PreCalculus

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Contents

Preface

There are very few good Calculus books, written in English, available to the American reader. Only [Har], [Kla], [Apo], [Olm], and [Spi] come to mind.

The situation in Precalculus is even worse, perhaps because Precalculus is a peculiar American animal: it is a review course of all that which should have been learned in High School but was not. A distinctive American slang is thus called to describe the situation with available Precalculus textbooks: they stink!

I have decided to write these notes with the purpose to, at least locally, for my own students, I could ameliorate this situation and provide a semi-rigorous introduction to precalculus.

These notes are in constant state of revision. I would greatly appreciate comments, additions, exercises, figures, etc., in order to help me enhance them. I would also like to begin translating them into Spanish. Any help would be appreciated.

David A. Santos

Legal Notice

To the Student

These notes are provided for your benefit as an attempt to organise the salient points of the course. They are a *very terse* account of the main ideas of the course, and are to be used mostly to refer to central definitions and theorems. The number of examples is minimal. The *motivation* or informal ideas of looking at a certain topic, the ideas linking a topic with another, the worked-out examples, etc., are given in class. Hence these notes are not a substitute to lectures: **you must always attend to lectures**. The order of the notes may not necessarily be the order followed in the class.

There is a certain algebraic fluency that is necessary for a course at this level. These algebraic prerequisites would be difficult to codify here, as they vary depending on class response and the topic lectured. If at any stage you stumble in Algebra, seek help! I am here to help you!

Tutoring can sometimes help, but bear in mind that whoever tutors you may not be familiar with my conventions. Again, I am here to help! On the same vein, other books may help, but the approach presented here is at times unorthodox and finding alternative sources might be difficult.

Here are more recommendations:

- Read a section before class discussion, in particular, read the definitions.
- Class provides the informal discussion, and you will profit from the comments of your classmates, as well as gain confidence by providing your insights and interpretations of a topic. **Don't be absent!**
- I encourage you to form study groups and to discuss the assignments. Discuss among yourselves and help each other but don't be *parasites!* Plagiarising your classmates' answers will only lead you to disaster!
- Once the lecture of a particular topic has been given, take a fresh look at the notes of the lecture topic.
- Try to understand a single example well, rather than ill-digest multiple examples.
- Start working on the distributed homework ahead of time.
- **Ask questions during the lecture.** There are two main types of questions that you are likely to ask.

  1. *Questions of Correction: Is that a minus sign there?* If you think that, for example, I have missed out a minus sign or wrote P where it should have been Q,[1] then by all means, ask. No one likes to carry an error till line XLV because the audience failed to point out an error on line I. Don't wait till the end of the class to point out an error. Do it when there is still time to correct it!
  2. *Questions of Understanding: I don't get it!* Admitting that you do not understand something is an act requiring utmost courage. But if you don't, it is likely that many others in the audience also don't. On the same vein, if you feel you can explain a point to an inquiring classmate, I will allow you time in the lecture to do so. The best way to ask a question is something like: "How did you get from the second step to the third step?" or "What does it mean to complete the square?" Asseverations like "I don't understand" do not help me answer your queries. If I consider that you are asking the same questions too many times, it may be that you need extra help, in which case we will settle what to do outside the lecture.

[1] My doctoral adviser used to say "I said A, I wrote B, I meant C and it should have been D!

- Don't fall behind! The sequence of topics is closely interrelated, with one topic leading to another.
- You will need square-grid paper, a ruler (preferably a T-square), some needle thread, and a compass.
- The use of calculators is allowed, especially in the occasional lengthy calculations. However, when graphing, you will need to provide algebraic/analytic/geometric support of your arguments. The questions on assignments and exams will be posed in such a way that it will be of no advantage to have a graphing calculator.
- Presentation is critical. Clearly outline your ideas. When writing solutions, outline major steps and write in complete sentences. As a guide, you may try to emulate the style presented in the scant examples furnished in these notes.

Notation

$\in$	Belongs to.
$\notin$	Does not belong to.
$\forall$	For all (Universal Quantifier).
$\exists$	There exists (Existential Quantifier).
$\varnothing$	Empty set.
$P \implies Q$	P implies Q.
$P \Leftrightarrow Q$	P if and only if Q.
$\mathbb{N}$	The Natural Numbers $\{0,1,2,3,\ldots\}$.
$\mathbb{Z}$	The Integers $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.
$\mathbb{Q}$	The Rational Numbers.
$\mathbb{R}$	The Real Numbers.
$\mathbb{C}$	The Complex Numbers.
A^n	The set of n-tuples $\{(a_1,a_2,\ldots,a_n)\vert a_k \in A\}$.
$]a;b[$	The open finite interval $\{x \in \mathbb{R} : a < x < b\}$.
$[a;b]$	The closed interval $\{x \in \mathbb{R} : a \leq x \leq b\}$.
$]a;b]$	The semi-open interval $\{x \in \mathbb{R} : a < x \leq b\}$.
$[a;b[$	The semi-closed interval $\{x \in \mathbb{R} : a \leq x < b\}$.
$]a;+\infty[$	The infinite open interval $\{x \in \mathbb{R} : x > a\}$.
$]-\infty;a]$	The infinite closed interval $\{x \in \mathbb{R} : x \leq a\}$.
$\sum_{k=1}^{n} a_k$	The sum $a_1 + a_2 + \cdots + a_{n-1} + a_n$.

Chapter 1

Numbers and Notation

This chapter introduces essential notation and terminology that will be used throughout these notes.

1.1 Sets and Notation

1 Definition We will mean by a *set* a collection of well defined members or *elements*. A *subset* is a sub-collection of a set. We denote that B is a subset of A by the notation $B \subseteq A$ or sometimes $B \subset A$.[1]

Some sets of numbers will be referred to so often that they warrant special notation. Here are some of the most common ones.

$\varnothing$	Empty set.
$\mathbb{N}$	The Natural Numbers $\{0, 1, 2, 3, \ldots\}$.[2]
$\mathbb{Z}$	The Integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[3]
$\mathbb{Q}$	The Rational Numbers.[4]
$\mathbb{R}$	The Real Numbers.
$\mathbb{C}$	The Complex Numbers.

Observe that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

From time to time we will also use the following notation, borrowed from set theory and logic.

$\in$	Belongs to. Is an element of.
$\notin$	Does not belong to. Is not an element of.
$\forall$	For all (Universal Quantifier).
$\exists$	There exists (Existential Quantifier).
$P \implies Q$	P implies Q.
$P \Leftrightarrow Q$	P if and only if Q.

2 Example $-1 \in \mathbb{Z}$ but $\frac{1}{2} \notin \mathbb{Z}$.

3 Definition Let A be a set. If a belongs to the set A, then we write $a \in A$, read "a is an element of A." If a does not belong to the set A, we write $a \notin A$, read "a is not an element of A." The set that has no elements, that is *empty set*, will be denoted by $\varnothing$.

1.2 Real Numbers

Why all these systems of numbers?

Let us start with the strictly positive natural numbers. Primitive societies needed to count objects, say, their cows or sheep. Though some societies, like the Yanomame indians in Brazil or members of the CCP English and Social Sciences

[1] There is no agreement relating the choice. Some use $\subset$ to denote strict containment, that is, $A \subseteq B$ but $A \neq B$. In the case when we want to denote strict containment we will simply write $A \subsetneq B$.

Department[5] cannot count above 3, the need for counting is indisputable. In fact, many of these societies were able to make the following abstraction: add to a pile one pebble (or stone) for every sheep, in other words, they were able to make one-to-one correspondences. In fact, the word *Calculus* comes from the Latin for "stone."

Breaking an object into almost equal parts (that is, *fractioning* it) justifies the creation of the positive rational numbers. In fact, most ancient societies did very well with just the strictly positive rational numbers. The problems of counting and of counting broken pieces were solved completely with these numbers.

As societies became more and more sophisticated, the need for new numbers arose. For example, it is believed that the introduction of negative quantities arose as an accounting problem in Ancient India. Fair enough, write $+1$ if you have a rupee—or whatever unit that ancient accountant used—in your favour. Write -1 if you owe one rupee. Write 0 if you are rupeeless.

Thus we have constructed $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. In $\mathbb{Q}$ we have, so far, a very elegant system of numbers which allows us to perform four arithmetic operations (addition, subtraction, multiplication, and division)[6]and that has the notion of "order."

Why create $\mathbb{R}$?

Enter the Pythagorean Society in the picture, whose founder, Pythagoras lived 582 to 500 BC. This loony sect of Greeks forbade their members to eat beans. But their lunacy went even farther. Rather than studying numbers to solve everyday "real world problems"—as some misguided pedagogues insist—they tried to understand the very essence of numbers, to study numbers in the abstract. At the beginning it seems that they thought that the "only numbers" were rational numbers. But one of them, Hipassus of Metapontum, was able to prove that the length of hypotenuse of a right triangle whose legs[7] had unit length could not be expressed as the ratio of two integers and hence, it was *irrational*.

4 Theorem [Hipassos of Metapontum]$\sqrt{2}$ is irrational.

Proof: *Assume there is $s \in \mathbb{Q}$ such that $s^2 = 2$. We can find integers $m, n \neq 0$ such that $s = \frac{m}{n}$. The crucial part of the argument is that we can choose m, n such that this fraction be in least terms, and hence, m, n must have opposite parity. Now, $m^2 s^2 = n^2$, that is $2m^2 = n^2$. This means that n^2 is even. But then n itself must be even, since the product of two odd numbers is odd. Thus $n = 2a$ for some non-zero integer a (since $n \neq 0$). This means that $2m^2 = (2a)^2 = 4a^2 \implies m^2 = 2a^2$. This means once again that m is even. But then we have a contradiction, since m and n were of opposite parity.* ❑

This is one of the very first theorems ever proved. It befits you, if you want to be called mathematically literate, to know its proof.

The shock caused by Hipassos' result was so great (remember the Pythagoreans were a cult), that they drowned him. Fortunately, mathematicians have matured since then and the task of burning people at the stake or flying planes into skyscrapers has fallen into other hands.

5 Example Write the infinitely repeating decimal $0.3\overline{45} = 0.345454545\ldots$ as the quotient of two natural numbers.

Solution: The trick is to obtain multiples of $x = 0.345454545\ldots$ so that they have the same infinite tail, and then subtract these tails, cancelling them out.[8] So observe that

$$10x = 3.45454545\ldots; 1000x = 345.454545\ldots \implies 1000x - 10x = 342 \implies x = \frac{342}{990} = \frac{19}{55}.$$

By mimicking the above example, the following should be clear: decimals whose decimal expansion terminates or repeats are rational numbers. Since we are too cowardly to prove the next statement, we prefer to call it a

[5]Among these, many are Philosophers, whom, though unsuccessful in finding their Philosopher's Stone, have found renal calculi.

[6] "Reeling and Writhing, of course, to begin with, "the Mock Turtle replied, "and the different branches of Arithmetic–Ambition, Distraction, Uglification, and Derision."

[7]The appropriate word here is "cathetus."

[8]That this cancellation is meaningful depends on the concept of *convergence*, of which we may talk more later.

6 Fact Real numbers are either rational or irrational. Rational numbers are those whose decimal expansion terminates or repeats. Irrational numbers are those whose decimal expansion is infinite and does not repeat.

