $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *be a partition of* $\left[a\,;b\right]$ *selected so that for all* j,

$$U(f_j, \mathscr{P}) - L(f_j, \mathscr{P}) < \frac{\varepsilon}{a_1 + a_2 + \cdots + a_m}.$$

Using the notation of the preceding theorem,

$$\begin{aligned} U(f, \mathscr{P}) - L(f, \mathscr{P}) &= Z(f, \mathscr{P}) \\ &= \sum_{k=1}^{n} \omega(f, \left[x_{k-1}\,;x_k\right])(x_k - x_{k-1}) \\ &\leq \sum_{k=1}^{n} \sum_{j=1}^{m} a_j \omega(f_j, \left[x_{k-1}\,;x_k\right])(x_k - x_{k-1}) \\ &= \sum_{j=1}^{m} a_j \sum_{k=1}^{n} \omega(f_j, \left[x_{k-1}\,;x_k\right])(x_k - x_{k-1}) \\ &= \sum_{j=1}^{m} a_j \left(U(f_j, \mathscr{P}) - L(f_j, \mathscr{P})\right) \\ &< \varepsilon, \end{aligned}$$

and the theorem follows from Theorem 666. ❑

670 THEOREM (Algebra of Riemann Integrable Functions) Let f and g be Riemann integrable functions on $\left[a\,;b\right]$ and let $\lambda \in \mathbb{R}$ be a constant. Then the following are also Riemann integrable

1. $f + \lambda g$
2. $|f|$
3. fg
4. provided $\inf_{x\in[a;b]} |g(x)| > 0$, also $\dfrac{1}{g}$
5. provided $\inf_{x\in[a;b]} |g(x)| > 0$, also $\dfrac{f}{g}$

Proof: *Since*

$$|f(x) + \lambda g(x) - f(t) - \lambda g(t)| \leq |f(x) - f(t)| + |\lambda|\,|g(x) - g(t)|, \qquad \text{and} \qquad \big||f(x) - |f(t)|\big|| \leq |f(x) - f(t)|,$$

we have

$$\omega(f + \lambda g, I) \leq \omega(f, I) + |\lambda|\,\omega(g, I) \qquad \text{and} \qquad \omega(|f|, I) \leq \omega(f, I),$$

from where the first two assertions follow, upon appealing to Theorem 669.

To prove the third assertion, put $a_1 = \sup_{u\in[a;b]} |f(u)|$ *and* $a_2 = \sup_{u\in[a;b]} |g(u)|$

$$\begin{aligned} |f(x)g(x) - f(t)g(t)| &= |f(x)(g(x) - g(t)) + g(t)(f(x) - f(t))| \\ &\leq |f(x)|\,|g(x) - g(t)| + |g(t)|\,|f(x) - f(t)| \\ &\leq \left(\sup_{u\in[a;b]} |f(u)|\right)|g(x) - g(t)| + \left(\sup_{u\in[a;b]} |g(u)|\right)|f(x) - f(t)| \\ &= a_1\,|g(x) - g(t)| + a_2\,|f(x) - f(t)|, \end{aligned}$$

which gives

$$\omega(fg, I) \leq a_1\omega(f, I) + a_2\omega(g, I),$$

and so the third assertion follows from Theorem 669.

To prove the fourth assertion, with $a = \inf_{x\in[a;b]} |g(x)| > 0$, *observe that we have*

$$\begin{aligned}\left|\frac{1}{g(x)} - \frac{1}{g(t)}\right| &= \frac{1}{|g(x)g(t)|}|g(x) - g(t)| \\ &\leq \frac{1}{a^2}|g(x) - g(t)|,\end{aligned}$$

and this gives $\omega(\frac{1}{g}, I) \leq \frac{1}{a^2}\omega(g, I)$. *The fourth assertion now follows by again appealing to Theorem 669.*

The fifth assertion follows from the third and the fourth.❑

671 THEOREM Let f and g be Riemann integrable functions on $\left[a\,;b\right]$ and let $\lambda \in \mathbb{R}$ be a constant. Then

$$\int_a^b (f(x) + \lambda g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \lambda \int_a^b g(x)\mathrm{d}x.$$

Proof: *Let* $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *be a partition of* $\left[a\,;b\right]$ *and choose* t_k *such that* $t_k \in \left[x_{k-1}\,;x_k\right]$. *Then for any* $\varepsilon > 0$ *there exist* $\delta > 0$ *and* $\delta' > 0$ *such that*

$$\left|\sum_{k=1}^n f(t_k)(x_k - x_{k-1}) - \int_a^b f(x)\mathrm{d}x\right| < \frac{\varepsilon}{2} \qquad \text{if } \left|\left|\mathscr{P}\right|\right| < \delta,$$

$$\left|\lambda \sum_{k=1}^n g(t_k)(x_k - x_{k-1}) - \lambda \int_a^b g(x)\mathrm{d}x\right| < \frac{\varepsilon}{2} \qquad \text{if } \left|\left|\mathscr{P}\right|\right| < \delta'.$$

Hence, if $\left|\left|\mathscr{P}\right|\right| < \min(\delta, \delta')$,

$$\begin{aligned}&\left|\sum_{k=1}^n \left(f(t_k) + \lambda g(t_k)\right)(x_k - x_{k-1}) - \int_a^b f(x)\mathrm{d}x - \lambda \int_a^b g(x)\mathrm{d}x\right| \\ &\leq \left|\sum_{k=1}^n f(t_k)(x_k - x_{k-1}) - \int_a^b f(x)\mathrm{d}x\right| + \left|\lambda \sum_{k=1}^n g(t_k)(x_k - x_{k-1}) - \lambda \int_a^b g(x)\mathrm{d}x\right| \\ &< \varepsilon\end{aligned}$$

proving the theorem.❑

672 THEOREM Let f and g be Riemann integrable functions on $\left[a\,;b\right]$ with $f(x) \leq g(x)$ for all $x \in \left[a\,;b\right]$. Then

$$\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x.$$

Proof: *The function* $h = g - f$ *is positive for all* $x \in [a;b]$ *and hence* $L(h, \mathscr{P}) \geq 0$ *for all partitions* $\mathscr{P}$. *It is also integrable by Theorem 671. Thus*

$$\int_a^b h(x)\mathrm{d}x = \underline{\int_a^b} h(x)\mathrm{d}x \geq 0.$$

But

$$\int_a^b h(x)\mathrm{d}x \geq 0 \implies 0 \leq \int_a^b (g(x) - f(x))\mathrm{d}x = \int_a^b g(x)\mathrm{d}x - \int_a^b f(x)\mathrm{d}x,$$

and so $\int_a^b f(x)\mathrm{d}x \leq \int_a^b g(x)\mathrm{d}x$, *as claimed.* ❑

673 THEOREM (Triangle Inequality for Integrals) Let f be a Riemann integrable function on $\left[a\,;b\right]$. Then

$$\left|\int_a^b f(x)\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

Proof: *By Theorem 670, $|f|$ is integrable. Now, since $-|f| \leq f \leq |f|$ we just need to apply Theorem 672 twice.* ❑

674 THEOREM (Chasles' Rule) Let f be a Riemann integrable function on $\left[a\,;b\right]$ and let $c \in \left]a\,;b\right[$. Then f is Riemann integrable function on $\left[a\,;c\right]$ and $\left[c\,;b\right]$. Moreover,

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Conversely, if $c \in \left]a\,;b\right[$ and f is Riemann integrable on $\left[a\,;c\right]$ and $\left[c\,;b\right]$ then f is Riemann integrable on $\left[a\,;b\right]$ and

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Proof: *Consider the partitions*

$$\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_m = c < x_{m+1} < \cdots < x_n = b\}, \quad \mathscr{P}' = \{a = x_0 < x_1 < \cdots < x_m = c\}, \quad \mathscr{P}'' = \{c = x_m < x_{m+1} < \cdots < x_n = b\}.$$

where by virtue of Theorem 666, given $\varepsilon > 0$, we choose $\mathscr{P}$ so that

$$U(f,\mathscr{P}) - L(f,\mathscr{P}) < \varepsilon.$$

It follows that

$$\left(U(f,\mathscr{P}') - L(f,\mathscr{P}')\right) + \left(U(f,\mathscr{P}'') - L(f,\mathscr{P}'')\right) = U(f,\mathscr{P}) - L(f,\mathscr{P}) < \varepsilon.$$

Hence f is Riemann-integrable over both $[a;c]$ and $[c;b]$. Observe that

$$0 \leq U(f,\mathscr{P}') - \int_a^c f(x)\mathrm{d}x < \varepsilon, \qquad 0 \leq U(f,\mathscr{P}'') - \int_c^b f(x)\mathrm{d}x < \varepsilon,$$

$$0 \leq \int_a^c f(x)\mathrm{d}x - L(f,\mathscr{P}') < \varepsilon, \qquad 0 \leq \int_c^b f(x)\mathrm{d}x - L(f,\mathscr{P}'') < \varepsilon,$$

and upon addition,

$$0 \leq U(f,\mathscr{P}) - \left(\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x\right) < 2\varepsilon,$$

$$0 \leq \left(\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x\right) - L(f,\mathscr{P}) < 2\varepsilon,$$

whence

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x,$$

as required. ❑

675 THEOREM (Converse of Chasles' Rule) Let f be a function defined on the interval $[a;b]$ and let $c \in]a;b[$. If f is Riemann-integrable on $[a;c]$ and $[c;b]$ then it is also Riemann integrable in $[a;b]$ and

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$$

Proof: *Since* f *is Riemann-integrable on both subintervals, it is bounded there, and so it is bounded on the larger subinterval. By Theorem 666, given* $\varepsilon > 0$ *there exist partitions* $\mathscr{P}'$ *and* $\mathscr{P}''$ *such that*

$$U_{[a;c]}(f, \mathscr{P}') - L_{[a;c]}(f, \mathscr{P}') < \varepsilon, \qquad U_{[c;b]}(f, \mathscr{P}'') - L_{[c;b]}(f, \mathscr{P}'') < \varepsilon.$$

The above inequalities also hold in the refinement $\mathscr{P} = \mathscr{P}' \cup \mathscr{P}''$, *and*

$$U(f, \mathscr{P}) = U_{[a;c]}(f, \mathscr{P}) + U_{[c;b]}(f, \mathscr{P}), \qquad L(f, \mathscr{P}) = L_{[a;c]}(f, \mathscr{P}) + L_{[c;b]}(f, \mathscr{P}).$$

We then deduce that

$$\begin{aligned} U(f, \mathscr{P}) - L(f, \mathscr{P}) &= \left(U_{[a;c]}(f, \mathscr{P}) + U_{[c;b]}(f, \mathscr{P})\right) - \left(L_{[a;c]}(f, \mathscr{P}) + L_{[c;b]}(f, \mathscr{P})\right) \\ &= \left(U_{[a;c]}(f, \mathscr{P}) - L_{[a;c]}(f, \mathscr{P})\right) - \left(U_{[a;c]}(f, \mathscr{P}) - L_{[c;b]}(f, \mathscr{P})\right) \\ &< 2\varepsilon, \end{aligned}$$

and so f *is Riemann integrable in* $[a;b]$ *by virtue of Theorem 666. Now*

$$\begin{aligned} \int_a^b f(x)\mathrm{d}x &\leq U(f, \mathscr{P}) \\ &< L(f, \mathscr{P}) + \varepsilon \\ &= L_{[a;c]}(f, \mathscr{P}) + L_{[c;b]}(f, \mathscr{P}) + \varepsilon \\ &\leq \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x + \varepsilon, \end{aligned}$$

and similarly

$$\begin{aligned} \int_a^b f(x)\mathrm{d}x &\geq L(f, \mathscr{P}) \\ &> U(f, \mathscr{P}) - \varepsilon \\ &= U_{[a;c]}(f, \mathscr{P}) + U_{[c;b]}(f, \mathscr{P}) - \varepsilon \\ &\geq \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x - \varepsilon, \end{aligned}$$

hence

$$\int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x - \varepsilon \leq \int_a^b f(x)\mathrm{d}x \leq \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x + \varepsilon$$

giving the desired equality between integrals.❑

676 THEOREM Let f be Riemann integrable over $[a;b]$ and let $g : \left[\inf_{u \in [a;b]} f(u)\,;\, \sup_{u \in [a;b]} f(u)\right] \to \mathbb{R}$ be continuous. Then $g \circ f$ is Riemann integrable on $[a;b]$.

Proof: *Since* g *is uniformly continuous on the compact interval* $\left[\inf_{u \in [a;b]} f(u)\,;\, \sup_{u \in [a;b]} f(u)\right]$, *for given* $\varepsilon > 0$ *we may find* δ' *such that*

$$(s, t) \in \left[\inf_{t \in [a;b]} f(t)\,;\, \sup_{u \in [a;b]} f(u)\right]^2; \quad |s - t| < \delta' \implies \left|f(s) - f(t)\right| < \varepsilon.$$

Let $\delta = \min(\delta', \varepsilon)$. *Since* f *is Riemann-integrable, we may choose a partition* $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *such that*

$$U(f, \mathscr{P}) - L(f, \mathscr{P}) < \delta^2, \tag{7.1}$$

by virtue of Theorem 666. Let

$$m_k = \inf_{x_{k-1}\le x\le x_k} f(x); \quad M_k = \sup_{x_{k-1}\le x\le x_k} f(x);$$

$$m_k^* = \inf_{x_{k-1}\le x\le x_k} (g\circ f)(x); \quad M_k^* = \sup_{x_{k-1}\le x\le x_k} (g\circ f)(x).$$

We split the set of indices $\{1,2,\ldots,n\}$ *into two classes:*

$$A = \{k : 1\le k\le n, M_k - m_k < \delta\}; \qquad B = \{k : 1\le k\le n, M_k - m_k \ge \delta\}.$$

If $k\in A$ *and* $x_{k-1}\le x\le y\le x_k$*, then*

$$\left|f(x)-f(y)\right| \le M_k - m_k < \delta \le \delta' \implies \left|(g\circ f)(x) - (f\circ g)(y)\right| < \varepsilon,$$

whence $M_k^* - m_k^* \le \varepsilon$*. Therefore*

$$\sum_{k\in A}\left(M_k^* - m_k^*\right)(x_k - x_{k-1}) \le \varepsilon \sum_{k=1}^{n} (x_k - x_{k-1}) = \varepsilon(b-a).$$

If $k\in B$ *then* $M_k - m_k \ge \delta$ *and by virtue of (7.1),*

$$\delta\sum_{k\in B}(x_k - x_{k-1}) \le \sum_{k\in B}(M_k - m_k)(x_k - x_{k-1}) \le \sum_{1\le k\le n}(M_k - m_k)(x_k - x_{k-1}) = U(f,\mathscr{P}) - L(f,\mathscr{P}) < \delta^2,$$

whence

$$\sum_{k\in B}(x_k - x_{k-1}) < \delta \le \varepsilon.$$

Upon assembling all these inequalities, and letting $M = \sup_{t\in\left[\inf_{u\in[a;b]} f(u)\,;\sup_{u\in[a;b]} f(u)\right]} \left|g(t)\right|$*, we obtain*

$$\begin{aligned} U(g\circ f,\mathscr{P}) - L(g\circ f,\mathscr{P}) &= \sum_{k\in A}\left(M_k^* - m_k^*\right)(x_k - x_{k-1}) + \sum_{k\in B}\left(M_k^* - m_k^*\right)(x_k - x_{k-1}) \\ &\le \varepsilon(b-a) + 2M\sum_{k\in B}(x_k - x_{k-1}) \\ &\le \varepsilon(b-a) + 2M\varepsilon \\ &= \varepsilon(b-a+2M), \end{aligned}$$

whence the result follows from Theorem 666. ❑

677 Definition If $b < a$ we define $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$. Also, $\int_a^a f(x)\mathrm{d}x = 0.$

678 Theorem A function f on $\left[a\,;b\right]$ is Riemann integrable on $\left[a\,;b\right]$ if and only if its set of discontinuities forms a set of Lebesgue measure 0.

Proof:

$\implies$ *Given* $\gamma > 0$ *and* $\delta > 0$*, put* $\varepsilon = \gamma\delta$*. Let* f *be Riemann integrable. There is a partition* $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ *such that*

$$U(f,\mathscr{P}) - L(f,\mathscr{P}) < \varepsilon.$$

Let $x \in \left]x_i\,;x_{i+1}\right[$ *be such that* $\omega(f,x)\ge\gamma$*. Then*

$$\sup_{\left]x_i\,;x_{i+1}\right[} f(x) - \inf_{\left]x_i\,;x_{i+1}\right[} f(x) \ge \gamma.$$

Now observe that

$$\{x \in [a;b] : \omega(f,x) \geq \delta\} = \left(\bigcup_{\sup f - \inf f \geq \gamma} \left]x_i\,;x_{i+1}\right[\right) \cup \{x_0, x_1, \ldots, x_n\}.$$

Hence

$$\begin{aligned}
\mu\left(\{x \in [a;b] : \omega(f,x) \geq \gamma\}\right) &\leq \sum_{\sup_{]x_i;x_{i+1}[} f(x) - \inf_{]x_i;x_{i+1}[} f(x) \geq \gamma} |x_{i+1} - x_i| \\
&\leq \frac{1}{\gamma} \sum_i |x_{i+1} - x_i| \left(\sup_{]x_i\,;x_{i+1}[} f(x) - \inf_{]x_i\,;x_{i+1}[} f(x) \right) \\
&\leq \frac{1}{\gamma} \left(U(f,\mathscr{P}) - L(f,\mathscr{P}) \right) \\
&< \frac{\varepsilon}{\gamma} \\
&= \delta.
\end{aligned}$$

Letting $\delta \to 0+$ *and* $\gamma \to 0+$ *we get* $\mu\left(\{x \in [a;b] : \omega(f,x) \geq 0\}\right) = 0$, *and in particular,* $\mu\left(\{x \in [a;b] : \omega(f,x) > 0\}\right) = 0$ *which means hat the set of discontinuities is a set of measure* 0.

$\Leftarrow$ *Conversely, assume* $\mu\left(\{x \in [a;b] : \omega(f,x) > 0\}\right) = 0$. *We can write*

$$\{x \in [a;b] : \omega(f,x) > 0\} = \bigcup_{K \geq 1} \{x \in [a;b] : \omega(f,x) > \frac{1}{K}\}.$$

Fix K *large enough so that* $\frac{1}{K} < \varepsilon$. *Since* $\mu\left(\{x \in [a;b] : \omega(f,x) \geq \frac{1}{K}\}\right) = 0$, *we can find open intervals* $I_j(K)$ *such that*

$$\{x \in [a;b] : \omega(f,x) \geq \frac{1}{K}\} \subseteqq \bigcup_{j \geq 1} I_j(K), \qquad \sum_{j \geq 1} \mu\left(I_j(K)\right) < \varepsilon.$$

It is easy to shew that $\left\{x \in [a;b] : \omega(f,x) > \frac{1}{K}\right\}$ *is closed and bounded and hence compact, so we may find a finite subcover with*

$$\{x \in [a;b] : \omega(f,x) > \frac{1}{K}\} \subseteqq I_{j_1} \cup I_{j_2} \cup \cdots \cup I_{j_N}.$$

Now

$$\left[a\,;b\right] \setminus \left(I_{j_1} \cup I_{j_2} \cup \cdots \cup I_{j_N}\right)$$

is a finite disjoint union of closed intervals, say $J_1 \cup J_2 \cup \cdots \cup J_M$. *If* $x \in J_i$ *then* $\omega(f,x) < \frac{1}{K}$. *Thus on each of the* J_i *we may find so fine a partition that* $\omega(f,L) < \frac{1}{K}$ *for every interval such partition. All these partitions and the endpoints of the* I_{j_k} *form a partition, say* $\mathscr{P}$. *Write* $\mathscr{S} = S_1 \cup S_2 \cup \cdots \cup S_M$ *for the intervals of the partition* $\mathscr{P}$ *that are not the* I_{j_k}. *Observe that* $\omega(f,S_k) < \frac{1}{K}$. *Then*

$$\begin{aligned}
U(f,\mathscr{P}) - L(f,\mathscr{P}) &= \sum_{I_{j_k}} \left(\sup_{I_{j_k}} f - \inf_{I_{j_k}} f \right) \left(\mu\left(I_{j_k}\right)\right) + \sum_{S_k} \left(\sup_{S_k} f - \inf_{S_k} f \right) \left(\mu\left(S_k\right)\right) \\
&\leq 2 \sup_{[a;b]} |f| \sum_{k=1}^{N} \mu\left(I_{j_k}\right) + \frac{1}{K} \sum_{S_k} \mu\left(S_k\right) \\
&\leq 2 \sup_{[a;b]} |f| \varepsilon + (b-a)\varepsilon \\
&= \left(2 \sup_{[a;b]} |f| + (b-a) \right) \varepsilon.
\end{aligned}$$

This proves the theorem.