After the discovery that $\sqrt{2}$ was irrational, suspicion arose that there were other irrational numbers. In fact, Archimedes suspected that π was irrational, a fact that wasn't proved till the XIX-th Century by Lambert. The "irrationalities" of $\sqrt{2}$ and π are of two entirely "different flavours," but we will need several more years of mathematical study[9] to even comprehend the meaning of that assertion.

Of course, by simply "looking" at the decimal expansion of a number we can't tell whether is irrational or rational without having more information. Your calculator probably gives about 9 decimal places when you try to compute $\sqrt{2}$, say, it says $\sqrt{2} \approx 1.414213562$. What happens after the final 2 is the interesting question. Do we have a pattern or do we not?

7 Example We expect a number like

$$0.1001000010000000001\ldots,$$

where there are $2, 4, 8, 16, \ldots$ zeroes between consecutive ones, to be irrational. Discuss!

By looking at decimal expansion of the real numbers, we may reach the following conclusion.

8 Fact Between any two different real numbers there is a rational number. Between any two real numbers there is an irrational number.

So what now? We know that the real numbers have all the nice arithmetical and order properties of $\mathbb{Q}$ and that also, contain all possible decimal expansions? What other properties do they have?

9 Definition A number u is an *upper bound* for a set of numbers A if for all $a \in A$ we have $a \leq u$. The smallest such upper bound is called the *supremum* of the set A. Similarly, a number l is a *lower bound* for a set of numbers B if for all $b \in B$ we have $l \leq b$. The largest such lower bound is called the *infimum* of the set B.

The real numbers have the following property, which further distinguishes them from the rational numbers.

10 Axiom (Completeness of $\mathbb{R}$) Any set of real numbers which is bounded above has a supremum. Any set of real numbers which is bounded below has a infimum.

Observe that the rational numbers are not complete. For example, there is no largest rational number in the set

$$\{x \in \mathbb{Q} : x^2 < 2\}$$

since $\sqrt{2}$ is irrational and for any good rational approximation to $\sqrt{2}$ we can always find a better one.

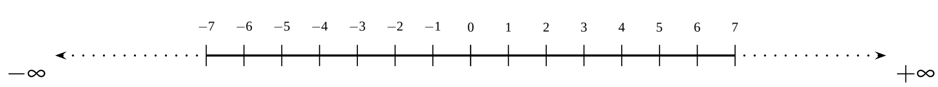

Figure 1.1: The Real Line.

[9]Or in the case of people in the English and the Social Sciences Departments, as many lifetimes as a cat.

Geometrically, each real number can be viewed as a point on a straight line. We make the convention that we orient the real line with 0 as the origin, the positive numbers increasing towards the right from 0 and the negative numbers decreasing towards the left of 0, as in figure 1.1. The Completeness Axiom says, essentially, that this line has no "holes."

We append the object $+\infty$, which is larger than any real number, and the object $-\infty$, which is smaller than any real number. Letting $x \in \mathbb{R}$, we make the following conventions.

$$(+\infty)+(+\infty) = +\infty \tag{1.1}$$

$$(-\infty)+(-\infty) = -\infty \tag{1.2}$$

$$x+(+\infty) = +\infty \tag{1.3}$$

$$x+(-\infty) = -\infty \tag{1.4}$$

$$x(+\infty) = +\infty \quad \text{if } x > 0 \tag{1.5}$$

$$x(+\infty) = -\infty \quad \text{if } x < 0 \tag{1.6}$$

$$x(-\infty) = -\infty \quad \text{if } x > 0 \tag{1.7}$$

$$x(-\infty) = +\infty \quad \text{if } x < 0 \tag{1.8}$$

$$\frac{x}{\pm\infty} = 0 \tag{1.9}$$

Observe that we leave the following undefined:

$$\frac{\pm\infty}{\pm\infty}, \quad (+\infty)+(-\infty), \quad 0(\pm\infty).$$

Homework

11 Problem Give examples, if at all possible, of the following.

❶ the sum of two rationals giving an irrational number.

❷ the sum of two irrationals giving an irrational number.

❸ the sum of two irrationals giving a rational number.

❹ the product of a rational and an irrational giving an irrational number.

❺ the product of a rational and an irrational giving a rational number.

❻ the product of two irrationals giving an irrational number.

❼ the product of two irrationals giving a rational number.

12 Problem Describe the following sets explicitly by either providing a list of their elements or an interval.

❶ $\{x \in \mathbb{R} : x^3 = 8\}$

❷ $\{x \in \mathbb{R} : |x|^3 = 8\}$

❸ $\{x \in \mathbb{R} : |x| = -8\}$

❹ $\{x \in \mathbb{R} : |x| < 4\}$

❺ $\{x \in \mathbb{Z} : |x| < 4\}$

❻ $\{x \in \mathbb{R} : |x| < 1\}$

❼ $\{x \in \mathbb{Z} : |x| < 1\}$

❽ $\{x \in \mathbb{Z} : x^{2002} < 0\}$

13 Problem Describe explicitly the set $\{x \in \mathbb{Z} : x < 0, 1000 < x^2 < 2003\}$ by listing its elements.

14 Problem Write the infinitely repeating decimal $0.\overline{123} = 0.123123123\ldots$ as the quotient of two positive integers.

1.3 The Square of a Real Number is Positive

Vocabulary Alert! *We will call a number x* positive *if $x \geq 0$ and* strictly positive *if $x > 0$. Similarly, we will call a number y* negative *if $y \leq 0$ and* strictly negative *if $y < 0$. This usage differs from most Anglo-American books, who prefer such terms as* non-negative *and* non-positive.

We start this section by recalling two very useful, but seldom exploited, algebraic identities. Let x and y be real numbers. Then

$$x^2 - y^2 = (x-y)(x+y) \tag{1.10}$$

$$(x \pm y)^2 = x^2 \pm 2xy + y^2. \tag{1.11}$$

Their proof can be obtained by simple multiplication.

15 Example Compute $(123456789)^2 - (123456791)(123456787)$ mentally.

Solution: We will see that this computation gives 4. Put $x = 123456789$. Then we want $x^2 - (x-2)(x+2) = x^2 - (x^2 - 4) = 4$. Remarkable! If you tried this in your calculator, it will probably have a memory overflow and return "ERROR."

The square of every real number x is positive[10], that is, for all real numbers x we have $x^2 \geq 0$. This simple fact gives us the following remarkable inequalities.

16 Theorem (Arithmetic Mean-Geometric Mean Inequality) Let a, b be positive real numbers. Then

$$\sqrt{ab} \leq \frac{a+b}{2}$$

with equality if and only if $a = b$.

Proof: *Since the square of any real number is positive,*

$$(\sqrt{a} - \sqrt{b})^2 \geq 0.$$

Expanding,

$$a - 2\sqrt{ab} + b \geq 0,$$

from where the desired inequality follows. Clearly, equality also follows if and only if $a = b$. ❑

17 Corollary (Harmonic Mean-Geometric Mean Inequality) Let a, b be strictly positive real numbers. Then

$$\frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab}$$

with equality if and only if $a = b$.

Proof: *By the Arithmetic Mean-Geometric Mean Inequality*

$$\sqrt{\frac{1}{a} \cdot \frac{1}{b}} \leq \frac{\frac{1}{a} + \frac{1}{b}}{2},$$

whence the desired result follows upon rearrangement. Clearly, equality also follows if and only if $a = b$. ❑

18 Example The sum of two positive real numbers is 50. Find their maximum product.

Let x, y be these positive real numbers with $x + y = 50$. By the Arithmetic Mean-Geometric Mean Inequality

$$\sqrt{xy} \leq \frac{x+y}{2} = 25.$$

Thus their maximum product is 625, since

$$xy \leq 25^2 = 625.$$

[10] We use the word *positive* to indicate a quantity ≥ 0, and use the term *strictly positive* for a quantity > 0. Similarly with *negative* (≤ 0) and *strictly negative* (< 0).

Homework

19 Problem Describe explicitly the set

$$\{x \in \mathbb{Z} : -13 \leq x \leq 15, x \text{ is divisible by } 3\}$$

by listing its elements.

20 Problem Prove that if a, b, c are positive real numbers then

$$(a+b)(b+c)(c+a) \geq 8abc.$$

21 Problem If a, b, c, d, are real numbers such that

$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da,$$

prove that $a = b = c = d$.

22 Problem The values of a, b, c, and d are $1, 2, 3$ and 4 but not necessarily in that order. What is the largest possible value of $ab + bc + cd + da$?

23 Problem Prove that if $r \geq s \geq t$ then

$$r^2 - s^2 + t^2 \geq (r - s + t)^2.$$

24 Problem Prove that

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

and that

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Deduce now the arithmetic-mean-geometric-mean inequality for three positive real numbers, that is, prove that if $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, then

$$\sqrt[3]{\alpha\beta\gamma} \leq \frac{\alpha + \beta + \gamma}{3}.$$

1.4 Intervals

25 Definition An *interval* I is a subset of the real numbers with the following property: if $s \in I$ and $t \in I$, and if $s < x < t$, then $x \in I$. In other words, intervals are those subsets of real numbers with the property that every number between two elements is also contained in the set. Since there are infinitely many decimals between two different real numbers, intervals with distinct endpoints contain infinitely many members. Table 1.1 shews the various types of intervals.

Observe that we indicate that the endpoints are included by means of shading the dots at the endpoints and that the endpoints are excluded by not shading the dots at the endpoints. [11]

Interval Notation	Set Notation	Graphical Representation
$[a;b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$ [12]	
$]a;b[$	$\{x \in \mathbb{R} : a < x < b\}$	
$[a;b[$	$\{x \in \mathbb{R} : a \leq x < b\}$	
$]a;b]$	$\{x \in \mathbb{R} : a < x \leq b\}$	
$]a;+\infty[$	$\{x \in \mathbb{R} : x > a\}$	
$[a;+\infty[$	$\{x \in \mathbb{R} : x \geq a\}$	
$]-\infty;b[$	$\{x \in \mathbb{R} : x < b\}$	
$]-\infty;b]$	$\{x \in \mathbb{R} : x \leq b\}$	
$]-\infty;+\infty[$	$\mathbb{R}$	

Table 1.1: Intervals.

[11] It may seem like a silly analogy, but think that in $[a;b]$ the brackets are "arms" "hugging" a and b, but in $]a;b[$ the "arms" are repulsed. "Hugging" is thus equivalent to including the endpoint, and "repulsing" is equivalent to excluding the endpoint.

We will often use the symbol $\iff$ for "if and only if", and the symbol $\implies$, "implies." The symbol $\approx$ means *approximately*. From time to time, we will also use the set theoretic notation below.

26 Definition The *union* of two sets A and B, is the set

$$A \cup B = \{x : (x \in A) \text{ or } (x \in B)\}.$$

This is read "A union B." See figure 1.2.

The *intersection* of two sets A and B, is

$$A \cap B = \{x : (x \in A) \text{ and } (x \in B)\}.$$

This is read "A intersection B." See figure 1.3.

The *difference* of two sets A and B, is

$$A \setminus B = \{x : (x \in A) \text{ and } (x \notin B)\}.$$

This is read "A set minus B." See figure 1.4.

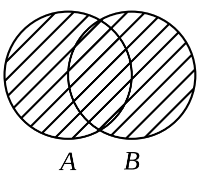

Figure 1.2: $A \cup B$

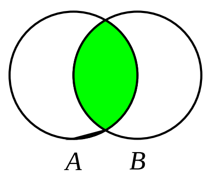

Figure 1.3: $A \cap B$

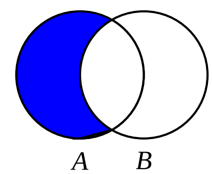

Figure 1.4: $A \setminus B$

27 Example Let $A = \{1,2,3,4,5,6\}$, and $B = \{1,3,5,7,9\}$. Then

$$A \cup B = \{1,2,3,4,5,6,7,9\}, \quad A \cap B = \{1,3,5\}, \quad A \setminus B = \{2,4,6\}, \quad B \setminus A = \{7,9\}.$$

28 Example If $A = [-10;2]$, $B =]-\infty;1[$, then

$$A \cap B = [-10;1[, \quad A \cup B =]-\infty;2], \quad A \setminus B = [1;2], \quad B \setminus A =]-\infty;-10[$$

29 Example Let $A = \left[1-\sqrt{3}\,;1+\sqrt{2}\right]$, $B = \left[\frac{\pi}{2}\,;\pi\right[$. By approximating the endpoints. To three decimal places $1-\sqrt{3} \approx -0.732$, $1+\sqrt{2} \approx 2.414$, $\frac{\pi}{2} \approx 1.571$, $\pi \approx 3.142$. Thus

$$A \cap B = \left[\frac{\pi}{2}\,;1+\sqrt{2}\right], \quad A \cup B = \left[1-\sqrt{3}\,;\pi\right[, \quad A \setminus B = \left[\,1-\sqrt{3};\frac{\pi}{2}\,\right[, \quad B \setminus A = \left]1+\sqrt{2}\,;\pi\right[.$$

Homework

30 Problem Let $A = \{a,b,c,d,e,f\}$ and $B = \{a,e,i,o,u\}$. Find $A \cup B$, $A \cap B$, $A \setminus B$ and $B \setminus A$.

31 Problem Let $C =]-5;5[$, $D =]-1;+\infty[$. Find $C \cap D$, $C \cup D$, $C \setminus D$, and $D \setminus C$.

32 Problem Let $C =]-5;3[$, $D = [4;+\infty[$. Find $C \cap D$, $C \cup D$, $C \setminus D$, and $D \setminus C$.

33 Problem Let $C = [-1;-2+\sqrt{3}[$, $D = [-0.5;\sqrt{2}-1]$. Find $C \cap D$, $C \cup D$, $C \setminus D$, and $D \setminus C$.

34 Problem Consider 101 different points $x_1, x_2, \ldots, x_{101}$ belonging to the interval $[0;1[$. Shew that there are at least two say x_i and $x_j, i \neq j$, such that

$$|x_i - x_j| \leq \frac{1}{100}$$

35 Problem (Dirichlet's Approximation Theorem) Shew that $\forall x \in \mathbb{R},\ \forall N \in \mathbb{N}, N > 1,\ \exists (h \in \mathbb{N} \wedge k \in \mathbb{N})$ with $0 < k \leq N$ such that

$$|x - \frac{h}{k}| < \frac{1}{Nk}.$$

36 Problem Prove that if x is irrational then there are infinitely many rational numbers $\frac{h}{k}$, $k \in \mathbb{N}$ such that

$$|x - \frac{h}{k}| < \frac{1}{k^2}.$$

1.5 Sets on the Plane

37 Definition Let $A_1, A_2, \ldots, A_n$, be sets. The *Cartesian Product* of these n sets is defined and denoted by

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_k \in A_k\},$$

that is, the set of all ordered n-tuples whose elements belong to the given sets.

In the particular case when all the A_k are equal to a set A, we write

$$A_1 \times A_2 \times \cdots \times A_n = A^n.$$

38 Example If $A = \{-1, -2\}$ and $B = \{-1, 2\}$ then

$$A \times B = \{(-1,-1), (-1,2), (-2,-1), (-2,2))\},$$

$$B \times A = \{(-1,-1), (-1,-2), (2,-1), (2,-2)\},$$

$$A^2 = \{(-1,-1), (-1,-2), (-2,-1), (-2,-2)\},$$

$$B^2 = \{(-1,-1), (-1,2), (2,-1), (2,2)\}.$$

Notice that these sets are all different, even though some elements are shared.

39 Example $(-1,2) \in \mathbb{Z}^2$ but $(-1, \sqrt{2}) \notin \mathbb{Z}^2$.

40 Example $(-1, \sqrt{2}) \in \mathbb{Z} \times \mathbb{R}$ but $(-1, \sqrt{2}) \notin \mathbb{R} \times \mathbb{Z}$.

41 Definition $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$—the *real Cartesian Plane*—- is the set of all ordered pairs (x, y) of real numbers.

We represent the elements of $\mathbb{R}^2$ graphically as follows. Intersect perpendicularly two copies of the real number line. These two lines are the *axes*. Their point of intersection—which we label $O = (0,0)$— is the *origin*. To each point P on the plane we associate an ordered pair $P = (x, y)$ of real numbers. Here x is the *abscissa*[13], which measures the horizontal distance of our point to the origin, and y is the *ordinate*, which measures the vertical distance of our point to the origin. The points x and y are the *coordinates* of P. This manner of dividing the plane and labelling its points is called the *Cartesian coordinate system.* The horizontal axis is called the *x-axis* and the vertical axis is called the *y-axis*. It is therefore sufficient to have two numbers x and y to completely characterise the position of a point $P = (x, y)$ on the plane $\mathbb{R}^2$.

42 Example Sketch the region $\mathscr{R} = \{(x, y) \in \mathbb{R}^2 : 1 < x < 3, \quad 2 < y < 4\}$.

Solution: The region is a square, excluding its boundary. The graph is shewn in figure 1.5, where we have dashed the boundary lines in order to represent their exclusion.

43 Example The region $\mathscr{R} = [1;3] \times [-3;+\infty[$ is the infinite half strip on the plane sketched in figure 1.6. The boundary lines are solid, to indicate their inclusion. The upper boundary line is toothed, to indicate that it continues to infinity.

[13] From the Latin *linea abscissa* or line cut-off.

44 Example The region $\mathscr{R} = \{(x,y) \in \mathbb{R}^2 : |x| \leq 2, \quad |y| \leq 2\}$ is the 4×4 square sketched in figure 1.7.

45 Example A quadrilateral has vertices at $A = (5,-9)$, $B = (2,3)$, $C = (0,2)$, and $D = (-8,4)$. Find the area, in square units, of quadrilateral $ABCD$.

Solution: Enclose quadrilateral $ABCD$ in right $\triangle AED$, and draw lines parallel to the y-axis in order to form trapezoids $AEFB$, $FBCG$, and right $\triangle GCD$, as in figure 1.8. The area $[ABCD]$ of quadrilateral $ABCD$ is thus given by

$$\begin{aligned} [ABCD] &= [AED]-[AEFB]-[FBCG]-[GCD] \\ &= \tfrac{1}{2}(AE)(DE)-\tfrac{1}{2}(FE)(FB+AE)- \\ &\qquad -\tfrac{1}{2}(GF)(GC+FB)-\tfrac{1}{2}(DG)(GC) \\ &= \tfrac{1}{2}(13)(13)-\tfrac{1}{2}(3)(13+1)-\tfrac{1}{2}(2)(2+1)-\tfrac{1}{2}(8)(2) \\ &= 84.5-21-3-8 \\ &= 52.5. \end{aligned}$$

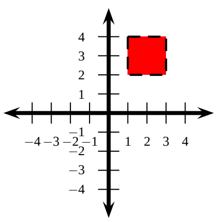

Figure 1.5: Example 42.

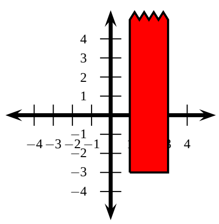

Figure 1.6: Example 43

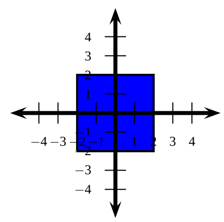

Figure 1.7: Example 44.

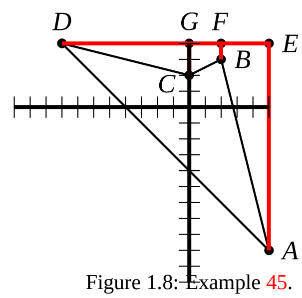

Figure 1.8: Example 45.

Homework

46 Problem Sketch the following regions on the plane.

❶ $R_1 = \{(x,y) \in \mathbb{R}^2 : x \leq 2\}$

❷ $R_2 = \{(x,y) \in \mathbb{R}^2 : y \geq -3\}$

❸ $R_3 = \{(x,y) \in \mathbb{R}^2 : |x| \leq 3, |y| \leq 4\}$

❹ $R_4 = \{(x,y) \in \mathbb{R}^2 : |x| \leq 3, |y| \geq 4\}$

❺ $R_5 = \{(x,y) \in \mathbb{R}^2 : x \leq 3, y \geq 4\}$

❻ $R_6 = \{(x,y) \in \mathbb{R}^2 : x \leq 3, y \leq 4\}$

47 Problem Find the area of $\triangle ABC$ where $A = (-1,0)$, $B = (0,4)$ and $C = (1,-1)$.

48 Problem Let $A = [-10;5]$, $B = \{5,6,11\}$ and $C =]-\infty;6[$. Answer the following true or false.

❶ $5 \in A$.

❷ $6 \in C$.

❸ $(0,5,3) \in A \times B \times C$.

❹ $(0,-5,3) \in A \times B \times C$.

❺ $(0,5,3) \in C \times B \times C$.

❻ $A \times B \times C \subseteq C \times B \times C$.

❼ $A \times B \times C \subseteq C^3$.

49 Problem True or false: $(\mathbb{R} \setminus \{0\})^2 = \mathbb{R}^2 \setminus \{(0,0)\}$.

50 Problem A *lattice point is a point* $(x,y) \in \mathbb{Z}^2$, that is, a point with integral coordinates. Prove that there is no equilateral triangle all whose coordinates are lattice points.

1.6 Neighbourhood of a point

Before stating the main definition of this section, let us consider the concept of "nearness." What does it mean for one point to be "near" another point? We could argue that 1 is near to 0, but, for some purposes, this distance could be "far." We could certainly see that 0.5 is closer to 0 than 1 is, but then again, for some purposes, even this distance could be "far." Mentioning a specific number "near" 0, like 1 or 0.5 fails in what we desire for "nearness" because mentioning a specific point immediately gives a “static” quality to “nearness”: once you mention a specific point, you could mention infinitely many more points which are closer than the point you mentioned. The points in the sequence

$$0.1, \qquad 0.01, \qquad 0.001, \qquad 0.0001, \qquad \ldots$$

get closer and closer to 0 with an arbitrary precision. Notice that this sequence approaches 0 through values >0. This arbitrary precision is what will be the gist of our concept of “nearness.” “Nearness” is dynamic: it involves the ability of getting closer to a point with any desired degree of accuracy. It is not static.