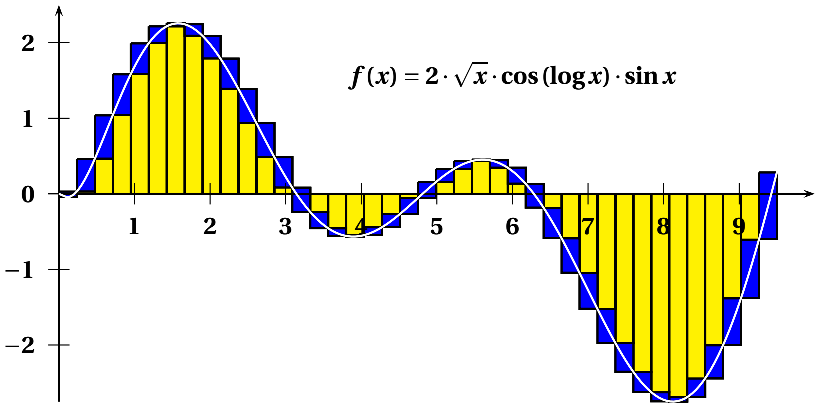
❑

679 Corollary Every continuous function f on $[a;b]$ is Riemann integrable on $\left[a\,;b\right]$.

Proof: *This is immediate from Theorem 678.* ❑

680 Corollary Every monotonic function f on $\left[a\,;b\right]$ is Riemann integrable on $\left[a\,;b\right]$.

Proof: *Since a countable set has measure* 0*, and since the set of discontinuities of a monotonic function is countable (Theorem 532), the result is immediate.*❑



Homework

681 Problem Let f be a bounded function on $\left[a\,;b\right]$. Then f is Riemann integrable if and only if $\forall\varepsilon>0$, $\exists\delta>0$ such that for all partitions $\mathscr{P}$ of $\left[a\,;b\right]$,

$$\left|\left|\mathscr{P}\right|\right|<\delta \implies U(f,\mathscr{P})-L(f,\mathscr{P})<\varepsilon.$$

682 Problem Let f be a bounded function on $\left[a\,;b\right]$. Then f is Riemann integrable on $\left[a\,;b\right]$ if and only if

$$\lim_{\left|\left|\mathscr{P}\right|\right|\to 0} S(f,\mathscr{P})$$

exists and is finite. In this case we write $\lim_{\left|\left|\mathscr{P}\right|\right|\to 0} S(f,\mathscr{P}) = \int_a^b f(x)\mathrm{d}x$.

683 Problem Let f be bounded on $\left[a\,;b\right]$. Then f is Riemann integrable on $\left[a\,;b\right]$ if and only if for every $\varepsilon>0, \varepsilon'>0$ there is a partition $\mathscr{P}$ of $\left[a\,;b\right]$ such that

$$\sum_{k=1}^{n}(x_k-x_{k-1})\chi_{\{x\in[a;b]:\omega(f,[x_{k-1};x_k])\geq\varepsilon'\}}<\varepsilon.$$

Here $\chi(.)$ is the *indicator function* defined on a set E as

$$\chi_E(x)=\begin{cases}1 & \text{if } x\in E\\ 0 & \text{if } x\not\in E\end{cases}.$$

7.2 Integration

684 Theorem (First Fundamental Theorem of Calculus) Let $f:\left[a\,;b\right]\to\mathbb{R}$ be Riemann integrable on $\left[a\,;b\right]$. If there exists a differentiable function $F:\left[a\,;b\right]\to\mathbb{R}$ such that $F'=f$ then

$$\int_a^b f(x)\mathrm{d}x = F(b)-F(a).$$

Proof: *Given* $\varepsilon>0$*, in view of Theorem 666, there is a partition* $\mathscr{P}=\{a=x_0<x_1<\cdots<x_n=b\}$ *such that*

$$U(f,\mathscr{P})-L(f,\mathscr{P})<\varepsilon.$$

Since F *is differentiable on* $[a;b]$*, it is continuous on* $[a;b]$*. Applying the Mean Value Theorem to each partition subinterval* $[x_{k-1};x_k]$*, we obtain* $c_k\in]x_{k-1};x_k[$ *such that*

$$F(x_k)-F(x_{k-1})=f(c_k)(x_k-x_{k-1}).$$

This gives

$$F(b)-F(a)=\sum_{1\le k\le n}(F(x_k)-F(x_{k-1}))=\sum_{1\le k\le n}f(c_k)(x_k-x_{k-1}),$$

and since $\inf_{u\in[x_{k-1};x_k]} f(u)\le f(c_k)\le \sup_{u\in[x_{k-1};x_k]} f(u)$, *we deduce that*

$$L(f,\mathscr{P})\le F(b)-F(a)\le U(f,\mathscr{P}).$$

Furthermore, we know that $L(f,\mathscr{P})\le \int_a^b f(x)\mathrm{d}x\le U(f,\mathscr{P})$. *Hence, combining these two last inequalities,*

$$\left|F(b)-F(a)-\int_a^b f(x)\mathrm{d}x\right|<\varepsilon,$$

and the theorem follows.❑

685 THEOREM (Second Fundamental Theorem of Calculus) Let $f:\left[a\,;b\right]\to\mathbb{R}$ be Riemann integrable on $\left[a\,;b\right]$, and let

$$F(x)=\int_a^x f(t)\mathrm{d}t,\qquad x\in\left[a\,;b\right].$$

Then F is continuous on $\left[a\,;b\right]$. Moreover, if f is continuous at $c\in\left]a\,;b\right[$, then F is differentiable at c and $F'(c)=f(c)$.

Proof: *There is* $M>0$ *such that* $\forall x\in[a;b],\quad |f(x)|\le M$. *Now, if* $a\le x<y\le b$ *with* $|x-y|<\frac{\varepsilon}{M}$, *then*

$$|F(y)-F(x)|=\left|\int_x^y f(t)\mathrm{d}t\right|\le\int_x^y|f(t)|\mathrm{d}t\le\int_x^y M\mathrm{d}t=M(y-x)<\varepsilon$$

Thus F *is continuous on* $[a;b]$ *and by Heine's Theorem, uniformly continuous on* $[a;b]$. *Now, take* $u\in]a;b[$, *and observe that*

$$x\neq u\implies\frac{F(x)-F(u)}{x-u}=\frac{1}{x-u}\int_u^x f(t)\mathrm{d}t.$$

Moreover,

$$f(u)=\frac{1}{x-u}\int_u^x f(u)\mathrm{d}t,$$

and therefore,

$$\frac{F(x)-F(u)}{x-u}-f(u)=\frac{1}{x-u}\int_u^x\left(f(t)-f(u)\right)\mathrm{d}t.$$

Since f *is continuous at* u, *there is* $\delta>0$ *such that*

$$t\in[a;b],|t-u|<\delta\implies|f(t)-f(u)|<\varepsilon.$$

This gives

$$\left|\frac{F(x)-F(u)}{x-u}-f(u)\right|<\varepsilon$$

for $x\in]a;b[$ *with* $|x-u|<\delta$. *From this it follows that* $F'(u)=f(u)$.❑

686 THEOREM (Young's Inequality for Integrals) Let f be a strictly increasing continuous function on $\left[0\,;+\infty\right[$ and let $f(0)=0$. If $A>0$ and $B>0$ then

$$AB\le\int_0^A f(x)\mathrm{d}x+\int_0^B f^{-1}(x)\mathrm{d}x.$$

Proof: *The inequality is evident from Figure 7.1. The rectangle of area* AB *fits nicely in the areas under the curves* $y=f(x), x\in[0;A]$ *and* $x=f^{-1}(y), y\in[0;B]$. ❑

687 THEOREM (Hölder's Inequality for Integrals) Let $p>1$ and put $\frac{1}{q}=1-\frac{1}{p}$. If f and g are Riemann integrable on $\left[a\,;b\right]$ then

$$\left|\int_a^b f(x)g(x)\mathrm{d}x\right| \leq \left(\int_a^b |f(x)|^p\,\mathrm{d}x\right)^{1/p}\left(\int_a^b |g(x)|^q\,\mathrm{d}x\right)^{1/q}.$$

Proof: *First observe that all of* $|fg|$, $|f|^p$ *and* $|g|^q$ *are Riemann-integrable, in view of Theorem 670. Now, with* $f(x)=x^{p-1}$ *in Young's Inequality (Theorem 686), we obtain,*

$$AB \leq \frac{A^p}{p}+\frac{B^{1/(p-1)+1}}{1/(p-1)+1}=\frac{A^p}{p}+\frac{B^q}{q}. \tag{7.2}$$

If any of the integrals in the statement of the theorem is zero, the result is obvious. Otherwise put $A^p=\int_a^b |f(x)|^p\,\mathrm{d}x$, $B^q=\int_a^b |g(x)|^p\,\mathrm{d}x$. *Then by (7.2),*

$$\frac{|f(x)g(x)|}{AB} \leq \frac{A^{-p}|f(x)|^p}{p}+\frac{B^{-q}|g(x)|^q}{q}.$$

Integrating throughout the above inequality,

$$\frac{1}{AB}\int_a^b |f(x)g(x)|\,\mathrm{d}x \leq \frac{1}{pA^p}\int_a^b |f(x)|^p\,\mathrm{d}x+\frac{1}{qB^q}\int_a^b |g(x)|^q\,\mathrm{d}x=\frac{1}{p}+\frac{1}{q}=1,$$

whence the theorem follows. ❑

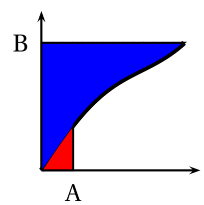


Figure 7.1: Young's Inequality (Theorem 686).

688 THEOREM Let $f:\left[a\,;b\right]\to\mathbb{R}$. Then

1. If f is continuous on $\left[a\,;b\right]$, $\forall x\in\left[a\,;b\right], f(x)\geq 0$, $\exists c\in\left[a\,;b\right]$ with $f(c)>0$ then $\int_a^b f(x)\mathrm{d}x>0$.

2. If f, g are continuous on $\left[a\,;b\right]$, $\forall x\in\left[a\,;b\right], f(x)\leq g(x)$, and $\exists c\in\left[a\,;b\right]$ with $f(c)<g(c)$ then $\int_a^b f(x)\mathrm{d}x<\int_a^b g(x)\mathrm{d}x$.

Proof: *The second part follows from the first by considering* $f-g$. *Let us prove the first part.*

Assume first that $c\in]a;b[$. *Then there is a neighourhood* $]c-\delta;c+\delta[\subseteq]a;b[$ *of* c, *with* $\delta>0$, *such that* $\forall x\in]c-\delta;c+\delta[, f(x)\geq\frac{f(c)}{2}$. *Therefore*

$$\int_a^b f(x)\mathrm{d}x \geq \int_{c-\delta}^{c+\delta} f(x)\mathrm{d}x > \int_{c-\delta}^{c+\delta}\frac{f(c)}{2}\mathrm{d}x=\delta f(c)>0.$$

If $c = a$ then we consider a neighbourhood of the form $]a;a+\delta[$, *and similarly if* $c = b$, *we consider a neighbourhood of the form* $]b-\delta;b[$ ❑

689 THEOREM (First Mean Value Theorem for Integrals) Let f,g be continuous on $\left[a;b\right]$, with g of constant sign on $\left[a;b\right]$. Then there exists $c \in \left]a;b\right[$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = f(c)\int_a^b g(x)\mathrm{d}x.$$

Proof: *If g is identically 0, there is nothing to prove. Similarly, if f is constant in $[a;b]$ there is nothing to prove. Otherwise, g is always strictly positive or strictly negative in the interval $[a;b]$. Let*

$$m = \inf_{x\in[a;b]} f(x); \quad M = \sup_{x\in[a;b]} f(x).$$

Then

$$m < \frac{\int_a^b f(x)g(x)\mathrm{d}x}{\int_a^b g(x)\mathrm{d}x} < M.$$

By the Intermediate Value Theorem, there is $c \in]a;b[$ such that

$$f(c) = \frac{\displaystyle\int_a^b f(x)g(x)\mathrm{d}x}{\displaystyle\int_a^b g(x)\mathrm{d}x},$$

proving the theorem.❑

690 THEOREM (Integration by Parts) Let f,g be differentiable functions on $[a;b]$ with f' and g' integrable on $[a;b]$. Then

$$\int_a^b f'(x)g(x)\mathrm{d}x + \int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)\Big|_a^b = f(b)g(b) - f(a)g(a).$$

Proof: *This follows at once from the Product Rule for Derivatives and the Second Fundamental Theorem of Calculus, since*

$$(fg)' = f'g + fg' \implies f(b)g(b) - f(a)g(a) = \int_a^b \big(f(x)g(x)\big)'\mathrm{d}x \int_a^b f'(x)g(x)\mathrm{d}x + \int_a^b f(x)g'(x)\mathrm{d}x.$$

❑

691 COROLLARY (Repeated Integration by Parts) Let $n \in \mathbb{N}$. If the $n+1$-th derivatives $f^{(n+1)}$ and $g^{(n+1)}$ are continuous on $[a;b]$ then

$$\int_a^b f(x)g^{(n+1)}(x)\mathrm{d}x = \left(f(x)g^{(n)}(x) - f'(x)g^{(n-1)}(x) + f''(x)g^{(n-1)}(x) - \cdots + (-1)^n f^{(n)}(x)g(x)\right)\Big|_a^b + (-1)^{n+1}\int_a^b f^{(n+1)}(x)g(x)\mathrm{d}x.$$

Proof: *Follows by inducting on n and applying Theorem 690.* ❑

692 THEOREM (Integration by Substitution) Let g be a differentiable function on an open interval I such that g' is continuous on I. If f is continuous on $g(I)$ then $f\circ g$ is continuous on I and $\forall (a,b) \in I^2$,

$$\int_a^b (f\circ g)(x)g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u.$$

Proof: *Fix* $c \in I$ *and put* $F(x) = \int_c^x f(u)\mathrm{d}u$. *By The Second Fundamental Theorem of Calculus,* $\forall x \in I, F'(x) = f(x)$. *Furthermore, let* $t(x) = F(g(x))$. *By The Chain Rule,* $t' = (F' \circ g)g' = (f \circ g)g'$. *Therefore*

$$\begin{aligned}\int_a^b (f \circ g)(x)g'(x)\mathrm{d}x &= \int_a^b t'(x)\mathrm{d}x \\ &= t(b) - t(a) \\ &= F(g(b)) - F(g(a)) \\ &= \int_c^{g(b)} f(u)\mathrm{d}u - \int_c^{g(a)} f(u)\mathrm{d}u \\ &= \int_{g(a)}^{g(b)} f(u)\mathrm{d}u,\end{aligned}$$

as was to be shewn. ❑

693 THEOREM (Second Mean Value Theorem for Integrals) Let f, g be continuous on $\left[a\,;b\right]$, with g monotonic on $\left[a\,;b\right]$. Then there exists $c \in \left]a\,;b\right[$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^c f(x)\mathrm{d}x + g(b)\int_c^b f(x)\mathrm{d}x.$$

Proof: *Put* $F(x) = \int_a^x f(t)\mathrm{d}t$. *Then* $F'(x) = f(x)$. *Hence*

$$\int_a^b f(x)g(x)\mathrm{d}x = \int_a^b F'(x)g(x)\mathrm{d}x = F(x)g(x)\Big|_a^b - \int_a^b F(x)g'(x)\mathrm{d}x$$

and therefore

$$\int_a^b f(x)g(x)\mathrm{d}x = F(b)g(b) - F(a)g(a) - \int_a^b F(x)g'(x)\mathrm{d}x.$$

By the First Mean Value Theorem for Integrals and by the First Fundamental Theorem of Calculus, there is a $c \in]a;b[$ *such that*

$$\int_a^b F(x)g'(x)\mathrm{d}x = F(c)\int_a^b g'(x)\mathrm{d}x = F(c)(g(b) - g(a)).$$

Assembling all the above,

$$\begin{aligned}\int_a^b f(x)g(x)\mathrm{d}x &= F(b)g(b) - F(a)g(a) - F(c)(g(b) - g(a)) \\ &= g(b)(F(b) - F(c)) + g(a)(F(c) - F(a)) \\ &= g(b)\int_c^b f(x)\mathrm{d}x + g(a)\int_a^c f(x)\mathrm{d}x,\end{aligned}$$

as desired.❑

694 THEOREM (Generalisation of the AM-GM Inequality) Let $a_i \geq 0$, $p_i \geq 0$ with $p_1 + p_2 + \cdots + p_n = 1$. Then

$$G = a_1^{p_1}a_2^{p_2}\cdots a_n^{p_n} \leq p_1a_1 + p_2a_2 + \cdots + p_na_n = A.$$

(Here we interpret $0^0 = 1$.)

Proof: *There is a subindex* k *such that* $a_k \leq G \leq a_{k+1}$. *Hence*

$$\sum_{i=1}^k p_i\int_{a_i}^G \left(\frac{1}{x} - \frac{1}{G}\right)\mathrm{d}x + \sum_{i=k+1}^n p_i\int_G^{a_i}\left(\frac{1}{G} - \frac{1}{x}\right)\mathrm{d}x \geq 0,$$

as all the integrands are ≥ 0. *Upon rearranging*

$$\sum_{i=1}^{n} p_i \int_{a_i}^{G} \frac{1}{x} \mathrm{d}x \leq \sum_{i=1}^{n} p_i \int_{G}^{a_i} \frac{1}{G} \mathrm{d}x \implies \sum_{i=1}^{n} p_i (\log a_i - \log G) \leq \sum_{i=1}^{n} p_i \cdot \frac{a_i - G}{G} \implies 0 \leq \frac{A}{G} - 1,$$

obtaining the inequality ❑

Homework

695 Problem Let p be a polynomial of degree at most 4 such that $p(-1) = p(1) = 0$ and $p(0) = 1$. If $p(x) \leq 1$ for $x \in [-1;1]$, find the largest value of $\int_{-1}^{1} p(x)\mathrm{d}x$.

696 Problem Compute $\int_0^3 x\lfloor x \rfloor \mathrm{d}x$.

697 Problem Let f be a differentiable function such that

$$f(x+h) - f(x) = e^{x+h} - h - e^x$$

and $f(0) = 3$. Find $f(x)$.

698 Problem Let f be a continuous function such that $f(x)f(a-x) = 1$ and let $a > 0$. Find $\int_0^a \frac{1}{f(x)+1}\mathrm{d}x$.

699 Problem Let f be a Riemann integrable function over every bounded interval and such that $f(a+b) = f(a) + f(b)$ for all $(a,b) \in \mathbb{R}^2$. Demonstrate that $f(x) = xf(1)$.

700 Problem Compute $\int_0^2 x\lfloor x^2 \rfloor \mathrm{d}x$.

701 Problem Find $\int_{-1}^{2} |x^2 - 1| \, \mathrm{d}x$.

702 Problem Let n be a fixed integer. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 2^n & \text{if } 2^n - 2^{n-2} < x \leq 2^{n+1} - 2^{n-1} \end{cases}$$

Prove that $\int_0^{2^n} f(x)\mathrm{d}x = \int_0^{2^n} x\mathrm{d}x = 2^{2^n - 1}$.

703 Problem (Putnam 1938) Evaluate the limit

$$\lim_{t \to 0} \frac{\int_0^t (1 + \sin 2x)^{1/x} \mathrm{d}x}{t}.$$

704 Problem Find the value of $\int_0^1 \max(x^2, 1-x)\mathrm{d}x$.

705 Problem Let $a > 0$. Let f be a continuous function on $\left[0\,;a\right]$ such that $f(x) + f(a-x)$ does not vanish on $\left[0\,;a\right]$. Evaluate $\int_0^a \frac{f(x)\mathrm{d}x}{f(x) + f(a-x)}$.

706 Problem Let $a > 0$. Let F be a differentiable function such that $\forall x \in \left[0\,;a\right]$ $F'(a-x) = F'(x)$. Evaluate $\int_0^a F(x)\mathrm{d}x$.

707 Problem Let $n \geq 0$ be an integer. Let a be the unique differentiable function such that $\forall x \in \mathbb{R}$

$$(a(x))^{2n+1} + a(x) = x.$$

Evaluate $\int_0^x a(t)\mathrm{d}t$.