Again, the points in the sequence

$$-\frac{1}{2}, \qquad -\frac{1}{4}, \qquad -\frac{1}{8}, \qquad -\frac{1}{16}, \qquad \ldots$$

are arbitrarily close to 0, but they “approach” 0 from the left. Once again, the sequence

$$+\frac{1}{2}, \qquad -\frac{1}{3}, \qquad +\frac{1}{4}, \qquad -\frac{1}{5}, \qquad \ldots$$

approaches 0 from both above and below. After this long preamble, we may formulate our first definition.

51 Definition The notation $x \longrightarrow a$, read “x tends to a,” means that x is very close, with an arbitrary degree of precision, to a. Here x can approach a through values smaller or larger than a. We write $x \longrightarrow a+$ (read “x tends to a from the right”) to mean that x approaches a through values larger than a and we write $x \longrightarrow a-$ (read “x tends to a from the left”) we mean that x approaches a through values smaller than a.

52 Definition A *neighbourhood of a point a* is an interval containing a.

Notice that the definition of neighbourhood does not rule out the possibility that a may be an endpoint of the the interval. Our interests will be mostly on arbitrarily small neighbourhoods of a point. Schematically we have a diagram like figure 1.9.

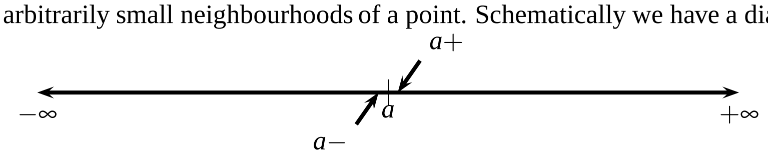

Figure 1.9: A neighbourhood of a.

Answers

11 1) this is impossible. 2) take both numbers to be $\sqrt{2}$. Their sum is $2\sqrt{2}$ which is also irrational. 3) take one number to be $\sqrt{2}$ and the other $-\sqrt{2}$. Their sum is 0, which is rational. 4) take the rational number to be 1 and the irrational to be $\sqrt{2}$. Their product is $1\cdot\sqrt{2} = \sqrt{2}$. 5) take the rational number to be 0 and the irrational to be $\sqrt{2}$. Their product is $0\cdot\sqrt{2} = 0$. 6) take one irrational number to be $\sqrt{2}$ and the other to be $\sqrt{3}$. Their product is $\sqrt{2}\cdot\sqrt{3} = \sqrt{6}$. 7) take one irrational number to be $\sqrt{2}$ and the other to be $\frac{1}{\sqrt{2}}$. Their product is $\sqrt{2}\cdot\frac{1}{\sqrt{2}} = 1$.

12 (i) $\{2\}$, (ii) $\{-2,2\}$, (iii) $\varnothing$, (iv) $]-4;4[$, (v) $\{-3,-2,-1,0,1,2,3\}$,(vi) $]-1;1[$, (vii) $\{0\}$, (viii) $\varnothing$

13 $\{-32,-33,-34,-35,-36,-37,-38,-39,-40,-41,-42,-43,-44\}$

14 If $x = 0.123123123\ldots$ then $1000x = 0.123123123\ldots$ giving $1000x - x = 123$, since the tails cancel out. This results in $x = \frac{123}{999} = \frac{41}{333}$.

19 $\{-12,-9,-6,-3,0,3,6,9,12,15\}$

20 The result quickly follows upon multiplying the three inequalities

$$a+b \geq 2\sqrt{ab},$$

$$b+c \geq 2\sqrt{bc},$$

$$c+a \geq 2\sqrt{ca}.$$

21 Transposing,

$$a^2 - ab + b^2 - bc + c^2 - dc + d^2 - da = 0,$$

or

$$\frac{a^2}{2} - ab + \frac{b^2}{2} + \frac{b^2}{2} - bc + \frac{c^2}{2} + \frac{c^2}{2} - dc + \frac{d^2}{2} + \frac{d^2}{2} - da + \frac{a^2}{2} = 0.$$

Factoring,

$$\frac{1}{2}(a-b)^2 + \frac{1}{2}(b-c)^2 + \frac{1}{2}(c-d)^2 + \frac{1}{2}(d-a)^2 = 0.$$

As the sum of positive quantities is zero only when the quantities themselves are zero, we obtain $a = b, b = c, c = d, d = a$, which proves the assertion.

22 25

30 $A\cup B = \{a,b,c,d,e,f,i,o,u\}, A\cap B = \{a,e\}, A\setminus B = \{b,c,d,f\}, B\setminus A = \{i,o,u\}$

31 $]-1;5[,]-5;+\infty[,]-5;-1], [5;+\infty[$

32 $\varnothing,]-5;3[\cup[4;+\infty[,]-5;3[, [4;+\infty[$

33 $[-0.5;-2+\sqrt{3}[, [-1;\sqrt{2}-1], [-1;-0.5[, [-2+\sqrt{3};\sqrt{2}-1]$

35 Hint: Consider the $N+1$ numbers $tx - \lfloor tx \rfloor, t = 0,1,2,\ldots,N$.

47 4.5 square units.

48 TFTFTTF

49 False. $(\mathbb{R}\setminus\{0\})^2$ consists of the plane minus the axes. $\mathbb{R}^2\setminus\{(0,0)\}$ consists of the plane minus the origin.

Chapter 2

Distance and Curves on the Plane

The main objective of this chapter is to introduce the distance formula for two points on the plane, and by means of this distance formula, the linking of certain equations with certain curves on the plane. Thus the main object of these notes, that of relating a graph to a formula, is partially answered.

2.1 Distance on the Real Line

53 Definition Let $x \in \mathbb{R}$. The *absolute value of x*—denoted by $|x|$—is defined by

$$\boxed{|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}}$$

The absolute value of a real number is thus the distance of that real number to 0, and hence $|x-y|$ is the distance between x and y on the real line. Below are some properties of the absolute value. Here x, y, t are all real numbers.

$$-|x| \leq x \leq |x|. \tag{2.1}$$

$$|x-y| = |y-x| \tag{2.2}$$

$$\sqrt{x^2} = |x| \tag{2.3}$$

$$|x|^2 = |x^2| = x^2 \tag{2.4}$$

$$|x| \leq t \iff -t \leq x \leq t \quad (t \geq 0) \tag{2.5}$$

$$|x| \geq t \iff x \leq -t \text{ or } x \geq t \quad (t \geq 0) \tag{2.6}$$

54 Example Write without absolute value signs:

❶ $|\sqrt{3}-2|$,

❷ $|\sqrt{7}-\sqrt{5}|$,

❸ $||\sqrt{7}-\sqrt{5}|-|\sqrt{3}-2||$

Solution: We have

❶ since $2 > 1.74 > \sqrt{3}$, we have $|\sqrt{3}-2| = 2-\sqrt{3}$.

❷ since $\sqrt{7} > \sqrt{5}$, we have $|\sqrt{7}-\sqrt{5}| = \sqrt{7}-\sqrt{5}$.

❸ by virtue of the above calculations,

$$||\sqrt{7}-\sqrt{5}|-|\sqrt{3}-2|| = |\sqrt{7}-\sqrt{5}-(2-\sqrt{3})| = |\sqrt{7}+\sqrt{3}-\sqrt{5}-2|.$$

The question we must now answer is whether $\sqrt{7}+\sqrt{3} > \sqrt{5}+2$. But $\sqrt{7}+\sqrt{3} > 4.38 > \sqrt{5}+2$ and hence

$$|\sqrt{7}+\sqrt{3}-\sqrt{5}-2| = \sqrt{7}+\sqrt{3}-\sqrt{5}-2.$$

55 Example If $x < -2$ prove that $|1-|1+x|| = -2-x$.

Solution: If $x < -2$ then $1+x < -1$ and hence $|1+x| = -(1+x) = -1-x$. Thus $|1-|1+x|| = |1-(-1-x)| = |2+x|$. But since $x < -2$, $x+2 < 0$ and so $|2+x| = -2-x$. We conclude that $|1-|1+x|| = -2-x$.

56 Example Find all real solutions to $|x+1|+|x+2|-|x-3| = 5$.

Solution: The vanishing points for the terms are $x = -1$, $x = -2$ and $x = 3$. We decompose $\mathbb{R}$ into (overlapping) intervals with endpoints at the places where each of the expressions in absolute values vanish. Thus we have

$$\mathbb{R} =]-\infty;-2] \cup [-2;-1] \cup [-1;3] \cup [3;+\infty[.$$

We examine the sign diagram

$x \in$	$]-\infty;-2]$	$[-2;-1]$	$[-1;3]$	$[3;+\infty[$
$\|x+2\| =$	$-x-2$	$x+2$	$x+2$	$x+2$
$\|x+1\| =$	$-x-1$	$-x-1$	$x+1$	$x+1$
$\|x-3\| =$	$-x+3$	$-x+3$	$-x+3$	$x-3$
$\|x+2\|+\|x+1\|-\|x-3\| =$	$-x-6$	$x-2$	$3x$	$x+6$

Thus on $]-\infty;-2]$ we need $-x-6 = 5$ from where $x = -11$. On $[-2;-1]$ we need $x-2 = 5$ meaning that $x = 7$. Since $7 \notin [-2;-1]$, this solution is spurious. On $[-1;3]$ we need $3x = 5$, and so $x = \frac{5}{3}$. On $[3;+\infty[$ we need $x+6 = 5$, giving the spurious solution $x = -1$. Upon assembling all this, the solution set is

$$\left\{-11, \frac{5}{3}\right\}.$$

We will now prove some fundamental inequalities satisfied by absolute values.

57 Theorem (Triangle Inequality) Let $(a,b) \in \mathbb{R}^2$. Then

$$\boxed{|a+b| \le |a|+|b|.} \tag{2.7}$$

Proof: *From 2.1, by addition,*

$$-|a| \le a \le |a|$$

to

$$-|b| \le b \le |b|$$

we obtain

$$-(|a|+|b|) \le a+b \le (|a|+|b|),$$

whence the theorem follows. ❑

58 Corollary Let $(a,b) \in \mathbb{R}^2$. Then

$$\boxed{||a|-|b|| \leq |a-b|}. \tag{2.8}$$

Proof: *We have*

$$|a| = |a-b+b| \leq |a-b|+|b|,$$

giving

$$|a|-|b| \leq |a-b|.$$

Similarly,

$$|b| = |b-a+a| \leq |b-a|+|a| = |a-b|+|a|,$$

gives

$$|b|-|a| \leq |a-b|.$$

The stated inequality follows from this. ❑

Homework

59 Problem Write without absolute values: $|\sqrt{3}-\sqrt{|2-\sqrt{15}|}\,|$

60 Problem Write without absolute values if $x > 2$: $|x-|1-2x||$.

61 Problem Find all the real solutions to $|5x-2| = |2x+1|$.

62 Problem Find all real solutions to $|x-2|+|x-3| = 1$.

63 Problem Find the set of solutions to the equation $|x|+|x-1| = 2$.

64 Problem Find the solution set to the equation $|x|+|x-1| = 1$.

65 Problem Find the solution set to the equation $|2x|+|x-1|-3|x+2| = 1$.

66 Problem Find the solution set to the equation $|2x|+|x-1|-3|x+2| = -7$.

67 Problem Find the solution set to the equation $|2x|+|x-1|-3|x+2| = 7$.

68 Problem If $x < 0$ prove that $\left|x-\sqrt{(x-1)^2}\right| = 1-2x$.

69 Problem Find the real solutions, if any, to $|x^2-3x| = 2$.

70 Problem Find the real solutions, if any, to $x^2-2|x|+1 = 0$.

71 Problem Find the real solutions, if any, to $x^2-|x|-6 = 0$.

72 Problem Find the real solutions, if any, to $x^2 = |5x-6|$.

73 Problem Prove that if $x \leq -3$, then $|x+3|-|x-4|$ is constant.

74 Problem Prove that $\max(a,b) = \dfrac{a+b+|a-b|}{2}$.

75 Problem Prove that $\min(a,b) = \dfrac{a+b-|a-b|}{2}$.

2.2 Distance on the Real Plane

We now turn our attention to the plane, which we denote by the symbol $\mathbb{R}^2$.