708 Problem Find $\int_0^{\pi/2} \frac{\sin x \mathrm{d}x}{\sin x + \cos x}$.

709 Problem Find $\int_0^{\pi/2} \frac{1\mathrm{d}x}{1 + (\tan x)^{\sqrt{2}}}$.

710 Problem Find $\int \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x$.

711 Problem Find $\int \frac{1}{1+\sqrt{x+1}}\mathrm{d}x$.

712 Problem Find $\int \frac{x^{1/2}}{x^{1/2} - x^{1/3}}\mathrm{d}x$.

713 Problem Find $\int \frac{a^{2x}}{\sqrt{a^x + 1}}\mathrm{d}x$, $a > 0$.

714 Problem Find $\int \frac{1}{(e^x - e^{-x})^2}\mathrm{d}x$.

715 Problem Prove that $\int_1^5 \frac{\lfloor x \rfloor}{x}\mathrm{d}x = 4\log(5) - 3\log(2) - \log(3)$.

716 Problem Find $\int e^{e^x + x} \, \mathrm{d}x$.

717 Problem Find $\int \tan x \log(\cos x) \, \mathrm{d}x$.

718 Problem Find $\int \frac{\log \log x}{x \log x} \, \mathrm{d}x$.

719 Problem Find $\int \frac{x^{18} - 1}{x^3 - 1} \, \mathrm{d}x$.

720 Problem Find $\int \frac{1}{x^8 + x} \, \mathrm{d}x$.

721 Problem Find $\int \frac{4^x}{2^x + 1} \, \mathrm{d}x$.

722 Problem Find $\int \frac{x^2}{(x+1)^{10}} \, \mathrm{d}x$.

723 Problem Find $\int \frac{1}{1+e^x} \, \mathrm{d}x$.

724 Problem Find $\int \frac{1}{1 - \sin x} \, \mathrm{d}x$.

725 Problem Find $\int \sqrt{1 + \sin 2x}\,\mathrm{d}x$.

726 Problem Find $\int \frac{x}{\sqrt{1-x^4}}\mathrm{d}x$.

727 Problem Find $\int \sec^4 x\mathrm{d}x$.

728 Problem Find $\int \sec^5 x\mathrm{d}x$.

729 Problem Find $\int e^{x^{1/3}}\mathrm{d}x$.

730 Problem Find $\int \log(x^2+1)\mathrm{d}x$.

731 Problem Find $\int xe^x\cos x\mathrm{d}x$.

732 Problem Find $\int x^{2/3}\log x\mathrm{d}x$.

733 Problem Find $\int \sin(\log x)\mathrm{d}x$.

734 Problem Find $\int \frac{\log\log x}{x}\mathrm{d}x$.

735 Problem ($\int \sec x\mathrm{d}x$ in three ways) A traditional indefinite integral is

$$\int \sec x\mathrm{d}x = \log(\tan x + \sec x) + C.$$

Justify this formula.

Now, prove that $\frac{1}{\cos x} = \frac{\cos x}{2(1+\sin x)} + \frac{\cos x}{2(1-\sin x)}$. Use this to find a second formula for $\int \sec x\mathrm{d}x$.

A third way is as follows. Using $\sin 2\theta = 2\sin\theta\cos\theta$ shew that $\int \csc x\mathrm{d}x = \log|\tan\frac{x}{2}| + C$. Now use $\csc(\frac{\pi}{2}+x) = \sec x$ to find yet another formula for $\int \sec x\mathrm{d}x$.

736 Problem Find $\int (\arcsin x)^2\mathrm{d}x$.

737 Problem Find $\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt{x-1}}$.

738 Problem $\int x\arctan x\mathrm{d}x$.

739 Problem Find $\int \sqrt{\tan x}\mathrm{d}x$.

740 Problem Find $\int \frac{2x+1}{x^2(x-1)}\mathrm{d}x$.

741 Problem Find $\int \log(x+\sqrt{x})\mathrm{d}x$.

742 Problem Find $\int \frac{1}{x^4+1}\mathrm{d}x$.

743 Problem Find $\int \frac{1}{x^3+1}\mathrm{d}x$.

744 Problem Demonstrate that for all strictly positive integers n,

$$\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{4n}\right) < e < \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right),$$

that is, e is contained in the second quarter of the interval $\left[\left(1+\frac{1}{n}\right)^n;\left(1+\frac{1}{n}\right)^{n+1}\right]$.

7.3 Riemann-Stieltjes Integration

7.4 Euler's Summation Formula

Chapter 8

Sequences and Series of Functions

8.1 Pointwise Convergence

745 Definition We say that a sequence of functions $\{f_n\}_{n=1}^{+\infty}$ $f_n : I \to \mathbb{R}$ defined on an interval $I \subseteqq \mathbb{R}$ *converges pointwise to the function f* if $\forall x \in I$, $\forall \varepsilon > 0$ $\exists N > 0$ (depending on ε and on x) such that

$$n \geq N \implies \left|f_n(x) - f(x)\right| < \varepsilon.$$

746 Example The sequence of functions $x \mapsto x^n, n = 1, 2, \ldots$ converges pointwise on the interval $\left[0\,;1\right]$ to the function $f : \left[0\,;1\right] \to \{0, 1\}$ with

$$f(x) = \begin{cases} 0 & \text{if } x \in \left[0\,;1\right[\\ 1 & \text{if } x = 1 \end{cases}$$

747 Definition We say that a sequence of functions $\{f_n\}_{n=1}^{+\infty}$ $f_n : I \to \mathbb{R}$ defined on an interval $I \subseteqq \mathbb{R}$ *converges uniformly to the function f* if $\forall x \in I$, $\forall \varepsilon > 0$ $\exists N > 0$ (depending only on ε) such that

$$n \geq N \implies \left|f_n(x) - f(x)\right| < \varepsilon.$$

748 THEOREM If the sequence of functions $\{f_n\}_{n=1}^{+\infty}$ $f_n : I \to \mathbb{R}$ defined on an open interval $I \subseteqq \mathbb{R}$ converges uniformly to f on I, then f is continuous on I.

8.2 Integrals and Derivatives of Sequences of Functions

8.3 Power Series

A *power series* about $x = a$ is a series of the form

$$f(x) = \sum_{n=0}^{+\infty} a_n (x - a)^n.$$

This is a function of x, and truncating it gives polynomial approximations to f. The goal is to approximate "decent" functions about a given point $x = a$.

These expansions don't necessarily make sense for all x. The region where the power series converges is called the *interval of convergence.*

749 Example Find the interval of convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{\sqrt{n}}$.

Solution: *By the ratio test, the series will converge if*

$$\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{2^n(x-3)^n}\right| = 2\sqrt{\frac{n}{n+1}}|x-3| \to r < 1,$$

that is when

$$2|x-3|<1 \implies \frac{5}{2}<x<\frac{7}{2}.$$

The series converges absolutely when $\frac{5}{2}<x<\frac{7}{2}$*. We must also test the endpoints. At* $x=\frac{5}{2}$ *the series is* $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$*, which converges conditionally by Leibniz's Test. At* $x=\frac{5}{2}$ *the series is* $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$*, which diverges.*

8.4 Maclaurin Expansions to know by inspection

- $$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$
- The sine is an odd function:
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$
- The cosine is an even function:
$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$
- If a is a real constant,
$$(1+x)^a = 1+ax+\frac{a(a-1)}{2!}x^2+\frac{a(a-1)(a-2)}{3!}x^3+\frac{a(a-1)(a-2)(a-3)}{4!}x^4+\cdots$$

750 Example Expand $f(x)=\cos x$ around $x=1$.

Solution: *We have*

$$\begin{aligned}\cos x &= \cos(x-1+1)\\ &= \cos(x-1)\cos 1-\sin(x-1)\sin 1\\ &= (\cos 1)\left(1-\frac{(x-1)^2}{2!}+\frac{(x-1)^4}{4!}-\cdots\right)-(\sin 1)\left((x-1)-\frac{(x-1)^3}{3!}+\frac{(x-1)^5}{5!}-\cdots\right)\end{aligned}$$

Homework

751 Problem Given a finite collection of closed squares of total area 3, prove that they can be arranged to cover the unit square.

752 Problem Given a finite collection of closed squares of total area $\frac{1}{2}$, prove that they can be arranged to cover the unit square, with no overlaps

753 Problem $\sum_{n=3}^{\infty}\frac{(-2)^n}{7^n}$.

754 Problem Shew that

$$\sum_{n=1}^{\infty}\arctan\frac{1}{1+n+n^2}=\frac{\pi}{4}.$$

755 Problem $\sum_{n=3}^{\infty}\frac{18}{(9n-1)(9n+8)}$

756 Problem Test $\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$ for convergence by comparing it to a suitable p-series. Use the direct comparison test.

757 Problem Test $\sum_{n=2}^{\infty}\frac{1}{n^{1+1/\log n}}$ for convergence by comparing it to a suitable p-series. Use the direct comparison test.

758 Problem Test $\sum_{n=2}^{\infty}\frac{1}{n^{1+1/\log\log n}}$ for convergence by comparing it to a suitable p-series. Use the direct comparison test.

759 Problem Test $\sum_{n=1}^{\infty} \frac{3^n}{n^{2n}}$ using both direct comparison and the root test.

760 Problem Test $\sum_{n=1}^{\infty} \frac{-\cos n\pi}{8n-1}$ for absolute or conditional convergence. Make sure to write your argument carefully, outlining the different tests you use.

761 Problem Use the comparison tests to shew that if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

8.5 Comparison Tests

Homework

762 Problem Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers satisfying $0 < a_n < 1$ for all n. Assume that $\sum_{n=1}^{\infty} a_n$ diverges but $\sum_{n=1}^{\infty} a_n^2$ converges. Let f be a function defined on $\left[0\,;1\right]$ whose second derivative exists and is bounded on $\left[0\,;1\right]$. Prove that if $\sum_{n=1}^{\infty} f(a_n)$ converges, so does $\sum_{n=1}^{\infty} \left|f(a_n)\right|$.

8.6 Taylor Polynomials

Homework

763 Problem Evaluate $\int_0^1 (\log x)(\log(1-x))\,\mathrm{d}x$.

764 Problem Evaluate the infinite series $\sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$.

765 Problem Find the sum of the infinite series

$$1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \cdots.$$

8.7 Abel's Theorem

Homework

766 Problem Put

$$a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{n} - \log 2.$$

Prove that $\sum_{n=1}^{\infty} a_n$ converges and find its sum.

767 Problem Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{(1}{n}}{n(n+1)}.$$

768 Problem Evaluate the sum

$$\sum_{n=0}^{\infty} \left(\frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2} \right).$$

769 Problem Evaluate the limit

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \cdot \int_0^{\pi} \tan(\alpha \sin x)\,\mathrm{d}x.$$

Appendix A

Answers and Hints

11 Observe that $A_n = \{0, n, 2n, 3n, \ldots\}$.

1. A_6.
2. $\mathbb{N}$.
3. $\{0\}$.

14 We have,

$$\begin{aligned} x \in (A \cup B) \cap C &\iff x \in (A \cup B) \text{ and } x \in C \\ &\iff (x \in A \text{ or } x \in B) \text{ and } x \in C \\ &\iff (x \in A \text{ and } x \in C) \text{ or } (x \in B \text{ and } x \in C) \\ &\iff (x \in A \cap C) \text{ or } (x \in B \cap C) \\ &\iff x \in (A \cap C) \cup (B \cap C), \end{aligned}$$

which establishes the equality.

23 We check the two statements

$$x \in A \times (B \setminus C) \iff x \in (A \times B) \setminus (A \times C).$$

Let us prove first $\Longrightarrow$. By definition of $\times$, $x = (a, b)$, where $a \in A, b \in B, b \notin C$. Thus $x \in A \times B$ but $x \notin A \times C$. By definition of $\setminus$ we are done. Now we prove the assertion $\Longleftarrow$. By definition of $\times$ and $\setminus$, $x = (a, b)$ where $a \in A, b \in B$. Since $x \notin A \times C$, we observe that $b \notin C$. Thus $a \in A, b \in B \setminus C$, and we gather that $x \in A \times (B \setminus C)$.

24 Attach a binary code to each element of the subset, 1 if the element is in the subset and 0 if the element is not in the subset. The total number of subsets is the total number of such binary codes, and there are 2^N in number.

42 There are $2^2 = 4$ such functions, namely:

❶ f_1 given by $f_1(a) = f_1(b) = c$. Observe that $\mathbf{Im}(f_1) = \{c\}$.

❷ f_2 given by $f_2(a) = f_2(b) = d$. Observe that $\mathbf{Im}(f_2) = \{d\}$.

❸ f_3 given by $f_3(a) = c, f_3(b) = d$. Observe that $\mathbf{Im}(f_3) = \{c, d\}$.

❹ f_4 given by $f_4(a) = d, f_4(b) = c$. Observe that $\mathbf{Im}(f_4) = \{c, d\}$.

43 Each of the n elements of A must be assigned an element of B, and hence there are $\underbrace{m \cdot m \cdots m}_{n \text{ factors}} = m^n$ possibilities, and thus m^n functions.If a function from A to B is injective then we must have $n \leq m$ in view of Theorem 30. If to different inputs we must assign different outputs then to the first element of A we may assign any of the m elements of B, to the second any of the $m-1$ remaining ones, to the third any of the $m-2$ remaining ones, etc., and so we have $m(m-1)\cdots(m-n+1)$ injective functions.

45 Rename the independent variable, say $h(1-s) = 2s$. Now, if $1 - s = 3x$ then $s = 1 - 3x$. Hence

$$h(3x) = h(1-s) = 2s = 2(1-3x) = 2 - 6x.$$

46 Put

$$p(x) = (1 - x^2 + x^4)^{2003} = a_0 + a_1 x + a_2 x^2 + \cdots + a_{8012} x^{8012}.$$

Then

❶ $a_0 = p(0) = (1 - 0^2 + 0^4)^{2003} = 1.$

❷ $a_0 + a_1 + a_2 + \cdots + a_{8012} = p(1) = (1 - 1^2 + 1^4)^{2003} = 1.$

❸

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 + \cdots - a_{8011} + a_{8012} &= p(-1) \\ &= (1 - (-1)^2 + (-1)^4)^{2003} \\ &= 1. \end{aligned}$$

❹ The required sum is $\dfrac{p(1) + p(-1)}{2} = 1$.

❺ The required sum is $\dfrac{p(1) - p(-1)}{2} = 0$.

47 We have

$$\begin{aligned}\left[f\left(\frac{27+y^3}{y^3}\right)\right]^{\sqrt{\frac{27}{y}}} &= \left[f\left(\left(\frac{3}{y}\right)^3+1\right)\right]^{3\sqrt{\frac{3}{y}}}\\ &= \left(\left[f\left(\left(\frac{3}{y}\right)^3+1\right)\right]^{\sqrt{\frac{3}{y}}}\right)^3\\ &= 5^3\\ &= 125.\end{aligned}$$

48 We have

$$\begin{aligned}f(2) &= (-1)^2 1-2f(1) &&= 1-2f(1)\\ f(3) &= (-1)^3 2-2f(2) &&= -2-2f(2)\\ f(4) &= (-1)^4 3-2f(3) &&= 3-2f(3)\\ f(5) &= (-1)^5 4-2f(4) &&= -4-2f(4)\\ &\vdots &&\vdots\\ f(999) &= (-1)^{999} 998-2f(998) &&= -998-2f(998)\\ f(1000) &= (-1)^{1000} 999-2f(999) &&= 999-2f(999)\\ f(1001) &= (-1)^{1001} 1000-2f(1000) &&= -1000-2f(1000)\end{aligned}$$

Adding columnwise,

$$f(2)+f(3)+\cdots+f(1001) = 1-2+3-\cdots+999-1000-2(f(1)+f(2)+\cdot+f(1000)).$$

This gives

$$2f(1)+3(f(2)+f(3)+\cdots+f(1000))+f(1001) = -500.$$

Since $f(1)=f(1001)$ we have $2f(1)+f(1001)=3f(1)$. Therefore

$$f(1)+f(2)+\cdots+f(1000) = -\frac{500}{3}.$$

49 Set $a=b=0$. Then $(f(0))^2 = f(0)f(0) = f(0+0) = f(0)$. This gives $f(0)^2 = f(0)$. Since $f(0)>0$ we can divide both sides of this equality to get $f(0)=1$.

Further, set $b=-a$. Then $f(a)f(-a) = f(a-a) = f(0) = 1$. Since $f(a)\neq 0$, may divide by $f(a)$ to obtain $f(-a) = \frac{1}{f(a)}$.

Finally taking $a=b$ we obtain $(f(a))^2 = f(a)f(a) = f(a+a) = f(2a)$. Hence $f(2a) = (f(a))^2$

50 To prove that f is injective, we prove that $f(a)=f(b) \implies a=b$. We have

$$\begin{aligned}f(a)=f(b) &\implies \frac{a-1}{a+1} = \frac{b-1}{b+1}\\ &\implies (a-1)(b+1) = (a+1)(b-1)\\ &\implies ab+a-b-1 = ab-a+b-1\\ &\implies 2a=2b\\ &\implies a=b,\end{aligned}$$

whence f is injective. To prove that f is surjective we must prove that any $y\in\mathbb{R}\setminus\{1\}$ has a pre-image $a\in\mathbb{R}\setminus\{-1\}$ such that $f(a)=y$. That is,

$$\frac{a-1}{a+1} = y \implies a-1 = ya+y \implies a-ya = 1+y \implies a(1-y) = 1+y \implies a = \frac{1+y}{1-y}.$$

Thus $f\left(\frac{1+y}{1-y}\right) = y$, and f is surjective. This also serves to prove that $f^{-1}(x) = \frac{1+x}{1-x}$.

51 We have $f^{[2]}(x) = f(x+1) = (x+1)+1 = x+2$, $f^{[3]}(x) = f(x+2) = (x+2)+1 = x+3$ and so, recursively, $f^{[n]}(x) = x+n$.

55 We have $f^{[2]}(x) = f(2x) = 2^2x$, $f^{[3]}(x) = f(2^2x) = 2^3x$ and so, recursively, $f^{[n]}(x) = 2^n x$.

56 Let $y=0$. Then $2g(x) = 2x^2$, that is, $g(x)=x^2$. Let us check that $g(x)=x^2$ works. We have

$$g(x+y)+g(x-y) = (x+y)^2+(x-y)^2 = x^2+2xy+y^2+x^2-2xy+y^2 = 2x^2+2y^2,$$

which is the functional equation given. Our choice of g works.

57 Let $x=1$. Then $f(y) = yf(1)$. Since $f(1)$ is a constant, we may let $k=f(1)$. So all the functions satisfying the above equation satisfy $f(y)=ky$.

58 From $f(x)+2f(\frac{1}{x}) = x$ we obtain $f(\frac{1}{x}) = \frac{x}{2}-\frac{1}{2}f(x)$. Also, substituting $1/x$ for x on the original equation we get

$$f(1/x)+2f(x) = 1/x.$$

Hence

$$f(x) = \frac{1}{2x}-\frac{1}{2}f(1/x) = \frac{1}{2x}-\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}f(x)\right),$$

which yields $f(x) = \frac{2}{3x}-\frac{x}{3}$.

59 We have

$$(f(x))^2 \cdot f\left(\frac{1-x}{1+x}\right) = 64x,$$

whence

$$(f(x))^4 \cdot \left(f\left(\frac{1-x}{1+x}\right)\right)^2 = 64^2x^2 \qquad (I)$$

Substitute x by $\frac{1-x}{1+x}$. Then

$$f\left(\frac{1-x}{1+x}\right)^2 f(x) = 64\left(\frac{1-x}{1+x}\right). \qquad (II)$$

Divide (I) by (II),

$$f(x)^3 = 64x^2\left(\frac{1+x}{1-x}\right),$$

from where the result follows.

60 We have (i) $f^{[2]}(x) = (f\circ f)(x) = f(f(x)) = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$.

(ii) $f^{[3]}(x) = (f\circ f\circ f)(x) = f(f^{[2]}(x))) = f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = x$.

(iii) Notice that $f^{[4]}(x) = (f\circ f^{[3]})(x) = f(f^{[3]}(x)) = f(x) = f^{[1]}(x)$. We see that f is cyclic of period 3, that is, $f^{[1]} = f^{[14]} = f^{[7]} = \ldots, f^{[2]} = f^{[5]} = f^{[8]} = \ldots, f^{[3]} = f^{[6]} = f^{[9]} = \ldots$. Hence $f^{[69]}(x) = f^{[3]}(x) = x$.