Consider two points $A = (x_1,y_1), B = (x_2,y_2)$ on the Cartesian plane, as in figure 2.1. Dropping perpendicular lines to C, as in the figure, we can find their Euclidean distance AB with the aid of the Pythagorean Theorem. For

$$AB^2 = AC^2+BC^2,$$

translates into

$$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

This motivates the following definition.

76 Definition Let $(x_1,y_1),(x_2,y_2)$ be points on the Cartesian plane. The *Euclidean distance* between them is given by

$$\mathbf{d}\langle (x_1,y_1),(x_2,y_2)\rangle = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}. \tag{2.9}$$

77 Example Find the Euclidean distance between $(-1,2)$ and $(-3,8)$.

Solution:

$$\mathbf{d}\langle (-1,2),(-3,-8)\rangle = \sqrt{(-1-(-3))^2+(2-8)^2} = \sqrt{40} = 2\sqrt{10} \approx 6.32.$$

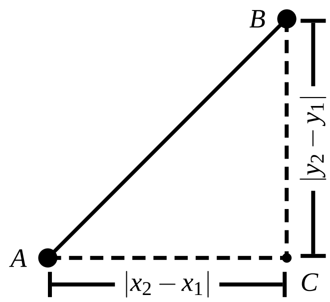

Figure 2.1: Distance between two points.

78 Example The point $(x,1)$ is at distance $\sqrt{11}$ from the point $(1,-x)$. Find the possible values of x.

Solution:

$$\begin{array}{rrcl} & \mathbf{d}\langle (x,1),(1,-x)\rangle & = & \sqrt{11} \\ \iff & \sqrt{(x-1)^2+(1+x)^2} & = & \sqrt{11} \\ \iff & (x-1)^2+(1+x)^2 & = & 11 \\ \iff & 2x^2+2 & = & 11. \end{array}$$

This yields $x = -\frac{3\sqrt{2}}{2}$ or $x = \frac{3\sqrt{2}}{2}$.

79 Example Find the point equidistant from $A=(-1,3), B=(2,4)$ and $C=(1,1)$.

Solution: Let (x,y) be the point sought. Then

$$\mathbf{d}\langle (x,y),(-1,3)\rangle = \mathbf{d}\langle (x,y),(2,4)\rangle \implies (x+1)^2+(y-3)^2 = (x-2)^2+(y-4)^2,$$
$$\mathbf{d}\langle (x,y),(-1,3)\rangle = \mathbf{d}\langle (x,y),(1,1)\rangle \implies (x+1)^2+(y-3)^2 = (x-1)^2+(y-1)^2.$$

This gives the two systems of linear equations

$$2x+1-6y+9 = -4x+4-8y+16,$$
$$2x+1-6y+9 = -2x+1-2y+1,$$

or

$$6x+2y = 10,$$
$$4x-4y = -8.$$

This system solves to $(x,y) = (\frac{3}{4},\frac{11}{4})$.

Given an interval $[a;b]$, its *midpoint* is $\dfrac{a+b}{2}$. By considering the similar triangles T_1 with vertices at $(x_1,y_1),(x_2,y_2),(x_2,y_1)$, T_2 with vertices at $(x_1,y_1),(\frac{x_1+x_2}{2},y_1),(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2})$, we motivate the following definition.

80 Definition The midpoint of the line segment joining the points $(x_1,y_1),(x_2,y_2)$ is

$$\boxed{\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right).}$$

81 Example Find the coordinates of the point (x,y) that is symmetric to the point $(1,2)$ with respect to the point $(-1,4)$.

Solution: The point $(-1,4)$ is the midpoint of the line segment joining $(1,2)$ and (x,y). Thus

$$\left(\frac{1+x}{2},\frac{2+y}{2}\right)=(-1,4),$$

whence $x=-3$ and $y=6$. The point sought is therefore $(x,y)=(-3,6)$.

Homework

82 Problem Find $\mathbf{d}\langle(-2,-5),(4,-3)\rangle$.

83 Problem If a and b are real numbers, find the distance between the points (a,a) and (b,b).

84 Problem Find the distance between the points (a^2+a,b^2+b) and $(b+a,b+a)$.

85 Problem A car is located at point $A=(-x,0)$ and an identical car is located at point $(x,0)$. Starting at time $t=0$, the car at point A travels downwards at constant speed, at a rate of $a>0$ units per second and simultaneously, the car at point B travels upwards at constant speed, at a rate of $b>0$ units per second. How many units apart are these cars after $t>0$ seconds?

86 Problem A bug starts at the point $(-1,-1)$ and wants to travel to the point $(2,1)$. In each quadrant, and on the axes, it moves with unit speed, except in quadrant II, where it moves with half the speed. Which route should the bug take in order to minimise its time? The answer is **not** a straight line from $(-1,-1)$ to $(2,1)$!

87 Problem Find the point equidistant from $(-1,0)$, $(1,0)$ and $(0,1/2)$.

88 Problem Find the coordinates of the point symmetric to (a,b) with respect to the point (b,a).

89 Problem Find the coordinates of the point which is a quarter of the way from $(2,3)$ to $(14,-11)$.

90 Problem Find the coordinates of the point which is three-quarters of the way from $(2,3)$ to $(14,-11)$.

91 Problem A fly starts at the origin and goes 1 unit up, $1/2$ unit right, $1/4$ unit down, $1/8$ unit left, $1/16$ unit up, etc., *ad infinitum*. In what coordinates does it end up?

92 Problem Find the coordinates of the point which is a quarter of the way from (a,b) to (b,a).

93 Problem Find the coordinates of the point symmetric to $(-a,b)$ with respect to: (i) the x-axis, (ii) the y-axis, (iii) the origin.

94 Problem (Minkowski's Inequality) Prove that if $(a,b),(c,d)\in\mathbb{R}^2$, then

$$\sqrt{(a+c)^2+(b+d)^2}\leq\sqrt{a^2+b^2}+\sqrt{c^2+d^2}.$$

Equality occurs if and only if $ad=bc$.

95 Problem Prove the following generalisation of Minkowski's Inequality. If $(a_k,b_k)\in(\mathbb{R}\setminus\{0\})^2, 1\leq k\leq n$, then

$$\sum_{k=1}^n\sqrt{a_k^2+b_k^2}\geq\sqrt{\left(\sum_{k=1}^n a_k\right)^2+\left(\sum_{k=1}^n b_k\right)^2}.$$

Equality occurs if and only if

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\cdots=\frac{a_n}{b_n}.$$

96 Problem (AIME 1991) Let $P=\{a_1,a_2,\ldots,a_n\}$ be a collection of points with

$$0<a_1<a_2<\cdots<a_n<17.$$

Consider

$$S_n=\min_P\sum_{k=1}^n\sqrt{(2k-1)^2+a_k^2},$$

where the minimum runs over all such partitions P. Shew that exactly one of $S_2,S_3,\ldots,S_n,\ldots$ is an integer, and find which one it is.

2.3 Circles

We now study our first curve on the plane: the circle. We will see that the *equation* of a circle on the plane is a consequence of the distance formula 2.9.

Here is a way to draw a circle on sand: using a string, tie it to what you wish to be the centre of the circle. Tighten up the string now and trace the path followed by the other extreme of the string. You now have a circle, whose radius is the length of the string. Notice then that every point on the circumference is at a fixed distance from the centre. This motivates the following.

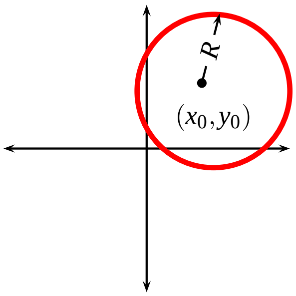

Figure 2.2: A circle with centre (x_0, y_0) and radius R.

97 Theorem A circle on the plane with radius R and centre (x_0, y_0) has equation

$$(x - x_0)^2 + (y - y_0)^2 = R^2, \tag{2.10}$$

called the *canonical equation* for a circle of radius R and centre (x_0, y_0). Conversely, the graph any equation of the form 2.10 is a circle.

Proof: *The point (x, y) belongs to circle of radius R and centre (x_0, y_0)*

$$\begin{array}{llcl} \iff & \mathbf{d}\langle (x,y),(x_0,y_0)\rangle & = & R \\ \iff & \sqrt{(x-x_0)^2+(y-y_0)^2} & = & R \quad , \\ \iff & (x-x_0)^2+(y-y_0)^2 & = & R^2 \end{array}$$

giving the desired result. ❑

98 Example The equation of the circle with centre $(-1, 2)$ and radius 6 is $(x+1)^2 + (y-2)^2 = 36$. Observe that the points $(-1 \pm 6, 2)$ and $(-1, 2 \pm 6)$ are on the circle. Thus $(-7, 2)$ is the left-most point on the circle, $(5, 2)$ is the right-most, $(-1, -4)$ is the lower-most, and $(-1, 8)$ is the upper-most. The circle is shewn in figure 2.3.

99 Example Rewrite the equation of the following circle in canonical form, and use it to find the centre and the radius of the circle:

$$x^2 + y^2 - 12x - 4y - 9 = 0.$$

Also, graph this circle

Solution: We complete squares:

$$x^2 - 12x + 36 + y^2 - 4y + 4 = 9 + 36 + 4$$

or $(x-6)^2 + (y-2)^2 = 49$. The centre is at $(6, 2)$ and the radius is 7. The points $(-1, 2)$, $(13, 2)$, $(6, -5)$, and $(6, 9)$ are on the circle. The graph is shewn in figure 2.4

100 Example A diameter of a circle has endpoints $(-2,-5)$ and $(4,3)$. Find the equation of this circle and graph it.

Solution: The centre of the circle lies on the midpoint of the diameter, thus the centre is $\left(\dfrac{-2+4}{2}, \dfrac{-5+3}{2}\right) = (1,-1)$. The equation of the circle is

$$(x-1)^2 + (y+1)^2 = R^2.$$

To find the radius, we observe that $(4,3)$ lies on the circle, thus

$$(4-1)^2 + (3+1)^2 = R^2 \implies R = 5.$$

The equation of the circle is finally

$$(x-1)^2 + (y+1)^2 = 25.$$

Observe that the points $(-2,-5)$, $(4,3)$, $(-4,-1)$, $(6,-1)$, $(1,4)$, and $(1,-6)$ all lie on the circle.

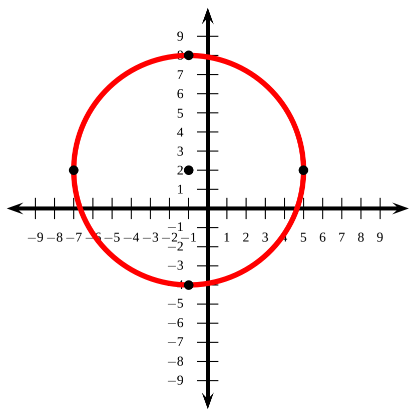

Figure 2.3: Example 98.

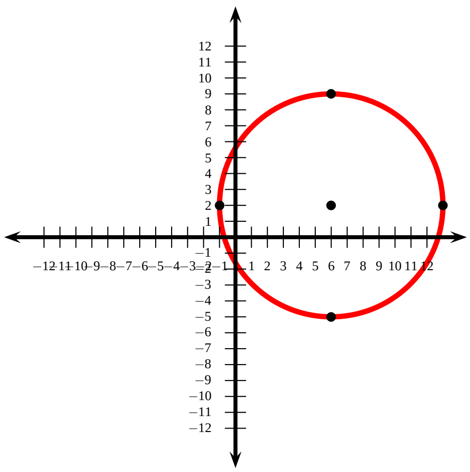

Figure 2.4: Example 99.

101 Example Find the equation of the circle passing through $(1,1)$, $(0,1)$ and $(1,2)$.

Solution: The centre (h,k) of the circle is the point equidistant to the given points, which can be found by solving

$$(h-1)^2 + (k-1)^2 = h^2 + (k-1)^2,$$

$$(h-1)^2 + (k-1)^2 = (h-1)^2 + (k-2)^2.$$

The first equation gives $h = \frac{1}{2}$, and the second equation gives $k = \frac{3}{2}$. The centre of the circle is thus $(h,k) = (\frac{1}{2}, \frac{3}{2})$. The radius of the circle is the distance from its centre to any point on the circle, say, to $(0,1)$:

$$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2} - 1\right)^2} = \frac{\sqrt{2}}{2}.$$

The equation sought is finally

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{1}{2}.$$

Homework

102 Problem Prove that the points $(4,2)$ and $(-2,-6)$ lie on the circle with centre at $(1,-2)$ and radius 5. Prove, moreover, that these two points are diametrically opposite.

103 Problem A diameter AB of a circle has endpoints $A=(1,2)$ and $B=(3,4)$. Find the equation of this circle.

104 Problem Find the equation of the circle with centre at $(-1,1)$ and passing through $(1,2)$.

105 Problem Rewrite the following circle equations in canonical form and find their centres C and their radius R. Draw the circles. Also, find at least four points belonging to each circle.

❶ $x^2+y^2-2y=35$,

❷ $x^2+4x+y^2-2y=20$,

❸ $x^2+4x+y^2-2y=5$,

❹ $2x^2-8x+2y^2=16$,

❺ $4x^2+4x+\frac{15}{2}+4y^2-12y=0$

❻ $3x^2+2x\sqrt{3}+5+3y^2-6y\sqrt{3}=0$

106 Problem Let

$$R_1=\{(x,y)\in\mathbb{R}^2|x^2+y^2\leq 9\},$$

$$R_2=\{(x,y)\in\mathbb{R}^2|(x+2)^2+y^2\leq 1\},$$

$$R_3=\{(x,y)\in\mathbb{R}^2|(x-2)^2+y^2\leq 1\},$$

$$R_4=\{(x,y)\in\mathbb{R}^2|x^2+(y+1)^2\leq 1\},$$

$$R_5=\{(x,y)\in\mathbb{R}^2||x|\leq 3,|y|\leq 3\},$$

$$R_6=\{(x,y)\in\mathbb{R}^2||x|\geq 2,|y|\geq 2\}.$$

Sketch the following regions.

❶ $R_1\setminus(R_2\cup R_3\cup R_4)$.

❷ $R_5\setminus R_1$

❸ $R_1\setminus R_6$

❹ $R_2\cup R_3\cup R_6$

107 Problem Find the equation of the circle passing through $(-1,2)$ and centre at $(1,3)$.

108 Problem Find the canonical equation of the circle passing through $(-1,1)$, $(1,-2)$, and $(0,2)$.

109 Problem Let a,b,c be real numbers with $a^2>4b$. Construct a circle with diameter at the points $(1,0)$ and $(-a,b)$. Shew that the intersection of this circle with the x-axis are the roots of the equation $x^2+ax+b=0$. Why must we impose $a^2>4b$?

2.4 Semicircles

Solving for y in $(x-x_0)^2+(y-y_0)^2=R^2$, we obtain

$$y=y_0\pm\sqrt{R^2-(x-x_0)^2}.$$

The choice of the $+$ sign gives the upper half of the circle (the upper semicircle) and the $-$ sign gives the lower semicircle.

110 Example Sketch the curve $y=\sqrt{1-x^2}$

Solution: Squaring, $y^2=1-x^2$. Hence $x^2+y^2=1$. This is the equation of a circle with centre at $(0,0)$ and radius 1. The original equation describes the upper semicircle (since $y\geq 0$). The graph is shewn in figure 2.5.

111 Example Sketch the curve $y=2-\sqrt{8-x^2-2x}$

Solution: We have $y-2=-\sqrt{8-x^2-2x}$. Squaring, $(y-2)^2=8-x^2-2x$. Hence, by completing squares,

$$x^2+2x+1+(y-2)^2=9 \implies (x+1)^2+(y-2)^2=9.$$

This is the equation of a circle with centre at $(-1,2)$ and radius 3. The original equation describes the lower semicircle (since $y\leq 2$). The graph is shewn in figure 2.6.

112 Example Sketch $y=-\sqrt{4-x^2}$

Solution: Squaring, $y^2=4-x^2$. Hence $x^2+y^2=4$. This is the equation of a circle with centre at $(0,0)$ and radius 2. The original equation describes the lower semicircle (since $y\leq 0$). The graph is shewn in figure 2.7.

STOP *The equation of the upper semicircle in example 112 is $y=\sqrt{4-x^2}$.*

Solving for x in $(x-x_0)^2+(y-y_0)^2=R^2$, we obtain

$$x = x_0 \pm \sqrt{R^2-(y-y_0)^2}.$$

The choice of the $+$ sign gives the right half of the circle and the $-$ sign gives the left semicircle.

113 Example Sketch the curve $x=\sqrt{4-y^2}$

Solution: Squaring, $x^2 = 4-y^2$. Hence $x^2+y^2=4$. This is the equation of a circle with centre at $(0,0)$ and radius 2. The original equation describes the right semicircle (since $x \geq 0$). The graph is shewn in figure 2.8.

The equation of the left semicircle in example 113 is $x=-\sqrt{4-y^2}$.

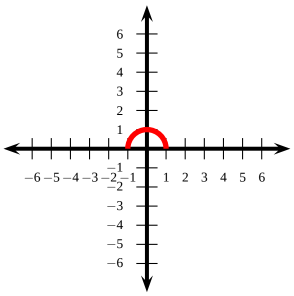

Figure 2.5: Example 110

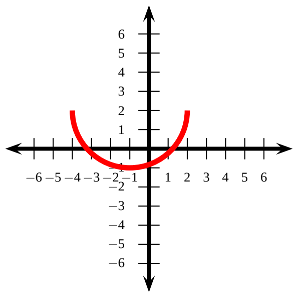

Figure 2.6: Example 111

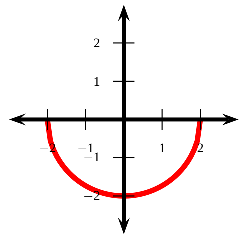

Figure 2.7: Example 112

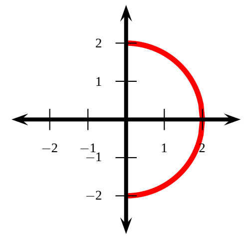

Figure 2.8: Example 113

114 Example Sketch the curve $x=1-\sqrt{8-2y-y^2}$

Solution: We have

$$\begin{aligned} x=1-\sqrt{8-2y-y^2} \quad &\implies \quad x-1=-\sqrt{8-2y-y^2} \\ &\implies \quad (x-1)^2 = 8-2y-y^2 \\ &\implies \quad (x-1)^2+y^2+2y=8 \\ &\implies \quad (x-1)^2+(y+1)^2=9. \end{aligned}$$

This is the equation of a circle with centre at $(1,-1)$ and radius 3. The original equation describes thus the left semicircle (since $x \leq 1$). The graph is shewn in figure 2.9.

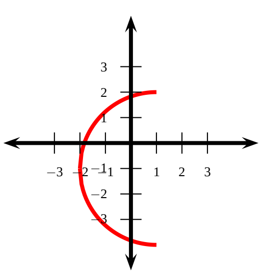

Figure 2.9: Example 114

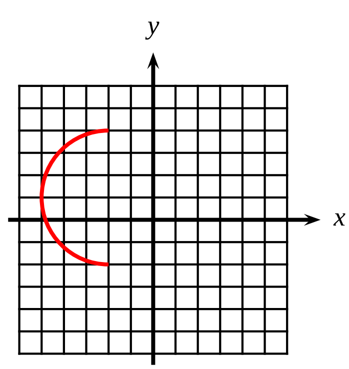

Figure 2.10: Example 115.

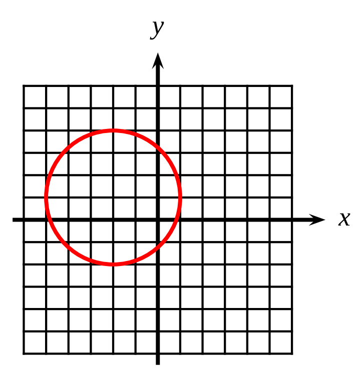

Figure 2.11: Example 115.

115 Example Find the equation of the semicircle depicted in figure 2.10.

Solution: Complete the semicircle to a full circle, as in figure 2.11. The equation of the circle is

$$(x+2)^2+(y+1)^2=9.$$

Since this is a left semi-circle, we solve for x and take the minus sign on the square root:

$$\begin{aligned}(x+2)^2+(y+1)^2=9 \quad &\implies \quad (x+2)^2=9-(y+1)^2\\ &\implies \quad x+2=-\sqrt{9-(y+1)^2}\\ &\implies \quad x=-2-\sqrt{9-(y+1)^2}.\end{aligned}$$

The equation sought is thus

$$x=-2-\sqrt{9-(y+1)^2}.$$

Homework

116 Problem Sketch the following curves.

❶ $y=\sqrt{16-x^2}$

❷ $x=-\sqrt{16-y^2}$

❸ $x=-\sqrt{12-4y-y^2}$

❹ $x=-5-\sqrt{12+4y-y^2}$

2.5 Lines

117 Definition Let a and b be real number constants. A *vertical line* on the plane is a set of the form

$$\{(x,y)\in\mathbb{R}^2 : x=a\}.$$

Similarly, a *horizontal line* on the plane is a set of the form

$$\{(x,y)\in\mathbb{R}^2 : y=b\}.$$

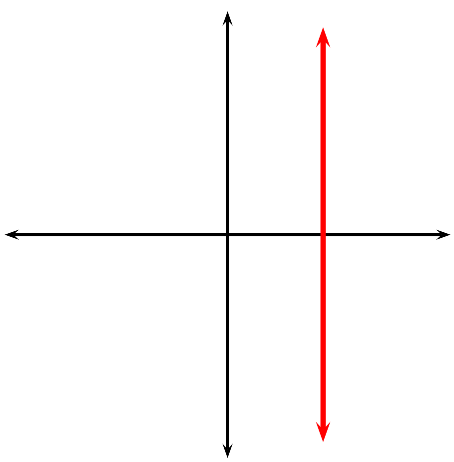

Figure 2.12: A vertical line.