61 To see (i) observe that

$$f(a) = f(b) \implies g(f(a)) = g(f(b)) \implies a = b,$$

whence f is injective. (The first implication is clear, the second implication follows because $g\circ f$ is injective.)

To see (ii), given $y \in C$, $\exists x \in A$ such that $g(f(x)) = y$, since $g\circ f$ is surjective. But then, letting $a = f(x) \in B$ we have $g(a) = y$ and g is surjective.

69 The map $f : [0;1] \to [a;b]$ $f(x) = \frac{x-a}{b-a}$ is a bijection.

70 The map $f : \left]-\infty;+\infty\right[\to \left]0;+\infty\right[$ $f(x) = e^x$ is a bijection.

84 Both answers are "no." If $a = -b = \sqrt{2}$, which we will prove later on to be irrational, we have $a + b = 0$, rational, and $ab = -2$, also rational.

85 Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then $\omega^2 + \omega + 1 = 0$ and $\omega^3 = 1$. Then

$$x = a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b + c\omega),$$
$$y = u^3 + v^3 + w^3 - 3uvw = (u+v+w)(u+\omega v+\omega^2 w)(u+\omega^2 v+\omega w).$$

Then

$$(a+b+c)(u+v+w) = au+av+aw+bu+bv+bw+cu+cv+cw,$$

$$\begin{aligned}(a+\omega b+\omega^2 c)(u+\omega v+\omega^2 w) &= au+bw+cv \\ &\quad +\omega(av+bu+cw) \\ &\quad +\omega^2(aw+bv+cu),\end{aligned}$$

and

$$\begin{aligned}(a+\omega^2 b+\omega c)(u+\omega^2 v+\omega w) &= au+bw+cv \\ &\quad +\omega(aw+bv+cu) \\ &\quad +\omega^2(av+bu+cw).\end{aligned}$$

This proves that

$$\begin{aligned}xy &= (au+bw+cv)^3 + (aw+bv+cu)^3 + (av+bu+cw)^3 \\ &\quad -3(au+bw+cv)(aw+bv+cu)(av+bu+cw),\end{aligned}$$

which proves that S is closed under multiplication.

86 We have

$$\begin{aligned} x\circ y &= (x\circ y)\circ(x\circ y) \\ &= [y\circ(x\circ y)]\circ x \\ &= [(x\circ y)\circ x]\circ y \\ &= [(y\circ x)\circ x]\circ y \\ &= [(x\circ x)\circ y]\circ y \\ &= (y\circ y)\circ(x\circ x) \\ &= y\circ x,\end{aligned}$$

proving commutativity.

87 By (1.4)

$$x*y = ((x*y)*x)*x.$$

By (1.4) again

$$((x*y)*x)*x = ((x*y)*((x*y)*y))*x.$$

By (1.3)

$$((x*y)*((x*y)*y))*x = (y)*x = y*x,$$

which is what we wanted to prove.

To shew that the operation is not necessarily associative, specialise $\mathscr{S} = \mathbb{Z}$ and $x*y = -x-y$ (the opposite of x minus y). Then clearly in this case $*$ is commutative, and satisfies (1.3) and (1.4) but

$$0*(0*1) = 0*(-0-1) = 0*(-1) = -0-(-1) = 1,$$

and

$$(0*0)*1 = (-0-0)*1 = (0)*1 = -0-1 = -1,$$

evincing that the operation is not associative.

88 1. Clearly, if a, b are rational numbers,

$$|a| < 1, |b| < 1 \implies |ab| < 1 \implies -1 < ab < 1 \implies 1 + ab > 0,$$

whence the denominator never vanishes and since sums, multiplications and divisions of rational numbers are rational, $\frac{a+b}{1+ab}$ is also rational. We must prove now that $-1 < \frac{a+b}{1+ab} < 1$ for $(a, b) \in]-1; 1[^2$. We have

$$\begin{aligned} -1 < \frac{a+b}{1+ab} < 1 & \Leftrightarrow & -1 - ab < a + b < 1 + ab \\ & \Leftrightarrow & -1 - ab - a - b < 0 < 1 + ab - a - b \\ & \Leftrightarrow & -(a+1)(b+1) < 0 < (a-1)(b-1). \end{aligned}$$

Since $(a, b) \in]-1; 1[^2$, $(a+1)(b+1) > 0$ and so $-(a+1)(b+1) < 0$ giving the sinistral inequality. Similarly $a - 1 < 0$ and $b - 1 < 0$ give $(a-1)(b-1) > 0$, the dextral inequality. Since the steps are reversible, we have established that indeed $-1 < \frac{a+b}{1+ab} < 1$.

2. Since $a \otimes b = \frac{a+b}{1+ab} = \frac{b+a}{1+ba} = b \otimes a$, commutativity follows trivially. Now

$$\begin{aligned} a \otimes (b \otimes c) & = & a \left(\frac{b+c}{1+bc} \right) \\ & = & \frac{a + \left(\frac{b+c}{1+bc} \right)}{1 + a \left(\frac{b+c}{1+bc} \right)} \\ & = & \frac{a(1+bc) + b + c}{1 + bc + a(b+c)} = \frac{a+b+c+abc}{1+ab+bc+ca}. \end{aligned}$$

One the other hand,

$$\begin{aligned} (a \otimes b) \otimes c & = & \left(\frac{a+b}{1+ab} \right) c \\ & = & \frac{\left(\frac{a+b}{1+ab} \right) + c}{1 + \left(\frac{a+b}{1+ab} \right) c} \\ & = & \frac{(a+b) + c(1+ab)}{1 + ab + (a+b)c} \\ & = & \frac{a+b+c+abc}{1+ab+bc+ca}, \end{aligned}$$

whence $\otimes$ is associative.

3. If $a \otimes e = a$ then $\frac{a+e}{1+ae} = a$, which gives $a + e = a + ea^2$ or $e(a^2 - 1) = 0$. Since $a \neq \pm 1$, we must have $e = 0$.

4. If $a \otimes b = 0$, then $\frac{a+b}{1+ab} = 0$, which means that $b = -a$, that is, $a^{-1} = -a$.

89 We must shew that $\forall (a, b) \in G^2$ we have $ab = ba$. But

$$\begin{aligned} ab & = & e(ab)e \\ & = & (b^2)(ab)(a^2) \\ & = & b((ba)(ba))a \\ & = & b(ba)^2 a \\ & = & b(e)a \\ & = & ba, \end{aligned}$$

whence the result follows.

90 We have

$$\begin{aligned} (ab)^3 = a^3 b^3 & \implies & ab(ab)ab = a(a^2 b^2)b \\ & \implies & baba = a^2 b^2 \\ & \implies & (ba)^2 = a^2 b^2. \end{aligned}$$

Similarly

$$(ab)^5 = a^5 b^5 \implies (ba)^4 = a^4 b^4.$$

But we also have

$$(ba)^4 = ((ba)^2)^2 = (a^2 b^2)^2 = a^2 (b^2 a^2) b^2,$$

and so

$$a^2 (b^2 a^2) b^2 = (ba)^4 = a^4 b^4 \implies b^2 a^2 = a^2 b^2.$$

We have shewn that $\forall (a, b) \in G^2$

$$((ba)^2 = a^2 b^2) \quad \text{and} \quad (b^2 a^2 = a^2 b^2).$$

Hence

$$\begin{aligned} (ba)^2 = a^2 b^2 = b^2 a^2 & \implies & baba = b^2 a^2 \\ & \implies & ab = ba, \end{aligned}$$

proving that the group is abelian.

91 Since
$$(ab)^{i+2} = \underbrace{(ab)(ab)\cdots(ab)}_{i+2 \text{ times}} = a(ba)^{i+1}b,$$
multiplying by a^{-1} on the left and by b^{-1} on the right the equality
$$(ab)^{i+2} = a^{i+2}b^{i+2} \tag{A.1}$$
we obtain
$$(ba)^{i+1} = (a)^{i+1}(b)^{i+1}. \tag{A.2}$$
By hypothesis
$$(ab)^{i+1} = (a)^{i+1}(b)^{i+1}. \tag{A.3}$$
Hence (A.2) and (A.3) yield
$$(ab)^{i+1} = (ba)^{i+1}. \tag{A.4}$$
Similarly, from (A.3) we obtain
$$(ab)^{i} = (ba)^{i}, \tag{A.5}$$
from which
$$(ab)^{-i} = (ba)^{-i}. \tag{A.6}$$
Multiplying (A.4) and (A.6) together, we deduce
$$ab = ba,$$
which is what we wanted to shew.

102 The first two follow immediately from the Binomial Theorem, the first by putting $x = y = 1$ and then $x = -y = 1$. The third follows by adding the first two and dividing by 2. The fourth follows by subtracting the second from the first and then dividing by 2.

103 If $a = 10^3, b = 2$ then
$$1002004008016032 = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5 = \frac{a^6 - b^6}{a - b}.$$
This last expression factorises as
$$\begin{aligned}\frac{a^6 - b^6}{a - b} &= (a+b)(a^2 + ab + b^2)(a^2 - ab + b^2) \\ &= 1002 \cdot 1002004 \cdot 998004 \\ &= 4 \cdot 4 \cdot 1002 \cdot 250501 \cdot k,\end{aligned}$$
where $k < 250000$. Therefore $p = 250501$.

105 From the Binomial Theorem,
$$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3 \implies A^3 + B^3 = (A+B)^3 - 3AB(A+B).$$
Then
$$\begin{aligned}a^3 + b^3 + c^3 - 3abc &= (a+b)^3 + c^3 - 3ab(a+b) - 3abc \\ &= (a+b+c)^3 - 3(a+b)c(a+b+c) - 3ab(a+b+c) \\ &= (a+b+c)((a+b+c)^2 - 3ac - 3bc - 3ab) \\ &= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca).\end{aligned}$$

106
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1}.$$

107
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)}{k(k-1)} \cdot \frac{(n-2)!}{(k-2)!(n-k)!} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \binom{n-2}{k-2}.$$

108 We use the identity $k\binom{n}{k} = n\binom{n-1}{k-1}$. Then
$$\begin{aligned}\sum_{k=1}^{n} k\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1}p^k(1-p)^{n-k} \\ &= \sum_{k=0}^{n-1} n\binom{n-1}{k}p^{k+1}(1-p)^{n-1-k} \\ &= np\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k} \\ &= np(p+1-p)^{n-1} \\ &= np.\end{aligned}$$

109 We use the identity
$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$
Then
$$\begin{aligned}\sum_{k=2}^{n} k(k-1)\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2}p^k(1-p)^{n-k} \\ &= \sum_{k=0}^{n-2} n(n-1)\binom{n-2}{k}p^{k+2}(1-p)^{n-1-k} \\ &= n(n-1)p^2\sum_{k=0}^{n-2}\binom{n-1}{k}p^k(1-p)^{n-2-k} \\ &= n(n-1)p^2(p+1-p)^{n-2} \\ &= n(n-1)p^2.\end{aligned}$$

110 We use the identity

$$(k-np)^2 = k^2 - 2knp + n^2p^2 = k(k-1) + k(1-2np) + n^2p^2.$$

Then

$$\begin{aligned}
\sum_{k=0}^{n}(k-np)^2\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=0}^{n}(k(k-1)+k(1-2np) \\
&\quad +n^2p^2)\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n}k(k-1)\binom{n}{k}p^k(1-p)^{n-k} \\
&\quad +(1-2np)\sum_{k=0}^{n}k\binom{n}{k}p^k(1-p)^{n-k} \\
&\quad +n^2p^2\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k} \\
&= n(n-1)p^2 + np(1-2np) + n^2p^2 \\
&= np(1-p).
\end{aligned}$$

112 Observe that the number of k-tuples with $\min((a_1, a_2, \ldots, a_k)) = t$ is $(n-t+1)^k - (n-t)^k$.

151 The given equalities entail t

$$\sum_{k=1}^{n}(x_k^2 - x_k)^2 = 0.$$

A sum of squares is 0 if and only if every term is 0. This gives the result.

152 The given equality entails that

$$\frac{1}{2}\left((x_1-x_2)^2 + (x_2-x_3)^2 + \cdots + (x_{n-1}-xn)^2 + (xn-x_1)^2\right) = 0.$$

A sum of squares is 0 if and only if every term is 0. This gives the result.

153 Since $aB < Ab$ one has $a(b+B) = ab + aB < ab + Ab = (a+A)b$ so $\frac{a}{b} < \frac{a+A}{b+B}$. Similarly $B(a+A) = aB + AB < Ab + AB = A(b+B)$ and so $\frac{a+A}{b+B} < \frac{A}{B}$.

We have

$$\frac{7}{10} < \frac{11}{15} \implies \frac{7}{10} < \frac{18}{25} < \frac{11}{15} \implies \frac{7}{10} < \frac{25}{35} < \frac{18}{25} < \frac{11}{15}.$$

Since $\frac{25}{35} = \frac{5}{7}$, we have $q \le 7$. Could it be smaller? Observe that $\frac{5}{6} > \frac{11}{15}$ and that $\frac{4}{6} < \frac{7}{10}$. Thus by considering the cases with denominators $q = 1, 2, 3, 4, 5, 6$, we see that no such fraction lies in the desired interval. The smallest denominator is thus 7.

154 We have

$$(r-s+t)^2 - t^2 = (r-s+t-t)(r-s+t+t) = (r-s)(r-s+2t).$$

Since $t - s \le 0$, $r - s + 2t = r + s + 2(t-s) \le r + s$ and so

$$(r-s+t)^2 - t^2 \le (r-s)(r+s) = r^2 - s^2$$

which gives

$$(r-s+t)^2 \le r^2 - s^2 + t^2.$$

155 Using the CBS Inequality (Theorem 147) on $\sum_{k=1}^{n}(a_kb_k)c_k$ once we obtain

$$\sum_{k=1}^{n}a_kb_kc_k \le \left(\sum_{k=1}^{n}a_k^2b_k^2\right)^{1/2}\left(\sum_{k=1}^{n}c_k^2\right)^{1/2}.$$

Using CBS again on $\left(\sum_{k=1}^{n}a_k^2b_k^2\right)^{1/2}$ we obtain

$$\begin{aligned}
\sum_{k=1}^{n}a_kb_kc_k &\le \left(\sum_{k=1}^{n}a_k^2b_k^2\right)^{1/2}\left(\sum_{k=1}^{n}c_k^2\right)^{1/2} \\
&\le \left(\sum_{k=1}^{n}a_k^4\right)^{1/4}\left(\sum_{k=1}^{n}b_k^4\right)^{1/4}\left(\sum_{k=1}^{n}c_k^2\right)^{1/2},
\end{aligned}$$

which gives the required inequality.

156 This follows directly from the AM-GM Inequality applied to $1, 2, \ldots, n$:

$$n!^{1/n}(1\cdot 2\cdots n)^{1/n} < \frac{1+2+\cdots+n}{n} = \frac{n+1}{2},$$

where strict inequality follows since the factors are unequal for $n > 1$.

157 First observe that for integer k, $1 < k < n$, $\quad k(n-k+1) = k(n-k) + k > 1(n-k) + k = n$. Thus

$$n!^2 = (1\cdot n)(2\cdot(n-1))(3\cdot(n-2))\cdots((n-1)\cdot 2)(n\cdot 1) > n\cdot n\cdot n\cdots n = n^n.$$

158 From the Binomial Theorem, for $n \ge 2$,

$$2^n = (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} > \binom{n}{2} = \frac{n(n-1)}{2} \implies 2^{n+1} > n(n-1).$$

This establishes the inequality for $n \ge 2$. For $n = 0$, $0 = 0(0-1) < 2^{0+1}$ and for $n = 1$, $0 = 1(1-1) < 2^{1+1}$, so the inequality is true for all natural numbers.

159 Assume without loss of generality that $a \geq b \geq c$. Then $a \geq b \geq c$ is similarly sorted as itself, so by the Rearrangement Inequality

$$a^2 + b^2 + c^2 = aa + bb + cc \geq ab + bc + ca.$$

This also follows directly from the identity

$$a^2 + b^2 + c^2 - ab - bc - ca = \left(a - \frac{b+c}{2}\right)^2 + \frac{3}{4}(b-c)^2.$$

One can also use the AM-GM Inequality thrice:

$$a^2 + b^2 \geq 2ab; \quad b^2 + c^2 \geq 2bc; \quad c^2 + a^2 \geq 2ca,$$

and add.

160 Assume without loss of generality that $a \geq b \geq c$. Then $a \geq b \geq c$ is similarly sorted as $a^2 \geq b^2 \geq c^2$, so by the Rearrangement Inequality

$$a^3 + b^3 + c^3 = aa^2 + bb^2 + cc^2 \geq a^2 b + b^2 c + c^2 a,$$

and

$$a^3 + b^3 + c^3 = aa^2 + bb^2 + cc^2 \geq a^2 c + b^2 a + c^2 b.$$

Upon adding

$$a^3 + b^3 + c^3 = aa^2 + bb^2 + cc^2 \geq \frac{1}{2}\left(a^2(b+c) + b^2(c+a) + c^2(a+b)\right).$$

Again, if $a \geq b \geq c$ then

$$ab \geq ac \geq bc,$$

thus

$$a^3 + b^3 + c^3 = \geq a^2 b + b^2 c + c^2 a = (ab)a + (bc)b + (ac)c \geq (ab)c + (bc)a + (ac)b = 3abc.$$

This last inequality also follows directly from the AM-GM Inequality, as

$$(a^3 b^3 c^3)^{1/3} \leq \frac{a^3 + b^3 + c^3}{3},$$

or from the identity

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca),$$

and the inequality of problem 159.

161 We apply n times the Rearrangement Inequality

$$\begin{array}{lllll}
\breve{a}_1 \hat{b}_1 + \breve{a}_2 \hat{b}_2 + \cdots + \breve{a}_n \hat{b}_n & \leq & a_1 b_1 + a_2 b_2 + \cdots + a_n b_n & \leq & \hat{a}_1 \hat{b}_1 + \hat{a}_2 \hat{b}_2 + \cdots + \hat{a}_n \hat{b}_n \\
\breve{a}_1 \hat{b}_1 + \breve{a}_2 \hat{b}_2 + \cdots + \breve{a}_n \hat{b}_n & \leq & a_1 b_2 + a_2 b_3 + \cdots + a_n b_1 & \leq & \hat{a}_1 \hat{b}_1 + \hat{a}_2 \hat{b}_2 + \cdots + \hat{a}_n \hat{b}_n \\
\breve{a}_1 \hat{b}_1 + \breve{a}_2 \hat{b}_2 + \cdots + \breve{a}_n \hat{b}_n & \leq & a_1 b_3 + a_2 b_4 + \cdots + a_n b_2 & \leq & \hat{a}_1 \hat{b}_1 + \hat{a}_2 \hat{b}_2 + \cdots + \hat{a}_n \hat{b}_n \\
 & & \vdots & & \\
\breve{a}_1 \hat{b}_1 + \breve{a}_2 \hat{b}_2 + \cdots + \breve{a}_n \hat{b}_n & \leq & a_1 b_n + a_2 b_1 + \cdots + a_n b_{n-1} & \leq & \hat{a}_1 \hat{b}_1 + \hat{a}_2 \hat{b}_2 + \cdots + \hat{a}_n \hat{b}_n
\end{array}$$

Adding we obtain the desired inequalities.

163 Use the fact that $(b-a)^2 = (\sqrt{b} - \sqrt{a})^2(\sqrt{b} + \sqrt{a})^2$.

164 Let

$$A = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{9999}{10000}$$

and

$$B = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{10000}{10001}.$$

Clearly, $x^2 - 1 < x^2$ for all real numbers x. This implies that

$$\frac{x-1}{x} < \frac{x}{x+1}$$

whenever these four quantities are positive. Hence

$$\begin{array}{ccc}
1/2 & < & 2/3 \\
3/4 & < & 4/5 \\
5/6 & < & 6/7 \\
\vdots & \vdots & \vdots \\
9999/10000 & < & 10000/10001
\end{array}$$

As all the numbers involved are positive, we multiply both columns to obtain

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{9999}{10000} < \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{10000}{10001},$$

or $A < B$. This yields $A^2 = A \cdot A < A \cdot B$. Now

$$A \cdot B = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{5}{6} \cdot \frac{6}{7} \cdot \frac{7}{8} \cdots \frac{9999}{10000} \cdot \frac{10000}{10001} = \frac{1}{10001},$$

and consequently, $A^2 < A \cdot B = 1/10001$. We deduce that $A < 1/\sqrt{10001} < 1/100$.