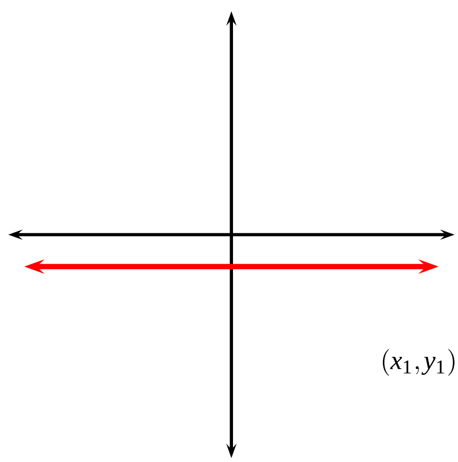

Figure 2.13: A horizontal line.

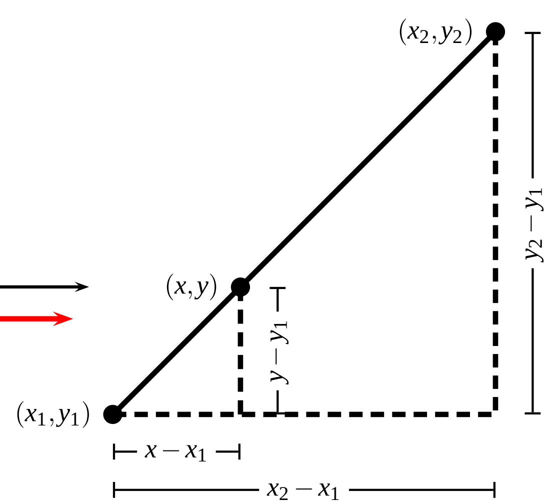

Figure 2.14: Theorem 118.

118 Theorem The equation of any non-vertical line on the plane can be written in the form $y=mx+k$, where m and k are real number constants. Conversely, any equation of the form $y=ax+b$, where a,b are fixed real numbers has as a line as a graph.

Proof: *If the line is parallel to the x-axis, that is, if it is horizontal, then it is of the form $y = b$, where b is a constant and so we may take $m = 0$ and $k = b$. Consider now a line non-parallel to any of the axes, as in figure 2.14, and let (x, y), (x_1, y_1), (x_2, y_2) be three given points on the line. By similar triangles we have*

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1},$$

which, upon rearrangement, gives

$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x - x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1,$$

and so we may take

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \quad k = -x_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) + y_1.$$

Conversely, consider real numbers $x_1 < x_2 < x_3$, and let $P = (x_1, ax_1 + b)$, $Q = (x_2, ax_2 + b)$, and $R = (x_3, ax_3 + b)$ be on the graph of the equation $y = ax + b$. We will shew that

$$\mathbf{d}\langle P, Q\rangle + \mathbf{d}\langle Q, R\rangle = \mathbf{d}\langle P, R\rangle.$$

Since the points P, Q, R are arbitrary, this means that any three points on the graph of the equation $y = ax + b$ are collinear, and so this graph is a line. Then

$$\mathbf{d}\langle P, Q\rangle = \sqrt{(x_2 - x_1)^2 + (ax_2 - ax_1)^2} = |x_2 - x_1|\sqrt{1 + a^2} = (x_2 - x_1)\sqrt{1 + a^2},$$

$$\mathbf{d}\langle Q, R\rangle = \sqrt{(x_3 - x_2)^2 + (ax_3 - ax_2)^2} = |x_3 - x_2|\sqrt{1 + a^2} = (x_3 - x_2)\sqrt{1 + a^2},$$

$$\mathbf{d}\langle P, Q\rangle = \sqrt{(x_3 - x_1)^2 + (ax_3 - ax_1)^2} = |x_3 - x_1|\sqrt{1 + a^2} = (x_3 - x_1)\sqrt{1 + a^2},$$

from where

$$\mathbf{d}\langle P, Q\rangle + \mathbf{d}\langle Q, R\rangle = \mathbf{d}\langle P, R\rangle$$

follows. This means that the points P, Q, and R lie on a straight line, which finishes the proof of the theorem. ❑

119 Definition The quantity $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ in Theorem 118 is the *slope or gradient of the line joining* (x_1, y_1) *and* (x_2, y_2). Since $y = m(0) + k$, the quantity k is the *y-intercept of the line joining* (x_1, y_1) *and* (x_2, y_2).

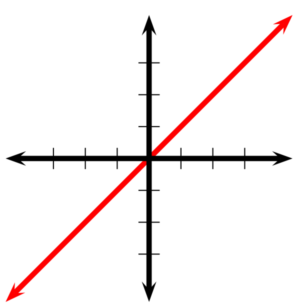

Figure 2.15: Example 120.

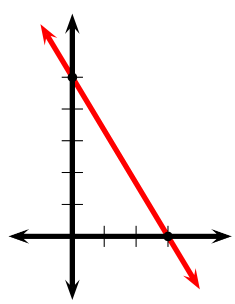

Figure 2.16: Example 121.

120 Example By Theorem 118, the equation $y = x$ represents a line with slope 1 and passing through the origin. Since $y = x$, the line makes a $45°$ angle with the x-axis, and bisects quadrants I and III. See figure 2.15

121 Example A line passes through $(-3, 10)$ and $(6, -5)$. Find its equation and draw it.

Solution: The equation is of the form $y = mx + k$. We must find the slope and the y-intercept. To find m we compute the ratio

$$m = \frac{10 - (-5)}{-3 - 6} = -\frac{5}{3}.$$

Thus the equation is of the form $y = -\frac{5}{3}x + k$ and we must now determine k. To do so, we substitute either point, say the first, into $y = -\frac{5}{3}x + k$ obtaining $10 = -\frac{5}{3}(-3) + k$, whence $k = 5$. The equation sought is thus $y = -\frac{5}{3}x + 5$. To draw the graph, first locate the y-intercept (at $(0,5)$). Since the slope is $-\frac{5}{3}$, move five units down (to $(0,0)$) and three to the right (to $(3,0)$). Connect now the points $(0,5)$ and $(3,0)$. The graph appears in figure 2.16.

122 Example Three points $(4,u)$, $(1,-1)$ and $(-3,-2)$ lie on the same line. Find u.

Solution: Since the points lie on the same line, any choice of pairs of points used to compute the gradient must yield the same quantity. Therefore

$$\frac{u - (-1)}{4 - 1} = \frac{-1 - (-2)}{1 - (-3)}$$

which simplifies to the equation

$$\frac{u + 1}{3} = \frac{1}{4}.$$

Solving for u we obtain $u = -\frac{1}{4}$.

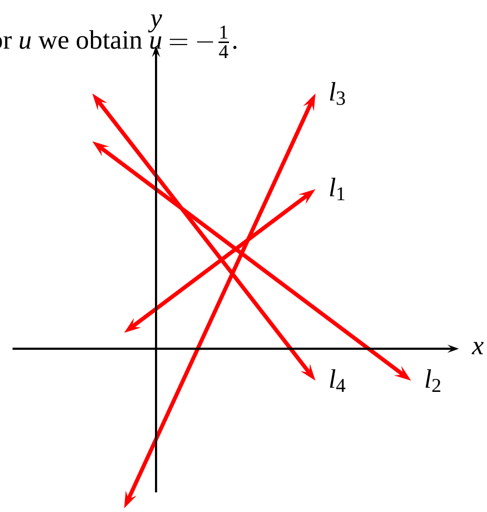

Figure 2.17: Problem 123.

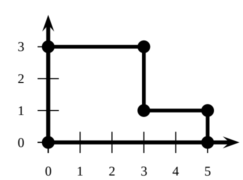

Figure 2.18: Problem 124.

Homework

123 Problem Assuming that the equations for the lines l_1, l_2, l_3, and l_4 in figure 2.17 below can be written in the form $y = mx + b$ for suitable real numbers m and b, determine which line has the largest value of m and which line has the largest value of b.

124 Problem (AHSME 1994) Consider the L-shaped region in the plane, bounded by horizontal and vertical segments with vertices at $(0,0)$, $(0,3)$, $(3,3)$, $(3,1)$, $(5,1)$ and $(5,0)$. Find the gradient of the line that passes through the origin and divides this area exactly in half.

125 Problem What is the slope of the line $\frac{x}{a} + \frac{y}{b} = 1$?

126 Problem If the point $(a, -a)$ lies on the line $-2x + 3y = 30$, find the value of a.

127 Problem Find the equation of the straight line joining $(3,1)$ and $(-5,-1)$.

128 Problem Let $(a,b) \in \mathbb{R}^2$. Find the equation of the straight line joining (a,b) and (b,a).

129 Problem Find the equation of the line that passes through (a, a^2) and (b, b^2).

130 Problem The points $(1, m), (2, 4)$ lie on a line with gradient m. Find m.

131 Problem Consider the following regions on the plane.

$$R_1 = \{(x, y) \in \mathbb{R}^2 | y \leq 1 - x\},$$
$$R_2 = \{(x, y) \in \mathbb{R}^2 | y \geq x + 2\},$$
$$R_3 = \{(x, y) \in \mathbb{R}^2 | y \leq 1 + x\}.$$

Sketch the following regions.

❶ $R_1 \setminus R_2$

❷ $R_2 \setminus R_1$

❸ $R_1 \cap R_2 \cap R_3$

❹ $R_2 \setminus (R_1 \cup R_2)$

132 Problem A vertical line divides the triangle with vertices $(0, 0), (1, 1)$ and $(9, 1)$ in the plane into two regions of equal area. Find the equation of this vertical line.

2.6 Parallel and Perpendicular Lines

133 Theorem Two lines are parallel if they have the same slope.

Proof:
❑

134 Example Find the equation of the line passing through $(4, 0)$ and parallel to the line joining $(-1, 2)$ and $(2, -4)$.

Solution: First we compute the slope of the line joining $(-1, 2)$ and $(2, -4)$:

$$m = \frac{2 - (-4)}{-1 - 2} = -2.$$

The line we seek is of the form $y = -2x + k$. We now compute the y-intercept, using the fact that the line must pass through $(4, 0)$. This entails solving $0 = -2(4) + k$, whence $k = 8$. The equation sought is finally $y = -2x + 8$.

135 Theorem Let $y = mx + k$ be a line non-parallel to the axes. If the line $y = m_1 x + k_1$ is perpendicular to $y = mx + k$ then $m_1 = -\frac{1}{m}$.