165 Observe that for $k \geq 1$, $(x+k)^2 > (x+k)(x+k-1)$ and so

$$\frac{1}{(x+k)^2} < \frac{1}{(x+k)(x+k-1)} = \frac{1}{x+k-1} - \frac{1}{x+k}.$$

Hence

$$\begin{array}{lll}
\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \cdots + \frac{1}{(x+n-1)^2} + \frac{1}{(x+n)^2} & < & \frac{1}{x(x+1)} + \frac{1}{(x+1)(x+2)} + \frac{1}{(x+2)((x+3))} + \cdots + \frac{1}{(x+n-2)(x+n-1)} + \frac{1}{(x+n-1)(x+n)} \\
 & = & \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+2} - \frac{1}{x+3} + \cdots + \frac{1}{x+n-2} - \frac{1}{x+n-1} + \frac{1}{x+n-1} - \frac{1}{x+n} \\
 & = & \frac{1}{x} - \frac{1}{x+n}.
\end{array}$$

166 For $1 \le i \le n$, we have

$$\left|\frac{2}{i}-1-\frac{1}{n}\right| \le 1-\frac{1}{n} \iff \left(\frac{2}{i}-\left(1+\frac{1}{n}\right)\right)^2 \le \left(1-\frac{1}{n}\right)^2 \iff \frac{4}{i^2}-\frac{4}{i}\left(1+\frac{1}{n}\right)+\frac{4}{n} \le 0 \iff \frac{(i-n)(i-1)}{i^2 n} \le 0.$$

Thus

$$\left|\sum_{i=1}^{n}\frac{x_i}{i}\right| = \frac{1}{2}\left|\sum_{i=1}^{n}\left(\frac{2}{i}-\left(1+\frac{1}{n}\right)\right)x_i\right|,$$

as $\sum_{i=1} x_i = 0$. Now

$$\left|\sum_{i=1}^{n}\left(\frac{2}{i}-\left(1+\frac{1}{n}\right)\right)x_i\right| \le \sum_{i=1}^{n}\left|\frac{2}{i}-1-\frac{1}{n}\right||x_i| \le \left(1-\frac{1}{n}\right)\sum_{i=1}^{n}|x_i| = \left(1-\frac{1}{n}\right).$$

167 Expanding the product

$$\prod_{k=1}^{n}(1+x_k) = 1+\sum_{k=1}^{n}x_k+\sum_{1\le i<j\le n}^{n}x_ix_j+\cdots \ge 1+\sum_{k=1}^{n}x_k,$$

since the $x_k \ge 0$. When $n=1$ equality is obvious. When $n>1$ equality is achieved when $\sum_{1\le i<j\le n}^{n} x_ix_j = 0$.

168 Assume $a \ge b \ge c$. Put $s = a+b+c$. Then

$$-a \le -b \le -c \implies s-a \le s-b \le s-c \implies \frac{1}{s-a} \ge \frac{1}{s-b} \ge \frac{1}{s-c}$$

and so the sequences a, b, c and $\frac{1}{s-a}, \frac{1}{s-b}, \frac{1}{s-c}$ are similarly sorted. Using the Rearrangement Inequality twice:

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c} \ge \frac{a}{s-c}+\frac{b}{s-a}+\frac{c}{s-b}; \qquad \frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c} \ge \frac{a}{s-b}+\frac{b}{s-c}+\frac{c}{s-a}.$$

Adding these two inequalities

$$2\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) \ge \frac{b+c}{s-a}+\frac{c+a}{s-b}+\frac{c+a}{s-c},$$

whence

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \ge 3,$$

from where the result follows.

169 Let

$$P(n):\ \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}}_{n \text{ radicands}} < \frac{1+\sqrt{4a+1}}{2}.$$

Let us prove $P(1)$, that is

$$\forall a>0,\quad \sqrt{a} < \frac{1+\sqrt{4a+1}}{2}.$$

To get this one, let's work backwards. If $a > \frac{1}{4}$

$$\begin{aligned}
\sqrt{a} < \frac{1+\sqrt{4a+1}}{2} &\iff 2\sqrt{a} < 1+\sqrt{4a+1}\\
&\iff 2\sqrt{a}-1 < \sqrt{4a+1}\\
&\iff (2\sqrt{a}-1)^2 < (\sqrt{4a+1})^2\\
&\iff 4a-4\sqrt{a}+1 < 4a+1\\
&\iff -2\sqrt{a} < 0.
\end{aligned}$$

all the steps are reversible and the last inequality is always true. If $a \le \frac{1}{4}$ then trivially $2\sqrt{a}-1 < \sqrt{4a+1}$. Thus $P(1)$ is true. Assume now that $P(n)$ is true and let's derive $P(n+1)$. From

$$\underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}}_{n \text{ radicands}} < \frac{1+\sqrt{4a+1}}{2} \implies \underbrace{\sqrt{a+\sqrt{a+\sqrt{a+\cdots+\sqrt{a}}}}}_{n+1 \text{ radicands}} < \sqrt{a+\frac{1+\sqrt{4a+1}}{2}}.$$

we see that it is enough to shew that

$$\sqrt{a+\frac{1+\sqrt{4a+1}}{2}} = \frac{1+\sqrt{4a+1}}{2}.$$

But observe that

$$(\sqrt{4a+1}+1)^2 = 4a+2\sqrt{4a+1}+2 \qquad \implies \qquad \frac{1+\sqrt{4a+1}}{2} = \sqrt{a+\frac{1+\sqrt{4a+1}}{2}},$$

proving the claim.

170 From the AM-GM Inequality,

$$a+b \ge 2\sqrt{ab}; \quad b+c \ge 2\sqrt{bc}; c+a \ge 2\sqrt{ca},$$

and the desired inequality follows upon multiplication of these three inequalities.

171 By the Rearrangement inequality

$$\sum_{k=1}^{n}\frac{a_k}{k^2} \ge \sum_{k=1}^{n}\frac{\breve{a}_k}{k^2} \ge \sum_{k=1}^{n}\frac{1}{k},$$

as $\breve{a}_k \ge k$, the a's being pairwise distinct positive integers.

172 By the AM-GM Inequality,

$$\left(\frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_n}\right)^{1/n} \leq \frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n},$$

whence the inequality.

173 By the CBS Inequality,

$$\left(1\cdot x_1+1\cdot x_2+\cdots+1\cdot x_n\right)^2 \leq \left(1^2+1^2+\cdots+1^2\right)\left(x_1^2+x_2^2+\cdots+x_n^2\right),$$

which gives the desired inequality.

174 Put

$$T_m = \sum_{1\leq k\leq m} a_k - \sum_{m<k\leq n} a_k.$$

Clearly $T_0 = -T_n$. Since the sequence $T_0, T_1, \ldots, T_n$ changes signs, choose an index p such that T_{p-1} and T_p have different signs. Thus either $T_{p-1} - T_p = 2|a_p|$ or $T_p - T_{p-1} = 2|a_p|$. We claim that

$$\min\left(\left|T_{p-1}\right|,\left|T_p\right|\right) = \leq \max_{1\leq k\leq n}\left|a_k\right|.$$

For

For, if contrariwise both $\left|T_{p-1}\right| > \max_{1\leq k\leq n}\left|a_k\right|$ and $\left|T_p\right| > \max_{1\leq k\leq n}\left|a_k\right|$, then $2|a_p| = |T_{p-1} - T_p| > 2\max_{1\leq k\leq n}\left|a_k\right|$, a contradiction.

175 It is enough to prove this in the case when a, b, c, d are all positive. To this end, put $O = (0,0)$, $L = (a,b)$ and $M = (a+c, b+d)$. By the triangle inequality $OM \leq OL + LM$, where equality occurs if and only if the points are collinear. But then

$$\sqrt{(a+c)^2+(b+d)^2} = OM \leq OL + LM = \sqrt{a^2+b^2}+\sqrt{c^2+d^2},$$

and equality occurs if and only if the points are collinear, that is $\frac{a}{b} = \frac{c}{d}$.

180 Use Minkowski's Inequality and the fact that $17^2 + 144^2 = 145^2$. The desired value is S_{12}.

199 We have

$$\begin{aligned}
\sum_{1\leq i<j\leq n}(x_j - x_i) &= \sum_{1\leq i<j\leq n} x_j - \sum_{1\leq i<j\leq n} x_i \\
&= \sum_{j=2}^{n}(j-1)x_j - \sum_{i=1}^{n-1}(n-1)x_i \\
&= -(n-1)x_1 + \sum_{k=2}^{n-1}((k-1)-(n-k))x_k + (n-1)x_n \\
&= -(n-1)x_1 - (n-3)x_2 - \cdots + (n-3)x_{n-1} + (n-1)x_n.
\end{aligned}$$

This sum is maximal when the negative coefficients of the x_i are 0 and the positive coefficients of the x_i are equal to 1. If n is even the maximum is

$$1+3+\cdots+(n-1).$$

If n is odd, the maximum coefficient is

$$2+4+\cdots+(n-1).$$

The result follows thus.

200 We claim that $3\lfloor 2t\rfloor - 2\lfloor 3t\rfloor = 0, \pm 1$ or -2. We can then take

$$P(x,y) = (3x-2y)(3x-2y-1)(3x-2y+1)(3x-2y+2).$$

In order to prove the claim, we observe that $\lfloor x\rfloor$ has unit period, so it is enough to prove the claim for $t \in [0,1)$. We divide $[0;1[$ as

$$[0,1[= [0;1/3[\cup[1/3;1/2[\cup[1/2;2/3[\cup[2/3;1[.$$

If $t \in [0,1/3[$, then both $\lfloor 2t\rfloor$ and $\lfloor 3t\rfloor$ are $=0$, and so $3\lfloor 2t\rfloor - 2\lfloor 3t\rfloor = 0.$ If $t \in [1/3;1/2[$ then $\lfloor 3t\rfloor = 1$ and $\lfloor 2t\rfloor = 0,$ and so $3\lfloor 2t\rfloor - 2\lfloor 3t\rfloor = -2$. If $t \in [1/2;2/3[,$ then $\lfloor 2t\rfloor = 1, \lfloor 3t\rfloor = 1$, and so $3\lfloor 2t\rfloor - 2\lfloor 3t\rfloor = 1.$ If $t \in [2/3;1[$, then $\lfloor 2t\rfloor = 1, \lfloor 3t\rfloor = 2,$ and $3\lfloor 2t\rfloor - 2\lfloor 3t\rfloor = -1.$

201 By the Binomial Theorem

$$(1+\sqrt{2})^n + (1-\sqrt{2})^n = 2\sum_{0\leq k\leq n/2}(2)^k\binom{n}{2k} := 2N,$$

an even integer. Since $-1 < 1-\sqrt{2} < 0$, it must be the case that $(1-\sqrt{2})^n$ is the fractional part of $(1+\sqrt{2})^n$ or $(1+\sqrt{2})^n + 1$ depending on whether n is odd or even, respectively. Thus for odd n, $(1+\sqrt{2})^n - 1 < (1+\sqrt{2})^n + (1-\sqrt{2})^n < (1+\sqrt{2})^n$, whence $(1+\sqrt{2})^n + (1-\sqrt{2})^n = \lfloor(1+\sqrt{2})^n\rfloor$, always even, and for n even $2N := (1+\sqrt{2})^n + (1-\sqrt{2})^n = \lfloor(1+\sqrt{2})^n\rfloor + 1$, and so $\lfloor(1+\sqrt{2})^n\rfloor = 2N-1,$ always odd for even $n.$

770 Example Prove that the first thousand digits after the decimal point in

$$(6+\sqrt{35})^{1980}$$

are all 9's.

Solution: Reasoning as in the preceding problem,

$$(6+\sqrt{35})^{1980} + (6-\sqrt{35})^{1980} = 2k,$$

an even integer. But $0 < 6-\sqrt{35} < 1/10$, (for if $\frac{1}{10} < 6-\sqrt{35}$, upon squaring $3500 < 3481$, which is clearly nonsense), and hence $0 < (6-\sqrt{35})^{1980} < 10^{-1980}$ which yields

$$2k - 1 + \underbrace{0.9\ldots9}_{1979\text{ nines}} = 2k - \frac{1}{10^{1980}} < (6+\sqrt{35})^{1980} < 2k,$$

This proves the assertion of the problem.

203 By squaring, it is easy to see that

$$\sqrt{4n+1} < \sqrt{n}+\sqrt{n+1} < \sqrt{4n+3}.$$

Neither $4n+2$ nor $4n+3$ are squares since squares are either congruent to 0 or 1 mod 4, so

$$\lfloor\sqrt{4n+2}\rfloor = \lfloor\sqrt{4n+3}\rfloor,$$

and the result follows.

204 Let T_n be the n-th non-square. There is a natural number m such that $m^2 < T_n < (m+1)^2$. As there are m squares less than T_n and n non-squares up to T_n, we see that $T_n = n+m$. We have then $m^2 < n+m < (m+1)^2$ or $m^2 - m < n < m^2 + m + 1$. Since $n, m^2 - m, m^2 + m + 1$ are all integers, these inequalities imply $m^2 - m + \frac{1}{4} < n < m^2 + m + \frac{1}{4}$, that is to say, $(m - 1/2)^2 < n < (m+1/2)^2$. But then $m = \lfloor\!\lfloor \sqrt{n} + \frac{1}{2} \rfloor\!\rfloor$. Thus the n-th non-square is $T_n = n + \lfloor\!\lfloor \sqrt{n} + 1/2 \rfloor\!\rfloor$.

205 Assume on the contrary that

$$\frac{(a+2b)^2}{(a+b)^2} \geq 2 \implies a^2 + 4ab + 4b^2 \geq 2(a^2 + 2ab + b^2) \implies 2b^2 \geq a^2 \implies \frac{a^2}{b^2} \geq 2,$$

a contradiction. By adding,

$$\frac{a^2}{b^2} < 2, \quad \frac{(a+2b)^2}{(a+b)^2} < 2 \implies \frac{a^2}{b^2} + \frac{(a+2b)^2}{(a+b)^2} < 4 \implies \frac{(a+2b)^2}{(a+b)^2} - 2 < 2 - \frac{a^2}{b^2}.$$

206 It needs to be proved that

$$\left|\frac{2x+5}{x+2} - \sqrt{5}\right| < \left|x - \sqrt{5}\right|.$$

207 Consider the set $E = \{x : x > 0, x^n < a\}$. Shew that E is bounded above with supremum $b = \sup E$. Then shew that $b^n = a$ by arguing by contradiction first against $b^n < a$ and then against $b^n > a$. In the first case it may be advantageous to prove $\left(b + \frac{a - b^n}{N}\right)^n < a$ for N large enough and use the Binomial Theorem to establish the inequality. In the second case consider $b^n \left(1 + \frac{b^n}{Ma}\right)^{-n} > a$, for integral M sufficiently large, again using the Binomial Theorem to establish the inequality.

216 $\left[500\,;501\right]$.

217 $\left[1\,;2\right]$.

218 $\mathbb{R}$.

219 $\{1\}$.

220 $\varnothing$.

221 $\varnothing$.

223 Closure is immediate. Most of the other axioms are inherited from the larger set $\mathbb{R}$. Observe $0_F = 0$, $1_F = 1$ and the multiplicative inverse of $a + \sqrt{2}b, (a,b) \neq (0,0)$ is

$$(a+\sqrt{2}b)^{-1} = \frac{1}{a+\sqrt{2}b} = \frac{a-\sqrt{2}b}{a^2-2b^2} = \frac{a}{a^2-2b^2} - \frac{\sqrt{2}b}{a^2-2b^2}.$$

Here $a^2 - 2b^2 \neq 0$ since $\sqrt{2}$ is irrational.

224 Assume $(a,b) \in \mathbb{R}^2$ with $a < b$. If $ab < 0$, then $0 \in D$ is between a and b. If $0 < a < b$ then $\sqrt{a} < \sqrt{b}$, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number r such that $\sqrt{a} < r < \sqrt{b} \implies a < r^2 < b$. If $a < b < 0$, then $\sqrt{-b} < \sqrt{-a}$, and since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number s such that $\sqrt{-b} < s < \sqrt{-a} \implies -b < s^2 < -a \implies a < -s^2 < b$.

225 Assume $(a,b) \in \mathbb{R}^2$ with $a < b$. There is a strictly positive integer n such that $n > \frac{1}{b-a}$. Thus

$$0 < \frac{1}{2^n} < \frac{1}{n} < b - a.$$

Put $m = \lfloor\!\lfloor 2^n a \rfloor\!\rfloor + 1$, and so by definition $m - 1 \leq 2^n x < m$. Hence

$$a < \frac{m}{2^n} \leq a + \frac{1}{2^n} < a + \frac{1}{n} < a + b - a = a.$$

261 For the proof of this let G be such a set (so that $x + y$ is in G if x, y are, and G is closed), and suppose that we are not in cases (i) or (ii). Then it is enough to show that G contains arbitrarily small positive numbers, for then multiples of these will be dense in $\mathbb{R}$, but G being closed forces $G = \mathbb{R}$. To achieve this let $\mathscr{I} = \inf\{x : x \in G, x > 0\}$. If $\mathscr{I} = 0$ we are done; but if $\mathscr{I} > 0$ there cannot be numbers $x \in G$ arbitrarily close to and greater than $\mathscr{I}$, for then $x - \mathscr{I}$ would run through small positive members of G, in particular smaller than $\mathscr{I}$, contradicting its definition. This means that $\mathscr{I}$ belongs itself to G, and from there it is easy to see that we are in case (ii) contrary to the assumption. Hence indeed $\mathscr{I} = 0$, $G = \mathbb{R}$.

291 No. Take $a_n = \frac{1}{n}$. Then $a_n > 0$ always, but $L = 0$.

299 We have for $n > 1$,

$$\frac{n^2}{n^2+n} = \underbrace{\frac{n}{n^2+n} + \cdots + \frac{n}{n^2+n}}_{n \text{ times}} < \sum_{i=1}^{n} \frac{n}{n^2+i} < \underbrace{\frac{n}{n^2+1} + \cdots + \frac{n}{n^2+1}}_{n \text{ times}} = \frac{n^2}{n^2+1},$$

and the result follows by the Sandwich Theorem since each of the sequences on the extremes converges to 1.

300 Evidently $n! \leq n^n$. By problem 157, if $n > 2$ then $n^{n/2} \leq n!$. Thus

$$\frac{1}{n} \leq \frac{1}{(n!)^{1/n}} \leq \frac{1}{n^{1/2}}$$

and the result follows by the Sandwich Theorem.

301 For $n \geq 2$ we have

$$\frac{2^n}{n!} = \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdots \frac{2}{n} \leq 2 \cdot 1 \cdot 1 \cdots 1 \cdot \frac{2}{n} = \frac{4}{n} \to 0.$$

302 There is a positive integer m with $m^2 \leq n < (m+1)^2$. Consider

$$\left|\frac{s_{m^2}}{m^2} - \frac{s_n}{n}\right|.$$

303 Since $-1 \le \sin n \le 1$, any possible limit must be finite. By way of contradiction assume that $\sin n \to a$ as $n \to +\infty$. Then

$$\lim_{n\to+\infty} \sin n = a \implies \lim_{n\to+\infty} \sin(n+2) = a,$$

whence

$$\lim_{n\to+\infty} (\sin(n+2) - \sin n) = a - a = 0.$$

Now,

$$\sin(n+2) - \sin n = 2(\sin 1)\cos(n+1) \implies \cos(n+1) \to 0, \quad \text{as } n \to +\infty.$$

From

$$\cos(n+1) = \cos n \cos 1 - \sin n \sin 1$$

we obtain

$$\sin n = \frac{1}{\sin 1}(\cos n \cos 1 - \cos(n+1)) \to \frac{1}{\sin 1}(0 \cdot \cos 1 - 0) = 0,$$

and so $a = 0$. But then

$$1 = \sin^2 n + \cos^2 n \to 0^2 + 0^2 = 0,$$

a contradiction.

304 By problem 157, $(n!)^{1/n} > \sqrt{n}$ for $n \ge 3$. Hence, for all $M > 0$, as long as $n > M^2$ we will have

$$(n!)^{1/n} > \sqrt{n} > M,$$

giving the result.

306 We have

$$\sqrt{n+1} - \sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}.$$

Hence, as long as $\frac{1}{2\sqrt{n}} < \varepsilon$ that is, as long as $n > \frac{1}{4\varepsilon^2}$ we will have

$$\left|\sqrt{n+1} - \sqrt{n}\right| < \frac{1}{2\sqrt{n}} < \varepsilon.$$

307 Write

$$\sum_{n=1}^{2^M} \frac{1}{n} = \sum_{m=1}^{M} \sum_{n=2^{m-1}+1}^{2^m} \frac{1}{n}.$$

Since $1/n \ge 1/N$ when $n \le N$, we gather that

$$\sum_{n=2^{m-1}+1}^{2^m} \frac{1}{n} \ge \sum_{n=2^{m-1}+1}^{2^m} 2^{-m} = (2^m - 2^{m-1})2^{-m} = \frac{1}{2}.$$

Thus

$$\sum_{n=1}^{2^M} \frac{1}{n} \ge \frac{M}{2}$$

and the sequence can be made arbitrarily large.

308 Observe that for $n \ge 2$,

$$\begin{aligned} & \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n-1})} - \frac{\sqrt{(n)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n})} \\ &= \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n-1})}\left(1 - \frac{\sqrt{n}}{1+\sqrt{n}}\right) \\ &= \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n})}. \end{aligned}$$

Therefore

$$\sum_{n=1}^{K} \frac{\sqrt{(n-1)!}}{(1+\sqrt{1})(1+\sqrt{2})(1+\sqrt{3})\cdots(1+\sqrt{n})} = 1 - \frac{\sqrt{K!}}{(1+\sqrt{1})(1+\sqrt{2})\cdots(1+\sqrt{K})}.$$

Now prove that $u_K = \frac{\sqrt{K!}}{(1+\sqrt{1})(1+\sqrt{2})\cdots(1+\sqrt{K})}$ decreases to 0.

309 Put $x_1 = 1, \quad x_{n+1} = \sqrt{1+x_n}, n \ge 0$. We claim that the sequence $\{x_n\}_{n=1}^{+\infty}$ is increasing and bounded above. By Theorem 284 the sequence must have a limit L. To prove that the sequence is increasing consider $x_{n+1} - x_n$ (fill in this gap). To prove that the sequence is bounded, we claim that for all $n \ge 1$, $x_n < 4$. For this is clearly true for $n = 1$. So assume that $x_n < 4$. Then

$$x_{n+1} = \sqrt{1+x_n} < \sqrt{1+4} = \sqrt{5} < 4,$$

and so the assertion follows by induction.

Since we have shewn that L exists we now may compute

$$L = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} \sqrt{1+x_n} = \sqrt{1+L} \implies L = \sqrt{1+L} \implies L^2 - L - 1 = 0 \implies L = \frac{1+\sqrt{5}}{2},$$

where we have chosen the positive root as the sequence is clearly strictly positive.

310 By Theorem 100, $1 + 2 + \cdots + n = \frac{n^2+n}{2}$, and the desired result follows.

311 $\frac{1}{3}$; 1; $\frac{1}{4}$.

312 Put $x_1 = 1, \quad x_{n+1} = \frac{1}{1+x_n}, n \geq 0$. We claim that the sequence $\{x_n\}_{n=1}^{+\infty}$ is increasing and bounded above. By Theorem 284 the sequence must have a limit L. To prove that the sequence is increasing consider $x_{n+1} - x_n$ (fill in this gap). To prove that the sequence is bounded, we claim that for all $n \geq 1$, prove by induction that $x_n < 4$ (fill in this gap).

Since we have shewn that L exists we now may compute

$$L = \lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} \frac{1}{1+x_n} = \frac{1}{1+L} \implies L = \frac{1}{1+L} \implies L^2 + L - 1 = 0 \implies L = \frac{\sqrt{5}-1}{2},$$

where we have chosen the positive root as the sequence is clearly strictly positive.

314 Assume that $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{+\infty}$ is increasing. Then

$$\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq \cdots \leq \frac{a_n}{b_n} \leq \frac{a_{n+1}}{b_{n+1}}.$$

Using Theorem 139,

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \frac{a_n}{b_n} \leq \frac{a_{n+1}}{b_{n+1}} \implies \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \frac{a_1 + a_2 + \cdots + a_{n+1}}{b_1 + b_2 + \cdots + b_{n+1}} \leq \frac{a_{n+1}}{b_{n+1}},$$

proving that $\left\{\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n}\right\}_{n=1}^{+\infty}$ is also increasing. If $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{+\infty}$ were decreasing, $\left\{-\frac{a_n}{b_n}\right\}_{n=1}^{+\infty}$ is increasing and we apply what we just have proved.

316 We have

$$\prod_{k=2}^{n} \frac{k^3-1}{k^3+1} = \prod_{k=2}^{n} \frac{k-1}{k+1} \prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1}.$$

Now

$$\prod_{k=2}^{n} \frac{k-1}{k+1} = \frac{(n-1)!}{\frac{(n+1)!}{2}} = \frac{2}{n(n+1)}.$$

By observing that $(k+1)^2 - (k+1) + 1 = k^2 + k + 1$, we gather that

$$\prod_{k=2}^{n} \frac{k^2-k+1}{k^2+k+1} = \frac{3^2+3+1}{2^2-2+1} \cdot \frac{4^2+4+1}{3^2+3+1} \cdot \frac{5^2+5+1}{4^2+4+1} \cdots \frac{n^2+n+1}{(n-1)^2+(n-1)+1} = \frac{n^2+n+1}{3}.$$

Thus

$$\prod_{k=2}^{n} \frac{k^3-1}{k^3+1} = \frac{2}{3} \cdot \frac{n^2+n+1}{n(n+1)} \to \frac{2}{3},$$

as $n \to +\infty$.

317 Clearly $x_n < x_n + \frac{1}{(n+1)^2} = x_{n+1}$, and so the sequence is strictly increasing. By shewing that $x_n < 2 - \frac{1}{n} < 2$ we will be shewing that it is bounded above, and hence convergent by Theorem 284. For $n = 1$, $x_1 = 1 = 2 - \frac{1}{1}$ and so the assertion is true. Assume that $x_n < 2 - \frac{1}{n}$. Then

$$x_{n+1} = x_n + \frac{1}{(n+1)^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 + \frac{n-(n+1)^2}{n(n+1)^2} = 2 - \frac{n^2+n+1}{n(n+1)^2} < 2 - \frac{n^2+n}{n(n+1)^2} = 2 - \frac{1}{n+1},$$

and the claimed inequality follows by induction. We will prove later on a result of Euler:

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots = \frac{\pi^2}{6}.$$

336 The product rule for limits only applies to a finite number of factors. Here the number of factors grows with n.

338 From Theorem 328, and since $x \mapsto \log x$ is increasing,

$$\left(1 + \frac{1}{k+1}\right)^{k+1} < e < \left(1 + \frac{1}{k}\right)^{k+1} \implies (k+1)\log\left(1 + \frac{1}{k+1}\right) < 1 < (k+1)\log\left(1 + \frac{1}{k}\right).$$

Rearranging,

$$\log\frac{k+2}{k+1} < \frac{1}{k+1} < \log\frac{k+1}{k}.$$

Summing from $k = n-1$ to $k = 2n-1$,

$$\begin{aligned} \sum_{k=n-1}^{2n-1} \log\frac{k+2}{k+1} < \sum_{k=n-1}^{2n-1} \frac{1}{k+1} < \sum_{k=n-1}^{2n-1} \log\frac{k+1}{k} \quad & \implies \quad \log\frac{2n+1}{n} < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < \log\frac{2n}{n-1} \\ & \implies \quad \log\left(2 + \frac{1}{n}\right) < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} < \log\left(2 + \frac{2}{n-1}\right) \end{aligned}$$

and the result follows from the Sandwich Theorem.

339 Observe that

$$\begin{aligned} & \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ & -2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\ = \quad & \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ & -2 \cdot \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ = \quad & \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ & -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ = \quad & \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \end{aligned}$$

and use the result of problem 338.

342 We begin by looking at the Taylor series for e^x:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

This converges for every $x \in \mathbb{R}$, so $e = \sum_{k=0}^{\infty} \frac{1}{k!}$ and $e^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}$. Arguing by contradiction, assume $ae^2 + be + c = 0$ for integers a, b and c. That is the same as $ae + b + ce^{-1} = 0$.
Fix $n > |a| + |c|$, then $a, c \mid n!$ and $\forall k \leq n$, $k! \mid n!$. Consider

$$\begin{aligned} 0 = n!(ae + b + ce^{-1}) &= an! \sum_{k=0}^{\infty} \frac{1}{k!} + b + cn! \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \\ &= b + \sum_{k=0}^{n} (a + c(-1)^k) \frac{n!}{k!} + \sum_{k=n+1}^{\infty} (a + c(-1)^k) \frac{n!}{k!} \end{aligned}$$

Since $k! \mid n!$ for $k \leq n$, the first two terms are integers. So the third term should be an integer. However,

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} (a + c(-1)^k) \frac{n!}{k!} \right| &\leq (|a| + |c|) \sum_{k=n+1}^{\infty} \frac{n!}{k!} \\ &= (|a| + |c|) \sum_{k=n+1}^{\infty} \frac{1}{(n+1)(n+2)\cdots k} \\ &\leq (|a| + |c|) \sum_{k=n+1}^{\infty} (n+1)^{n-k} \\ &= (|a| + |c|) \sum_{t=1}^{\infty} (n+1)^{-t} \\ &= (|a| + |c|) \frac{1}{n} \end{aligned}$$

is less than 1 by our assumption that $n > |a| + |c|$. Since there is only one integer which is less than 1 in absolute value, this means that $\sum_{k=n+1}^{\infty} (a + c(-1)^k) \frac{1}{k!} = 0$ for every sufficiently large n which is not the case because

$$\sum_{k=n+1}^{\infty} (a + c(-1)^k) \frac{1}{k!} - \sum_{k=n+2}^{\infty} (a + c(-1)^k) \frac{1}{k!} = (a + c(-1)^{n+1}) \frac{1}{(n+1)!}$$

is not identically zero. The contradiction completes the proof.

344 Apply Problem 261 We can apply this to the stated problem by observing that for a fixed d, a positive integer without square factors, the numbers $a + b\sqrt{d}$ are quadratic integers if a, b are rational integers, and that the set of such numbers is an additive group of reals. Clearly the closure of this group (it, together with its set of limit points) is a group too, for if $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$. The new group is not of form (i) or (ii), hence must be all reals, and the proof (of a slightly stronger theorem) is complete.

388 $a_n = o\left(n^2\right)$ does, since this says that $\lim_{n\to+\infty} \frac{a_n}{n^2} = 0$, whereas $a_n = O\left(n^2\right)$ says that $\frac{a_n}{n^2}$ is bounded by some positive constant.

389 False. Take $a_n = 2n$, for example. Then $a_n << n$, $\frac{a_n}{n} = 2$, and so $\frac{a_n}{n} \not\to 0$.

390 True. $\frac{a_n}{n} \to 0$ and so by Theorem 361, $a_n << n$.

391 False. Take $a_n = n^{3/2}$. Then $\frac{a_n}{n} \to 0$ but $a_n \neq O(n)$.

392 True. $\frac{a_n}{n} \to 0$ and so by Theorem 361, $a_n << n$. Since $n << n^2$, the assertion follows by transitivity.

435 Put $a_n = \frac{1}{\left(2n - \frac{1}{2}\right)\pi}$, $b_n = \frac{1}{\left(2n + \frac{1}{2}\right)\pi}$ for integer $n \geq 1$. Then $a_n \to 0$ and $b_n \to 0$, but $\sin \frac{1}{a_n} \to -1$ and $\sin \frac{1}{b_n} \to +1$, so the limit does not exist in view of Proposition 424.

459 $f(0) = 0$, but for $x > 0$, $f(x) = \frac{1 + \sqrt{1 + 4x}}{2}$, so f is not right-continuous at $x = 0$.

476 Consider a unit circle and take any point P on the circumference of the circle.
Drop the perpendicular from P to the horizontal line, M being the foot of the perpendicular and Q the reflection of P at M. (refer to figure)
Let $x = \angle POM.$
For x to be in $[0, \frac{\pi}{2}]$, the point P lies in the first quadrant, as shown.
The length of line segment PM is $\sin(x)$. Construct a circle of radius MP, with M as the center.
Length of line segment PQ is $2\sin(x)$.
Length of arc PAQ is $2x$.
Length of arc PBQ is $\pi \sin(x)$.
Since $PQ \leq$ length of arc PAQ (equality holds when $x = 0$) we have $2\sin(x) \leq 2x$. This implies

$$\sin(x) \leq x$$

Since length of arc PAQ is $\leq$ length of arc PBQ (equality holds true when $x = 0$ or $x = \frac{\pi}{2}$), we have $2x \leq \pi \sin(x)$. This implies

$$\frac{2}{\pi} x \leq \sin(x)$$

Thus we have

$$\frac{2}{\pi} x \leq \sin(x) \leq x, \forall x \in [0, \frac{\pi}{2}]$$

510 If p had odd degree, then, by the Intermediate Value Theorem it would have a real root. Let α be its largest real root. Then

$$0 = p(\alpha)q(\alpha) = p(\alpha^2 + \alpha + 1)$$

meaning that $\alpha^2 + \alpha + 1 > \alpha$ is a real root larger than the supposedly largest real root α, a contradiction.

511 Observe that $f(1000)f(f(1000))=1 \implies f(999)=\frac{1}{999}$. So the range of f include all numbers from $\frac{1}{999}$ to 999. By the intermediate value theorem, there is a real number a such that $f(a)=500$. Thus

$$f(a)f(f(a))=1 \implies f(500)=\frac{1}{500}.$$

514

519 If either $f(0)=1$ or $f(1)=0$, we are done. So assume that $0 \geq f(0)<1$ and $0<f(1)\leq 1$. Put $g(x)=f(x)+x-1$. Then $g(0)=f(0)-1<0$ and $g(1)=f(1)>0$. By Bolzano's Theorem there is a $c\in \left]0\,;1\right[$ such that $g(c)=0$, that is, $f(c)+c-1=0$, as required.

520 Consider $g(x)=f(x)-f(x+1/n)$, which is clearly continuous. If g is never 0 in $\left[0\,;1\right]$ then by Corollary 506 g must be either strictly positive or strictly negative. But then

$$0=f(0)-f(1)=\left(f(0)-f\left(\frac{1}{n}\right)\right)+\left(f\left(\frac{1}{n}\right)-f\left(\frac{2}{n}\right)\right)+\left(f\left(\frac{2}{n}\right)-f\left(\frac{3}{n}\right)\right)+\cdots+\left(f\left(\frac{n-1}{n}\right)-f\left(\frac{n}{n}\right)\right).$$

The sum of each parenthesis on the right is strictly positive or strictly negative and hence never 0, a contradiction.

521 Consider the function $f:\left[0\,;1\right]\to\left[0\,;1\right]$, $x\mapsto \dfrac{\sin\frac{2\pi x}{a}}{\sin\frac{2\pi}{a}}-x$.

566 Observe that that

$$\frac{1}{x-1}-\frac{1}{x+1}=\frac{(x+1)-(x-1)}{(x-1)(x+1)}=\frac{2}{x^2-1}.$$

If $f(x)=(x-1)^{-1}$ then

$$f'(x)=-1(x-1)^{-2};f''(x)=(-1)(-2)(x-1)^{-3};(-1)(-2)(-3)(x-1)^{-4};\ldots;f^{(100)}(x)=100!(x-1)^{-101}.$$

Similarly, if $g(x)=(x+1)^{-1}$ then

$$g'(x)=-1(x+1)^{-2};g''(x)=(-1)(-2)(x+1)^{-3};(-1)(-2)(-3)(x+1)^{-4};\ldots;g^{(100)}(x)=100!(x+1)^{-101}.$$

Hence

$$\frac{\mathrm{d}^{100}}{\mathrm{d}x^{100}}\frac{2}{x^2-1}=f^{(100)}(x)-g^{(100)}(x)=100!(x-1)^{-101}-100!(x+1)^{-101}.$$

567 We use Leibniz's Rule and the observation that the third derivative of $x\mapsto x^2$ is 0. Also $(\sin x)^{(4n)}=\sin x$, $(\sin x)^{(4n+2)}=-\sin x$, $(\sin x)^{(4n+1)}=\cos x$, and $(\sin x)^{(4n+3)}=-\cos x$, Then

$$\frac{\mathrm{d}^{100}}{\mathrm{d}x^{100}}x^2\sin x=\binom{100}{0}x^2(\sin x)^{(100)}+\binom{100}{1}(x^2)'(\sin x)^{(99)}+\binom{100}{2}(x^2)''(\sin x)^{(98)}=x^2\sin x-200x\cos x-9900\sin x.$$

576 Put $f(x)=x^5-2x^2+x$. Then $f(0)=f(1)=0$ and by Rolle's Theorem there is $c\in]0;1[$ such that $f'(c)=5c^4-4c+1=0$.

577 Set

$$f(x)=a_0x+\frac{a_1x^2}{2}+\frac{a_2x^3}{3}+\cdots+\frac{a_nx^{n+1}}{n+1},$$

and use Rolle's Theorem.

579 Set $g(x)=f(x)^2f(1-x)$. Since $g(0)=g(1)=0$, g satisfies the hypotheses of Rolle's Theorem. There is a $c\in\left]0\,;1\right[$ such that

$$g'(c)=0\implies 2f'(c)f(c)f(1-c)-f(c)^2f'(1-c)=0.$$

Since by assumption $f(c)f(1-c)\neq 0$ we must have, upon dividing by every term by $f(c)^2f(1-c)$, the assertion.

580 For $0\leq k\leq n-1$, consider the interval $\left[\frac{k}{n}\,;\frac{k+1}{n}\right]$. By the Mean Theorem, there are $a_k\in\left]\frac{k}{n}\,;\frac{k+1}{n}\right[$ such that

$$f'(a_k)=\frac{f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)}{\frac{1}{n}}=n\left(f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right).$$

Summing from $k=0$ to $k=n-1$ and noting that the dextral side telescopes,

$$\sum_{k=0}^{n-1}f'(a_k)=n\sum_{k=0}^{n-1}\left(f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right)=n(f(1)-f(0))=n.$$

581 Let $k_i\in\left[0\,;1\right]$ be the smallest number such that $f(k_i)=\frac{i}{n}$, $1\leq i\leq n-1$. Put $k_0=0, k_n=1$. The existence of the k_i is guaranteed by the Intermediate Value Theorem. Moreover, since the k_i are chosen to be the first time f is $\frac{i}{n}$, once again, by the Intermediate Value Theorem we must have

$$0<k_1<k_2<\cdots<k_{n-1}<1.$$

Hence, by the Mean Value Theorem, there exists $a_i\in\left]k_i\,;k_{i+1}\right[$, $0\leq i\leq n-1$, such that

$$f'(a_i)=\frac{f(k_{i+1})-f(k_i)}{k_{i+1}-k_i}=\frac{1}{n(k_{i+1}-k_i)}\implies\frac{1}{f'(a_i)}=n(k_{i+1}-k_i).$$

Summing,

$$\sum_{k=0}^{n-1}\frac{1}{f'(a_k)}=n\sum_{k=0}^{n-1}(k_{i+1}-k_i)=n(k_n-k_0)=n.$$

594 We have $f'(x)=x^x(\log x+1)$ whence $f'(x)=0\implies x=e^{-1}$. Since $f'(x)<0$ for $0<x<e^{-1}$ and $f'(x)>0$ for $x>e^{-1}$, $x=e^{-1}$ is a local (relative) minimum. Thus $f(x)\geq f(e^{-1})=\left(\frac{1}{e}\right)^{1/e}$.

607 Let $0 < k < 1$, and consider the function

$$f: \begin{array}{ccc} [0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & x^k - k(x-1) \end{array}.$$

Then $0 = f'(x) = kx^{k-1} - k \Leftrightarrow x = 1$. Since $f''(x) = k(k-1)x^{k-2} < 0$ for $0 < k < 1, x \geq 0$, $x = 1$ is a maximum point. Hence $f(x) \leq f(1)$ for $x \geq 0$, that is $x^k \leq 1 + k(x-1)$. Letting $k = \frac{1}{p}$ and $x = \frac{a^p}{b^q}$ we deduce

$$\frac{a}{b^{q/p}} \leq 1 + \frac{1}{p}\left(\frac{a^p}{b^q} - 1\right).$$

Rearranging gives

$$ab \leq b^{1+p/q} + \frac{a^p b^{1+p/q-p}}{p} - \frac{b^{1+p/q}}{p}$$

from where we obtain the inequality.

618 We have:

1. Put $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x-1} - x$. Clearly $f(1) = e^0 - 1 = 0$. Now,

$$f'(x) = e^{x-1} - 1,$$
$$f''(x) = e^{x-1}.$$

If $f'(x) = 0$ then $e^{x-1} = 1$ implying that $x = 1$. Thus f has a single minimum point at $x = 1$. Thus for all real numbers x

$$0 = f(1) \leq f(x) = e^{x-1} - x,$$

which gives the desired result.

2. Easy Algebra!

3. Easy Algebra!

4. By the preceding results, we have

$$A_1 \leq \exp(A_1 - 1),$$
$$A_2 \leq \exp(A_2 - 1),$$
$$\vdots$$
$$A_n \leq \exp(A_n - 1).$$

Since all the quantities involved are positive, we may multiply all these inequalities together, to obtain,

$$A_1 A_2 \cdots A_n \leq \exp(A_1 + A_2 + \cdots + A_n - n).$$

In view of the observations above, the preceding inequality is equivalent to

$$\frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n} \leq \exp(n - n) = e^0 = 1.$$

We deduce that

$$G_n \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n,$$

which is equivalent to

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now, for equality to occur, we need each of the inequalities $A_k \leq \exp(A_k - 1)$ to hold. This occurs, in view of the preceding lemma, if and only if $A_k = 1$, $\forall k$, which translates into $a_1 = a_2 = \ldots = a_n$. This completes the proof.

631 $(\log\log x)^{\log x} = \exp((\log x)(\log\log\log x))$ and $(\log x)^{\log\log x} = \exp((\log\log x)^2)$. Now, lexicographically,

$$(\log\log x)^2 << (\log x)(\log\log\log x) \implies \exp((\log\log x)^2) << \exp((\log x)(\log\log\log x))$$

and thus $(\log\log x)^{\log x}$ is faster.

681 $\Leftarrow$ This follows directly from Theorem 666.

$\Longrightarrow$ If f is Riemann integrable, let $\varepsilon > 0$ and let $\mathscr{P}' = \{a = y_0 < y_1 < \ldots < y_m = b\}$ be a partition with $m + 1$ points such that

$$U(f, \mathscr{P}') - L(f, \mathscr{P}') < \frac{\varepsilon}{2}.$$

As f is bounded, there is $M > 0$ such that $\forall x \in \left[a\,;b\right]$, $\left|f(x)\right| \leq M$. Take $\delta = \frac{\varepsilon}{8mM}$ and consider now an arbitrary partition $\mathscr{P} = \{a = x_0 < x_1 < \ldots < x_n = b\}$ with norm $\left|\left|\mathscr{P}\right|\right| < \delta$. Put $\mathscr{P}'' = \mathscr{P} \cup \mathscr{P}'$. Arguing as in Theorem 663, we obtain

$$L(f, \mathscr{P}'') - L(f, \mathscr{P}) < 2mM\left|\left|\mathscr{P}\right|\right| < 2mM\delta = \frac{\varepsilon}{4}.$$

Since by Theorem 664 $L(f, \mathscr{P}') \leq L(f, \mathscr{P}'')$ we gather

$$L(f, \mathscr{P}') - L(f, \mathscr{P}) < \frac{\varepsilon}{4}.$$

In a similar fashion we establish that

$$U(f, \mathscr{P}) - U(f, \mathscr{P}') < \frac{\varepsilon}{4},$$

and upon assembling the inequalities,

$$U(f, \mathscr{P}) - L(f, \mathscr{P}) < U(f, \mathscr{P}') - L(f, \mathscr{P}') + \frac{\varepsilon}{2} < \varepsilon,$$

since we had assumed that $U(f, \mathscr{P}') - L(f, \mathscr{P}') < \frac{\varepsilon}{2}$.

682 $\Longrightarrow$ Assume f is Riemann-integrable. For $\varepsilon > 0$ let $\delta > 0$ be chosen so that the conditions of Theorem **??** be fulfilled. By definition of a Riemann sum,

$$L(f,\mathscr{P}) \leq S(f,\mathscr{P}) \leq U(f,\mathscr{P}),$$

and therefore

$$U(f,\mathscr{P}) < L(f,\mathscr{P}) + \varepsilon \leq \underline{\int_a^b} f(x)\mathrm{d}x + \varepsilon = \int_a^b f(x)\mathrm{d}x + \varepsilon$$

and

$$L(f,\mathscr{P}) > U(f,\mathscr{P}) - \varepsilon \geq \overline{\int_a^b} f(x)\mathrm{d}x - \varepsilon = \int_a^b f(x)\mathrm{d}x - \varepsilon.$$

These inequalities give

$$\left| S(f,\mathscr{P}) - \int_a^b f(x)\mathrm{d}x \right| < \varepsilon,$$

whence $\lim_{\left\|\mathscr{P}\right\| \to 0} S(f,\mathscr{P}) = \int_a^b f(x)\mathrm{d}x.$

$\Leftarrow$ Suppose that $\lim_{\left\|\mathscr{P}\right\| \to 0} S(f,\mathscr{P}) = L$, existing and finite. Given $\varepsilon > 0$ there is $\delta > 0$ such that $\left\|\mathscr{P}\right\| < \delta$ implies

$$L - \frac{\varepsilon}{3} < S(f,\mathscr{P}) < A + \frac{\varepsilon}{3}. \tag{A.7}$$

Now, choose $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$. By letting t_k range over $\left[x_{k-1}\,;x_k\right]$ we gather, from (A.7)

$$L - \frac{\varepsilon}{3} \leq L(f,\mathscr{P}) \leq U(f,\mathscr{P}) \leq L + \frac{\varepsilon}{3},$$

whence

$$U(f,\mathscr{P}) - L(f,\mathscr{P}) \leq \frac{2}{3}\varepsilon < \varepsilon,$$

meaning that f is Riemann-integrable over $\left[a\,;b\right]$ by Theorem 666. Thus

$$L(f,\mathscr{P}) \leq \int_a^b f(x)\mathrm{d}x \leq U(f,\mathscr{P}),$$

and so $\lim_{\left\|\mathscr{P}\right\| \to 0} S(f,\mathscr{P}) = \int_a^b f(x)\mathrm{d}x$.

683 $\Longrightarrow$ Let $\mathscr{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $\left[a\,;b\right]$. Set

$$Z(f,\mathscr{P}) = \sum_{k=1}^{n} \omega(f,\left[x_{k-1}\,;x_k\right])(x_k - x_{k-1}) = U(f,\mathscr{P}) - L(f,\mathscr{P}), \qquad \Omega = \sup_{x\in[a;b]} f(x) - \inf_{x\in[a;b]} f(x).$$

Let

$$\delta = \sum_{k=1}^{n} (x_k - x_{k-1})\chi_{\{x\in[a;b]:\omega(f,[x_{k-1};x_k])\geq\varepsilon'\}}.$$

Then $Z(f,\mathscr{P}) \geq \delta\varepsilon'$. Since we are assuming that f is Riemann-integrable, there exists a partition $\mathscr{P}$ (by Theorem 666) such that

$$Z(f,\mathscr{P}) \leq \varepsilon'\varepsilon.$$

Thus we have $\delta\varepsilon' < \varepsilon\varepsilon'$ from where $\delta < \varepsilon$.

$\Leftarrow$ Assume there is a partition $\mathscr{P}$ for which $\delta < \varepsilon$. In the intervals $I = [x_{k-1}\,;x_k]$ where $\omega(f,I) \geq \varepsilon'$ the oscillation of f is at most Ω, and in the remaining intervals (the sum of which is $b - a - \delta$, the oscillation is less than ε'. Hence

$$Z(f,\mathscr{P}) \leq \delta\Omega + (b - a - \delta)\varepsilon'.$$

Choose now

$$\varepsilon' = \frac{\varepsilon''}{2(b-a)}, \qquad \delta = \frac{\varepsilon''}{2\Omega}.$$

Since $b - a - \delta \leq b - a$,

$$Z(f,\mathscr{P}) \leq \delta\Omega + (b - a - \delta)\varepsilon' \leq \frac{\varepsilon''}{2} + \frac{\varepsilon''}{2} = \varepsilon'',$$

whence f is Riemann-integrable by Theorem 666.

695 $\frac{8}{5}$

696

$$\begin{aligned}
\int_0^3 x\lfloor x\rfloor\,\mathrm{d}x &= \int_0^1 x\lfloor x\rfloor\,\mathrm{d}x + \int_1^2 x\lfloor x\rfloor\,\mathrm{d}x + \int_2^3 x\lfloor x\rfloor\,\mathrm{d}x \\
&= 0\int_0^1 x\,\mathrm{d}x + 1\int_1^2 x\,\mathrm{d}x + 2\int_2^3 x\,\mathrm{d}x \\
&= \frac{x^2}{2}\Big|_1^2 + x^2\Big|_2^3 \\
&= (2 - \frac{1}{2}) + (9 - 4) \\
&= \frac{13}{2}.
\end{aligned}$$

697 We have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - h - e^x}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} - \lim_{h\to 0} \frac{h}{h} = e^x - 1,$$

whence $f(x) = e^x - x + C$. Since $3 = f(0) = e^0 - 0 + C \implies C = 2$, we deduce that $f(x) = e^x - x + 2$.

698 Put $I = \int_0^a \frac{1}{f(x)+1}\mathrm{d}x$. We have

$$I = \int_0^a \frac{1}{f(u)+1}\mathrm{d}u = \int_0^a \frac{f(u)f(a-u)}{f(u)+f(u)f(a-u)}\mathrm{d}u = \int_0^a \frac{f(a-u)}{1+f(a-u)}\mathrm{d}u = -\int_a^0 \frac{f(v)}{1+f(v)}\mathrm{d}v = \int_0^a \frac{f(u)}{1+f(u)}\mathrm{d}u,$$

whence

$$2I = \int_0^a \frac{f(u)}{1+f(u)}\mathrm{d}u + \int_0^a \frac{f(a-u)}{1+f(a-u)}\mathrm{d}u = \int_0^a \frac{2+f(u)+f(a-u)}{2+f(u)+f(a-u)}\mathrm{d}u = a,$$

and so $I = \frac{a}{2}$.

699 Observe first that $f(0+0) = f(0)+f(0)$ and so $f(0) = 0$. Integrate $f(u+y) = f(u)+f(y)$ for $u \in [0;x]$keeping y constant, getting

$$\int_0^x f(u+y)\mathrm{d}u = \int_0^x f(u)\mathrm{d}u + \int_0^x f(y)\mathrm{d}u = \int_0^x f(u)\mathrm{d}u + xf(y).$$

Also, by substitution,

$$\int_0^x f(u+y)\mathrm{d}u = \int_y^{y+x} f(u)\mathrm{d}u = \int_0^{y+x} f(u)\mathrm{d}u - \int_0^y f(u)\mathrm{d}u.$$

Hence

$$xf(y) = \int_0^{y+x} f(u)\mathrm{d}u - \int_0^y f(u)\mathrm{d}u - \int_0^x f(u)\mathrm{d}u. \tag{A.8}$$

Exchanging x and y:

$$yf(x) = \int_0^{y+x} f(u)\mathrm{d}u - \int_0^x f(u)\mathrm{d}u - \int_0^y f(u)\mathrm{d}u. \tag{A.9}$$

From (A.8) and (A.9) we gather that $xf(y) = xf(y)$. If $xy \neq 0$ then $\frac{f(x)}{x} = \frac{f(y)}{y}$. This means that for $\frac{f(x)}{x}$ is constant, and so for $x \neq 0$, $f(x) = cx$ for some constant c. Since $f(0) = 0$, $f(x) = cx$ for all x. Taking $x = 1$, $f(1) = c$.

701 We have

$$\begin{aligned}
\int_{-1}^{2} |x^2-1|\ \mathrm{d}x &= \int_{-1}^{1} (1-x^2)\ \mathrm{d}x + \int_1^2 (x^2-1)\ \mathrm{d}x \\
&= (x - \frac{x^3}{3})\Big|_{-1}^{1} + (\frac{x^3}{3} - x)\Big|_1^2 \\
&= 2(1-\frac{1}{3}) + (\frac{8}{3}-2) - (\frac{1}{3}-1) \\
&= \frac{4}{3} + \frac{2}{3} + \frac{2}{3} \\
&= \frac{8}{3}
\end{aligned}$$

710 Put $u = \sqrt{x^2-1}; u^2 = x^2-1$ so that $2u\mathrm{d}u = 2x\mathrm{d}x$ and $\frac{\mathrm{d}x}{x} = \frac{x\mathrm{d}x}{x^2} = \frac{u\mathrm{d}u}{u^2+1}$. Thus

$$\int \frac{1}{x\sqrt{x^2-1}}\mathrm{d}x = \int \frac{u}{(u^2+1)u}\mathrm{d}u = \int \frac{1}{u^2+1}\mathrm{d}u = \arctan u + C = \arctan\sqrt{x^2-1} + C.$$

711 Put $u = \sqrt{x+1}; u^2 = x+1$; from where $\mathrm{d}x = 2u\mathrm{d}u$. Whence

$$\int \frac{1}{1+\sqrt{x+1}}\mathrm{d}x = \int \frac{2u}{1+u}\ \mathrm{d}u = \int \left(2 - \frac{2}{1+u}\right)\mathrm{d}u = 2u - 2\log|1+u| + C = 2\sqrt{1+x} - 2\log|1+\sqrt{1+x}| + C.$$

712 Put $x = u^6; \mathrm{d}x = 6u^5\mathrm{d}u$, giving

$$\begin{aligned}
\int \frac{x^{1/2}}{x^{1/2}-x^{1/3}}\mathrm{d}x &= \int \frac{(u^3)(6u^5)}{u^3-u^2}\mathrm{d}u \\
&= \int \frac{6u^6}{u-1}\mathrm{d}u \\
&= 6\int \left(u^5+u^4+u^3+u^2+u+1+\frac{1}{u-1}\right)\mathrm{d}u \\
&= 6\left(\frac{u^6}{6}+\frac{u^5}{5}+\frac{u^4}{4}+\frac{u^3}{3}+\frac{u^2}{2}+u+\log|u-1|\right)+C \\
&= x + \frac{6x^{5/6}}{5} + \frac{3x^{2/3}}{2} + 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + 6\log|x^{1/6}-1| + C.
\end{aligned}$$

713 Put $u^2 = a^x + 1; 2u\mathrm{d}u = (\log a)a^x\mathrm{d}x$ and so

$$\int \frac{a^{2x}}{\sqrt{a^x+1}}\mathrm{d}x = \int \frac{2u(u^2-1)}{u\log a}\mathrm{d}u = \int \frac{2u^2-2}{\log a}\mathrm{d}u = \frac{2u^3}{3\log a} - \frac{2u}{\log a} + C = \frac{2(a^x+1)^{3/2}}{3\log a} - \frac{2(a^x+1)^{1/2}}{\log a} + C.$$

714 Observe that $(e^x - e^{-x})^2 = (e^{-x}(e^{2x}-1))^2 = e^{-2x}(e^{2x}-1)^2$, and so

$$\int \frac{1}{(e^x-e^{-x})^2}\mathrm{d}x = \int \frac{e^{2x}}{(e^{2x}-1)^2}\mathrm{d}x = \int \frac{1}{2u^2}\mathrm{d}u = -\frac{1}{2u} + C = -\frac{1}{2(e^{2x}-1)} + C,$$

on putting $u = e^{2x} - 1$.

715 We have

$$\begin{aligned}
\int_1^5 \frac{\lfloor x\rfloor}{x}\mathrm{d}x &= \int_1^2 \frac{\lfloor x\rfloor}{x}\mathrm{d}x + \int_2^3 \frac{\lfloor x\rfloor}{x}\mathrm{d}x + \int_3^4 \frac{\lfloor x\rfloor}{x}\mathrm{d}x + \int_4^5 \frac{\lfloor x\rfloor}{x}\mathrm{d}x \\
&= \int_1^2 \frac{1}{x}\mathrm{d}x + \int_2^3 \frac{2}{x}\mathrm{d}x + \int_3^4 \frac{3}{x}\mathrm{d}x + \int_4^5 \frac{4}{x}\mathrm{d}x \\
&= (\log 2 - \log 1) + 2(\log 3 - \log 2) + 3(\log 4 - \log 3) + 4(\log 5 - \log 4) \\
&= 4\log(5) - 3\log(2) - \log(3).
\end{aligned}$$

716 Put $u = e^x$, etc.

$$\int e^{e^x+x}\mathrm{d}x = \int e^x e^{e^x}\mathrm{d}x = \int e^{e^x}\mathrm{d}e^x = e^{e^x} + C$$

717 Put $u = \log(\cos x)$, etc.

$$\int \tan x \log(\cos x)\,dx = \int (\log(\cos x))\,d(-\log(\cos(x))) = -\frac{(\log(\cos x))^2}{2} + C$$

718 Put $u = \log\log x$, etc.

$$\int \frac{\log\log x}{x\log x}\,dx = \int \log\log x\, d\left(\log\log x\right) = \frac{\log\log x}{2} + C$$

719 Carry out the long division.

$$\int \frac{x^{18}-1}{x^3-1}\,dx = \int (x^{15}+x^{12}+x^9+x^6+x^3+1)\,dx = \frac{x^{16}}{16}+\frac{x^{13}}{13}+\frac{x^{10}}{10}+\frac{x^7}{7}+\frac{x^4}{4}+x+C$$

720 After an algebraic trick, put $u = 1 + x^{-7}$, etc.

$$\int \frac{1}{x^8+x}\,dx = \int \frac{x^{-8}}{1+x^{-7}}\,dx = -\frac{1}{7}\int \frac{d(1+x^{-7})}{1+x^{-7}} = -\frac{1}{7}\log|1+x^{-7}| + C$$

721 Put $u = 2^x + 1$

$$\int \frac{2^x 2^x}{2^x+1}\,dx = \frac{1}{\log 2}\int \frac{2^x}{2^x+1}\,d(2^x+1) = \frac{1}{\log 2}\int \frac{u-1}{u}\,du = \frac{1}{\log 2}\left(u - \log|u|\right) + C = \frac{1}{\log 2}\left(2^x+1-\log|2^x+1|\right) + C$$

722 Put $u = x+1$. Then $x^2 = (u-1)^2 = u^2 - 2u + 1$, and hence

$$\begin{aligned}
\int \frac{x^2}{(x+1)^{10}}\,dx &= \int \frac{u^2-2u+1}{u^{10}}\,du \\
&= \int u^{-8} - 2u^{-9} + u^{-10}\,du \\
&= -\frac{u^{-7}}{7} + \frac{u^{-8}}{4} - \frac{u^{-9}}{9} + C \\
&= -\frac{(x+1)^{-7}}{7} + \frac{(x+1)^{-8}}{4} - \frac{(x+1)^{-9}}{9} + C
\end{aligned}$$

723 Algebraic trick, and then $u = e^{-x} + 1$, etc.

$$\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx = -\int \frac{1}{e^{-x}+1}\,d(e^{-x}+1) = -\log|e^{-x}+1| + C$$

724

$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1+\sin x}{1-\sin^2 x}\,dx = \int \frac{1+\sin x}{\cos^2 x}\,dx = \int \sec^2 x + \sec x \tan x\,dx = \tan x + \sec x + C$$

725

$$\begin{aligned}
\int \sqrt{1+\sin 2x}\,dx &= \int \sqrt{\sin^2 x + 2\sin x\cos x + \cos^2 x}\,dx \\
&= \int \sqrt{(\sin x + \cos x)^2}\,dx \\
&= \int |\sin x + \cos x|\,dx \\
&= \mp\cos x \pm \sin x + C
\end{aligned}$$

726 Put $u = x^2$, etc.

$$\int \frac{x}{\sqrt{1-(x^2)^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin x^2 + C$$

727 We have

$$\begin{aligned}
\int \sec^4 x\,dx &= \int \sec^2 x(\tan^2 x + 1)\,dx \\
&= \int \sec^2 x \tan^2 x\,dx + \int \sec^2 x\,dx \\
&= \int (\tan x)^2\,d(\tan x) + \int \sec^2 x\,dx \\
&= \frac{\tan^3 x}{3} + \tan x + C.
\end{aligned}$$

728 We have

$$\begin{aligned}
\int \sec^5 x\,dx &= \int \sec^3 x \sec^2 x\,dx \\
&= \int \sec^3 x\,d(\tan x) \\
&= \sec^3 x \tan x - \int \tan x\,d(\sec^3 x) \\
&= \sec^3 x \tan x - 3\int \tan^2 x \sec^2 x \sec x\,dx \\
&= \sec^3 x \tan x - 3\int (\sec^2 x - 1)\sec^3 x\,dx \\
&= \sec^3 x \tan x - 3\int \sec^5 x\,dx + 3\int \sec^3 x\,dx
\end{aligned}$$

The above implies that

$$\begin{aligned}
\int \sec^5 x\,dx &= \frac{\tan x \sec^3 x}{4} + \frac{3}{4}\int \sec^3 x\,dx \\
&= \frac{\tan x \sec^3 x}{4} + \frac{3\tan x \sec x}{8} + \frac{3}{8}\log|\sec x + \tan x| + C,
\end{aligned}$$

upon recalling from class that

$$\int \sec^3 x\,dx = \frac{\tan x \sec x}{2} + \frac{1}{2}\log|\sec x + \tan x| + C$$

729 First put $t = x^{1/3}$, then $t^3 = x \Longrightarrow 3t^2\,dt = dx$. Thus

$$\begin{aligned}
\int e^{x^{1/3}}\,dx &= \int 3t^2 e^t\,dt \\
&= 3t^2 e^t - 6te^t - 6e^t + C \\
&= 3x^{2/3}e^{x^{1/3}} - 6x^{1/3}e^{x^{1/3}} - 6e^{x^{1/3}} + C,
\end{aligned}$$

where the penultimate step results from tabular integration by parts.

730 We have

$$\begin{aligned}\int \log(x^2+1)\,\mathrm{d}x &= x\log(x^2+1) - \int x\mathrm{d}(\log(x^2+1)) \\ &= x\log(x^2+1) - 2\int \frac{x^2}{x^2+1}\,\mathrm{d}x \\ &= x\log(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,\mathrm{d}x \\ &= x\log(x^2+1) - 2\int \left(1-\frac{1}{x^2+1}\right)\mathrm{d}x \\ &= x\log(x^2+1) - 2(x-\arctan x) + C\end{aligned}$$

731 Put

$$I = \int xe^x\cos x := (Ax+B)e^x\cos x + (Cx+D)e^x\sin x + K.$$

Differentiating both sides,

$$xe^x\cos x = Ae^x\cos x + (Ax+B)e^x\cos x - (Ax+B)e^x\sin x + Ce^x\sin x + (Cx+D)e^x\sin x + (Cx+D)e^x\cos x.$$

Equating coefficients,

$$\begin{aligned} xe^x\cos x &: \quad 1 = A + C \\ xe^x\sin x &: \quad 0 = -A + C \\ e^x\cos x &: \quad 0 = A + B + D \\ e^x\sin x &: \quad 0 = -B + C + D \end{aligned}$$

From the first two equations $C = \frac{1}{2}, A = \frac{1}{2}$. Then the third and fourth equations become $-\frac{1}{2} = B + D; -\frac{1}{2} = -B + D$, whence $D = -\frac{1}{2}$, and $B = 0$. We conclude that

$$\int xe^x\cos x = \frac{x}{2}e^x\cos x + \left(\frac{x-1}{2}\right)e^x\sin x + K.$$

732 We will do this one two ways: first, by making the substitution

$$t = \log x \implies e^t = x \implies e^t\,\mathrm{d}t = \mathrm{d}x.$$

Observe also that $x^{2/3} = e^{2t/3}$. Then

$$\begin{aligned}\int x^{2/3}\log x\mathrm{d}x &= \int te^{2t/3}e^t\,\mathrm{d}t \\ &= \frac{3t}{5}e^{5t/3} - \frac{9}{25}e^{5t/3} + C \\ &= \frac{3(\log x)}{5}x^{5/3} - \frac{9}{25}x^{5/3} + C.\end{aligned}$$

Aliter: By directly integrating by parts,

$$\begin{aligned}\int x^{2/3}\log x\mathrm{d}x &= \int \log x\mathrm{d}\left(\frac{3x^{5/3}}{5}\right) \\ &= \frac{3x^{5/3}}{5}\log x - \frac{3}{5}\int x^{5/3}\mathrm{d}(\log x) \\ &= \frac{3(\log x)}{5}x^{5/3} - \frac{3}{5}\int x^{2/3}\mathrm{d}x \\ &= \frac{3(\log x)}{5}x^{5/3} - \frac{9}{25}x^{5/3} + C,\end{aligned}$$

as before.

733 This integral can be done multiple ways. For example, you may integrate by parts directly and then "solve" for the integral. Another way is the following. Start by putting

$$t = \log x \implies e^t = x \implies e^t\,\mathrm{d}t = \mathrm{d}x.$$

Then

$$\int \sin(\log x)\mathrm{d}x = \int e^t\sin t\,\mathrm{d}t,$$

an integral that we found in class. We will find it again, using a method similar of problem 731. Put

$$I = \int e^t\cos t\mathrm{d}t := Ae^t\cos t + Be^t\sin t + K.$$

Differentiating both sides

$$e^t\cos t = Ae^t\cos t - Ae^t\sin t + Be^t\sin t + Be^t\cos t.$$

Equating coefficients,

$$\begin{aligned} e^t\cos t &: \quad 1 = A + B \\ e^t\sin t &: \quad 0 = -A + B \end{aligned}$$

and so $A = B = \frac{1}{2}$. We have thus

$$\begin{aligned}\int \sin(\log x)\mathrm{d}x &= \int e^t\sin t\mathrm{d}t \\ &= \frac{1}{2}e^t\cos t + \frac{1}{2}e^t\sin t + K \\ &= \frac{1}{2}x\cos\log x + \frac{1}{2}x\sin\log x + K.\end{aligned}$$

734 Put $t = \log\log x \implies e^{e^t} = x \implies e^te^{e^t}\mathrm{d}t = \mathrm{d}x$. Hence

$$\begin{aligned}\int \frac{\log\log x}{x}\mathrm{d}x &= \int \frac{te^te^{e^t}}{e^{e^t}}\mathrm{d}t \\ &= te^t - e^t + C \\ &= (\log x)(\log\log x) - (\log x) + C,\end{aligned}$$

where the penultimate equality follows from a tabular integration by parts.

735 Observe that

$$\int \sec x \mathrm{d}x = \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x}\mathrm{d}x = \int \mathrm{d}\left(\log(\tan x + \sec x)\right) = \log(\tan x + \sec x) + C,$$

For the second way, simple algebra will yield the identity. We have

$$\begin{aligned}\int \sec x \mathrm{d}x &= \int \frac{\cos x}{2(1+\sin x)}\mathrm{d}x + \int \frac{\cos x}{2(1-\sin x)}\mathrm{d}x \\ &= \frac{1}{2}\log|1+\sin x| - \frac{1}{2}\log|1-\sin x| + C \\ &= \frac{1}{2}\log\left|\frac{1+\sin x}{1-\sin x}\right| + C\end{aligned}.$$

For the third way, we have

$$\begin{aligned}\int \csc x \mathrm{d}x &= \int \frac{1}{\sin x}\mathrm{d}x \\ &= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}}\mathrm{d}x \\ &= \int \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos^2\frac{x}{2}}\mathrm{d}x \\ &= \int \frac{\sec^2\frac{x}{2}}{2\tan\frac{x}{2}}\mathrm{d}x \\ &\overset{u=\tan\frac{x}{2}}{=} \int \frac{\mathrm{d}u}{u} \\ &= \log|\tan\frac{x}{2}| + C.\end{aligned}$$

Thus

$$\int \sec x \mathrm{d}x = \int \csc(\frac{\pi}{2}+x)\mathrm{d}x = \int \csc(\frac{\pi}{2}+x)\mathrm{d}(\frac{\pi}{2}+x) = \log\left|\tan(\frac{\pi}{4}+\frac{x}{2})\right| + C.$$

736 Putting $t = \arcsin x$ we have

$$\sin t = x \implies \cos t\mathrm{d}t = \mathrm{d}x,$$

whence

$$\begin{aligned}\int (\arcsin x)^2\mathrm{d}x &= \int t^2\cos t\mathrm{d}t \\ &= t^2\sin t + 2t\cos t - 2\sin t + C \\ &= (\arcsin x)^2 x + 2(\arcsin x)\cos(\arcsin x) - 2x + C \\ &= (\arcsin x)^2 x + 2(\arcsin x)\sqrt{1-x^2} - 2x + C\end{aligned}$$

737 We have

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt{x-1}} &= \int \frac{(\sqrt{x+1}-\sqrt{x-1})\mathrm{d}x}{2} \\ &= \frac{1}{3}(x+1)^{3/2} - \frac{1}{3}(x-1)^{3/2} + C\end{aligned}.$$

738 We have

$$\begin{aligned}\int x\arctan x\mathrm{d}x &= \int \arctan x\mathrm{d}\left(\frac{x^2}{2}\right) \\ &= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\mathrm{d}(\arctan x) \\ &= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\frac{x^2}{1+x^2}\mathrm{d}x \\ &= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\frac{x^2+1-1}{1+x^2}\mathrm{d}x \\ &= \frac{x^2}{2}\arctan x - \frac{x}{2} + \frac{1}{2}\arctan x + C\end{aligned}.$$

739 Put $u = \sqrt{\tan x}$ and so $u^2 = \tan x$, $2u\mathrm{d}u = \sec^2 x\mathrm{d}x = (\tan^2 x + 1)\mathrm{d}x = (u^4+1)\mathrm{d}x$. Hence the integral becomes

$$\int \sqrt{\tan x}\mathrm{d}x = 2\int \frac{u^2}{u^4+1}\mathrm{d}u.$$

To decompose the above fraction into partial fractions observe (Sophie Germain's trick) that $u^4+1 = u^4+2u^2+1-2u^2 = (u^2+u\sqrt{2}+1)(u^2-u\sqrt{2}+1)$ and hence

$$\begin{aligned}\int \sqrt{\tan x}\mathrm{d}x &= 2\int \frac{u^2}{u^4+1}\mathrm{d}u \\ &= -\frac{\sqrt{2}}{2}\int \frac{u}{u^2+u\sqrt{2}+1}\mathrm{d}u + \frac{\sqrt{2}}{2}\int \frac{u}{u^2-u\sqrt{2}+1}\mathrm{d}u \\ &= -\frac{\sqrt{2}}{4}\log(u^2+u\sqrt{2}+1) + \frac{\sqrt{2}}{4}\log(u^2-u\sqrt{2}+1) + \frac{\sqrt{2}}{2}\arctan(\sqrt{2}u+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2}u+1) + C \\ &= -\frac{\sqrt{2}}{4}\log(\tan x + \sqrt{2\tan x}+1) + \frac{\sqrt{2}}{4}\log(\tan x - \sqrt{2\tan x}+1) \\ &\quad + \frac{\sqrt{2}}{2}\arctan(\sqrt{2\tan x}+1) - \frac{\sqrt{2}}{2}\arctan(-\sqrt{2\tan x}+1) + C\end{aligned}$$

740 Put

$$\frac{2x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \implies 2x+1 = Ax(x-1) + B(x-1) + Cx^2.$$

Letting $x=1$ we get $3=C$. Letting $x=0$ we get $1=-B \implies B=-1$. To get A observe that equating the coefficients of x^2 on both sides we get $0 = A+C$, whence $A=-3$. Thus

$$\begin{aligned}\int \frac{2x+1}{x^2(x-1)}\mathrm{d}x &= -3\int \frac{1}{x}\mathrm{d}x - \int \frac{1}{x^2}\mathrm{d}x + 3\int \frac{1}{x-1}\mathrm{d}x \\ &= -3\log|x| + \frac{1}{x} + 3\log|x-1| + C \\ &= 3\log\left|\frac{x-1}{x}\right| + \frac{1}{x} + C.\end{aligned}$$

741 Integrating by parts,

$$\begin{aligned}
\int \log(x+\sqrt{x})\mathrm{d}x &= x\log(x+\sqrt{x}) - \int x\mathrm{d}\log(x+\sqrt{x}) \\
&= x\log(x+\sqrt{x}) - \int \frac{x(1+\frac{1}{2\sqrt{x}})}{x+\sqrt{x}}\mathrm{d}x \\
&= x\log(x+\sqrt{x}) - \int \left(1-\frac{1}{2}\cdot\frac{\sqrt{x}}{x+\sqrt{x}}\right)\mathrm{d}x \\
&= x\log(x+\sqrt{x}) - x + \frac{1}{2}\int \frac{\sqrt{x}}{x+\sqrt{x}}\mathrm{d}x \\
&\overset{u=\sqrt{x}}{=} x\log(x+\sqrt{x}) - x + \int \frac{u^2}{u^2+u}\mathrm{d}u \\
&\overset{u=\sqrt{x}}{=} x\log(x+\sqrt{x}) - x + \int 1-\frac{1}{u+1}\mathrm{d}u \\
&= x\log(x+\sqrt{x}) - x + u - \log(u+1) + C \\
&= x\log(x+\sqrt{x}) - x + \sqrt{x} - \log(\sqrt{x}+1) + C
\end{aligned}$$

742 We use Sophie Germain's trick to factor

$$x^4+1 = x^4+2x^2+1-2x^2 = (x^2+1)^2-2x^2 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1),$$

and seek the partial fraction decomposition

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1} \implies 1 = (Ax+B)(x^2+\sqrt{2}x+1)+(Cx+D)(x^2-\sqrt{2}x+1).$$

Equating coefficients

$$\begin{aligned}
x^3 &: \quad 0 = A+C \\
x^2 &: \quad 0 = B+D+\sqrt{2}(A-C) \\
x &: \quad 0 = A+C+\sqrt{2}(B-D) \\
x^0 &: \quad 1 = B+D
\end{aligned}$$

From the first and third equation it follows that $A=-C$ and that $B=D$. From the fourth equation $B=D=\frac{1}{2}$ and from the second equation $A=-\frac{1}{2\sqrt{2}}=-C$. Hence we must integrate

$$\begin{aligned}
\int \frac{1}{x^4+1}\mathrm{d}x &= \int \frac{\sqrt{2}x+2}{4(x^2+\sqrt{2}x+1)}\mathrm{d}x - \int \frac{\sqrt{2}x-2}{4(x^2-\sqrt{2}x+1)}\mathrm{d}x \\
&= \frac{\sqrt{2}}{8}\int \frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\mathrm{d}x + \frac{1}{4}\int \frac{1}{x^2+\sqrt{2}x+1}\mathrm{d}x - \frac{\sqrt{2}}{8}\int \frac{2x+\sqrt{2}}{x^2-\sqrt{2}x+1}\mathrm{d}x + \frac{1}{4}\int \frac{1}{x^2-\sqrt{2}x+1}\mathrm{d}x \\
&= \frac{\sqrt{2}}{8}\log(x^2+x\sqrt{2}+1) - \frac{\sqrt{2}}{8}\log(x^2-x\sqrt{2}+1) + \frac{1}{2}\int \frac{\mathrm{d}x}{(x\sqrt{2}+1)^2+1} + \frac{1}{2}\int \frac{\mathrm{d}x}{(-x\sqrt{2}+1)^2+1} \\
&= \frac{\sqrt{2}}{8}\log(x^2+x\sqrt{2}+1) - \frac{\sqrt{2}}{8}\log(x^2-x\sqrt{2}+1) + \frac{\sqrt{2}}{4}\arctan(x\sqrt{2}+1) - \frac{\sqrt{2}}{4}\arctan(-x\sqrt{2}+1) + C
\end{aligned}$$

743 We begin by observing that

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \implies 1 = A(x^2-x+1)+(Bx+C)(x+1).$$

Letting $x=-1$ we obtain $1=3A \implies A=\frac{1}{3}$. Letting $x=0$ we obtain $1=A+C \implies C=1-A=\frac{2}{3}$. Finally, we must have $A+B=0$, since the coefficient of x^2 must be zero. thus $B=-\frac{1}{3}$. We must then integrate

$$\begin{aligned}
\int \frac{\mathrm{d}x}{3(x+1)} - \int \frac{x-2}{3(x^2-x+1)}\mathrm{d}x &= \frac{1}{3}\log|x+1| - \int \frac{x-\frac{1}{2}}{3(x-\frac{1}{2})^2+\frac{3}{4}} + \frac{1}{2}\int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} \\
&= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|(x-\frac{1}{2})^2+\frac{3}{4}| + \frac{2}{3}\int \frac{1}{\frac{4}{3}(x-\frac{1}{2})^2+1} \\
&= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|(x-\frac{1}{2})^2+\frac{3}{4}| + \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\arctan(x-\frac{1}{2}) \\
&= \frac{1}{3}\log|x+1| - \frac{1}{6}\log|x^2-x+1| + \frac{\sqrt{3}}{3}\arctan\frac{2}{\sqrt{3}}(x-\frac{1}{2})
\end{aligned}$$

753 Geometric series with first term $a=\frac{(-2)^3}{7^3}=-\frac{8}{343}$ and common ratio $r=\frac{-2}{7}$. Hence

$$\sum_{n=3}^{\infty}\frac{(-2)^n}{7^n} = \frac{-\frac{8}{343}}{1-\frac{-2}{7}} = -\frac{8}{441}.$$

755 By partial fractions

$$\frac{18}{(9n-1)(9n+8)} = \frac{2}{9n-1} - \frac{2}{9n+8}$$

and this

$$\implies \sum_{n=3}^{\infty}\frac{18}{(9n-1)(9n+8)} = \left(\frac{2}{26}-\frac{2}{35}\right)+\left(\frac{2}{35}-\frac{2}{44}\right)+\left(\frac{2}{44}-\frac{2}{53}\right)+\cdots = \frac{2}{26} = \frac{1}{13}.$$

756 By induction $n<2^n \implies n^{1/n}<2$ and so $n^{1+1/n}<2n \implies \frac{1}{2n}<\frac{1}{n^{1+/1n}}$. So the series diverges by direct comparison to $\sum_{n=1}^{\infty}\frac{1}{2n}$.

757 $n=e^{\log n} \implies n^{\frac{1}{\log n}}=e$ and so $n^{1+1/\log n}=en, \quad n>1.$. So the series diverges by direct comparison to $\sum_{n=2}^{\infty}\frac{1}{en}$.

758 This one is really tricky! $e^x>\frac{x^2}{2}$ for $x>0$ as can be seen by considering the monotonicity of $f(x)=e^x-\frac{x^2}{2}$ or considering the Maclaurin expansion of e^x. Now,

$$n^{1/\log\log n} = e^{\log n^{1/\log\log n}} = e^{\frac{\log n}{\log\log n}} > \frac{(\log n)^2}{2(\log\log n)^2}.$$

This gives

$$\frac{2(\log\log n)^2}{n(\log n)^2} > \frac{1}{n^{1+\frac{1}{\log\log n}}}.$$

Now,

$$\sum_{n=2}^{+\infty} \frac{2(\log\log n)^2}{n(\log n)^2}$$

can be shewn to converge by comparing to a series in the De Morgan logarithmic scale.

759 By the root test

$$a_n^{1/n} = \left(\frac{3^n}{n^{2n}}\right)^{1/n} = \frac{3}{n} \to 0 < 1,$$

and the series converges. By direct comparison, for $n \geq 3$ we have

$$\frac{3^n}{n^{2n}} = \frac{3^n}{n^n} \cdot \frac{1}{n^n} \leq 1\frac{1}{n^n} \leq \frac{1}{n^3},$$

and the series converges by direct comparison to $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

760 $-\cos n\pi = (-1)^n$, so the series is clearly alternating. Since $\frac{1}{8n-1}$ decreases to 0, the series converges. But it is clear that the series of absolute values diverges by comparing to the harmonic series. Conclusion: the series is conditionally convergent.

761 Only the fact that $\frac{a_n}{n} \leq a_n$ is needed here.

766

767

768

769

Bibliography

[Apo] Apostol, T. M., **Calculus**, Vol 1 & 2, 2nd ed., Waltham: Xerox, 1967.

[Har] Hardy, G. H., **Pure Mathematics**, 10th ed., New York: Cambridge University Press, 1952.

[Kla] Klambauer, Gabriel, **Aspects of Calculus**, New York: Springer-Verlag, 1986.

[Lan] Landau, E., **Differential and Integral Calculus**, New York, Chelsea Publishing Company, 1950.

[Olm] Olmstead, J. M. H., **Calculus with Analytic Geometry**, Vol 1 & 2, New York: Appleton-Century-Crofts, 1966.

[Spi] Spivak, Michael **Calculus**, 3rd ed., Houston, Texas: Publish or Perish, Inc., 1994.