Proof: *Refer to figure 2.19. Since we may translate lines without affecting the angle between them, we assume without loss of generality that both $y = mx + k$ and $y = m_1 x + k_1$ pass through the origin, giving thus $k = k_1 = 0$. Now, the line $y = mx$ meets the vertical line $x = 1$ at $(1, m)$ and the line $y = m_1 x$ meets this same vertical line at $(1, m_1)$ (see figure 2.19). By the Pythagorean Theorem*

$$(m - m_1)^2 = (1 + m^2) + (1 + m_1^2).$$

Upon simplifying we gather that $mm_1 = -1$, which proves the assertion. ❑

136 Example Find the equation of the line passing through $(4, 0)$ and perpendicular to the line joining $(-1, 2)$ and $(2, -4)$.

Solution: The slope of the line joining $(-1, 2)$ and $(2, -4)$ is -2. The slope of any line perpendicular to it

$$m_1 = -\frac{1}{m} = \frac{1}{2}.$$

The equation sought has the form $y = \frac{x}{2} + k$. We find the y-intercept by solving $0 = \frac{4}{2} + k$, whence $k = -2$. The equation of the perpendicular line is thus $y = \frac{x}{2} - 2$.

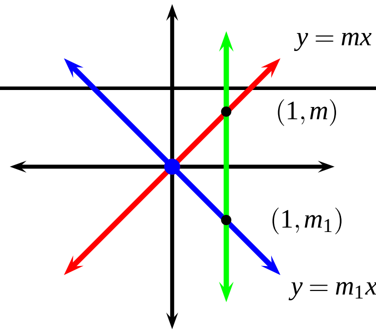

Figure 2.19: Theorem 135.

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137 Example For a given real number t, associate the straight line L_t with the equation

$$L_t: \quad (4-t)y = (t+2)x + 6t.$$

❶ Determine t so that the point $(1,2)$ lies on the line L_t and find the equation of this line.

❷ Determine t so that the L_t be parallel to the x-axis and determine the equation of the resulting line.

❸ Determine t so that the L_t be parallel to the y-axis and determine the equation of the resulting line.

❹ Determine t so that the L_t be parallel to the line $-5y = 3x - 1$.

❺ Determine t so that the L_t be perpendicular to the line $-5y = 3x - 1$.

❻ Is there a point (a,b) belonging to every line L_t regardless of the value of t?

Solution:

❶ If the point $(1,2)$ lies on the line L_t then we have

$$(4-t)(2) = (t+2)(1) + 6t \implies t = \frac{2}{3}.$$

The line sought is thus

$$L_{2/3}: \quad (4 - \frac{2}{3})y = (\frac{2}{3} + 2)x + 6\left(\frac{2}{3}\right)$$

or $y = \frac{4}{5}x + \frac{6}{5}$.

❷ We need $t + 2 = 0 \implies t = -2$. In this case

$$(4 - (-2))y = -12 \implies y = -2.$$

❸ We need $4 - t = 0 \implies t = 4$. In this case

$$0 = (4+2)x + 24 \implies x = -4.$$

❹ The slope of L_t is

$$\frac{t+2}{4-t},$$

and the slope of the line $-5y = 3x - 1$ is $-\frac{3}{5}$. Therefore we need

$$\frac{t+2}{4-t} = -\frac{3}{5} \implies -3(4-t) = 5(t+2) \implies t = -11.$$

❺ In this case we need

$$\frac{t+2}{4-t} = \frac{5}{3} \implies 5(4-t) = 3(t+2) \implies t = \frac{7}{4}.$$

❻ Yes. From above, the obvious candidate is $(-4,-2)$. To verify this observe that

$$(4-t)(-2) = (t+2)(-4)+6t,$$

regardless of the value of t.

138 Theorem The point (b,a) is symmetric about the line $y = x$ to the point (a,b).

Proof: *The line joining (b,a) to (a,b) is $y = -x+a+b$. This line is perpendicular to the line $y = x$ and meets it when $x = -x+a+b$, or $x = \frac{a+b}{2} = y$. But $(\frac{a+b}{2}, \frac{a+b}{2})$ is the midpoint of (a,b) and (b,a), which means that (a,b) and (b,a) are at equal distances from the line $y = x$, establishing the theorem.* ❑

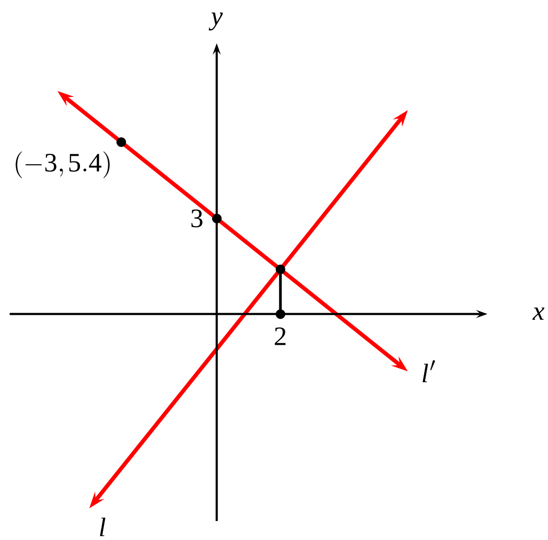


Figure 2.20: Problem 147.

Homework

139 Problem Find the equation of the straight line parallel to the line $8x-2y = 6$ and passing through $(5,6)$.

140 Problem Let $(a,b) \in (\mathbb{R}\setminus\{0\})^2$. Find the equation of the line passing through (a,b) and parallel to the line $\frac{x}{a}-\frac{y}{b} = 1$.

141 Problem Find the equation of the straight line normal to the line $8x-2y = 6$ and passing through $(5,6)$.

142 Problem Let a,b be strictly positive real numbers. Find the equation of the line passing through (a,b) and perpendicular to the line $\frac{x}{a}-\frac{y}{b} = 1$.

143 Problem Find the equation of the line passing through $(12,0)$ and parallel to the line joining $(1,2)$ and $(-3,-1)$.

144 Problem Find the equation of the line passing through $(12,0)$ and normal to the line joining $(1,2)$ and $(-3,-1)$.

145 Problem Find the equation of the straight line tangent to the circle $x^2+y^2 = 1$ at the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$.

146 Problem Consider the line L passing through (a,a^2) and (b,b^2). Find the equations of the lines L_1 parallel to L and L_2 normal to L, if L_1 and L_2 must pass through $(1,1)$.

147 Problem Find the equation of the line l in figure 2.20 given that line l' is perpendicular to it.

148 Problem For any real number t, associate the straight line L_t having equation

$$(2t-1)x+(3-t)y-7t+6 = 0.$$

In each of the following cases, find an t satisfying the stated conditions.

❶ L_t passes through $(1,1)$.

❷ L_t passes through the origin $(0,0)$.

❸ L_t is parallel to the x-axis.

➍ L_t is parallel to the y-axis.

➎ L_t is parallel to the line of equation $3x-2y-6=0$.

➏ L_t is normal to the line of equation $y=4x-5$.

➐ L_t has gradient -2.

➑ Is there a point (x_0,y_0) belonging to L_t no matter which real number t be chosen?

149 Problem For any real number t, associate the straight line L_t having equation

$$(t-2)x+(t+3)y+10t-5=0.$$

In each of the following cases, find an t and the resulting line satisfying the stated conditions.

➊ L_t passes through $(-2,3)$.

➋ L_t is parallel to the x-axis.

➌ L_t is parallel to the y-axis.

➍ L_t is parallel to the line of equation $x-2y-6=0$.

➎ L_t is normal to the line of equation $y=-\frac{1}{4}x-5$.

➏ Is there a point (x_0,y_0) belonging to L_t no matter which real number t be chosen?

150 Problem Shew that the four points $A=(-2,0)$, $B=(4,-2)$, $C=(5,1)$, and $D=(-1,3)$ form the vertices of a rectangle.

2.7 Linear Absolute Value Curves

To draw the graph of the absolute value curve $y=|x|$ we simply draw the line $y=x$ for $x\geq 0$ and the line $y=-x$ for $x<0$. This gives the curve depicted in figure 2.21.

151 Example Draw the graph of the equation $y=|2x-1|$.

Solution: For $2x-1\geq 0 \implies x\geq 1/2$ we have $y=2x-1$ and for $x<1/2$ we have $y=1-2x$. The graph appears in figure 2.22.

152 Example Draw the graph of the equation $y=|x+3|+2$.

Solution: For $x+3\geq 0 \implies x\geq -3$ we have $y=x+3+2=x+5$ and for $x<-3$ we have $y=-(x+3)+2=-1-x$. The graph appears in figure 2.23.

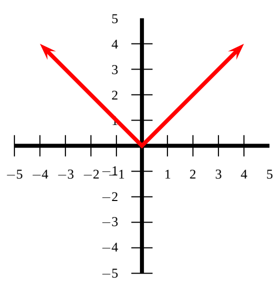


Figure 2.21: Absolute value $y=|x|$.

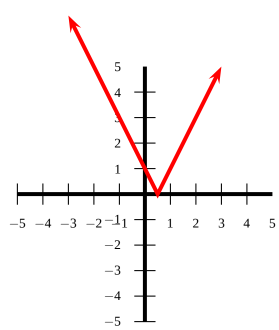


Figure 2.22: Example 151.

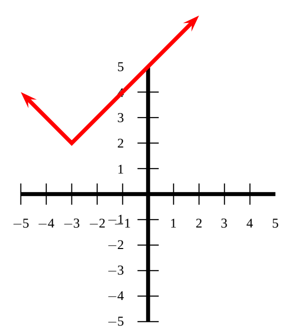


Figure 2.23: Example 152.

In curves with more than one absolute value term, the following tabular method might facilitate computations.

153 Example Draw the graph of the equation $y=|x-1|+|x+3|$.

Solution: The quantities in absolute values vanish for $x = 1$ and $x = -3$. We make a sign diagram as follows.

$x \in$	$]-\infty;-3]$	$[-3;1]$	$[1;+\infty[$
$\lvert x-1\rvert =$	$1-x$	$1-x$	$x-1$
$\lvert x+3\rvert =$	$-x-3$	$x+3$	$x+3$
$\lvert x-1\rvert + \lvert x+3\rvert =$	$-2x-2$	4	$2x+2$

Thus for $x \in]-\infty;-3]$, $y = -2x-2$, for $x \in [-3;1]$, $y = 4$, and for $x \in [1;+\infty[$, $y = 2x+2$. The graph is shewn in figure 2.24.

154 Example Draw the graph of the equation $y = |x+2| - 2|x-3|$.

Solution: The quantities in absolute values vanish for $x = -2$ and $x = 3$. We make a sign diagram as follows.

$x \in$	$]-\infty;-2]$	$[-2;3]$	$[3;+\infty[$
$\lvert x+2\rvert =$	$-x-2$	$x+2$	$x+2$
$\lvert x-3\rvert =$	$3-x$	$3-x$	$x-3$
$\lvert x+2\rvert - 2\lvert x-3\rvert =$	$x-8$	$3x-4$	$-x+8$

Thus for $x \in]-\infty;-3]$, $y = -2x-2$, for $x \in [-3;1]$, $y = 4$, and for $x \in [1;+\infty[$, $y = 2x+2$. The graph is shewn in figure 2.25.

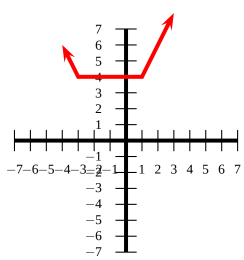


Figure 2.24: Example 153.

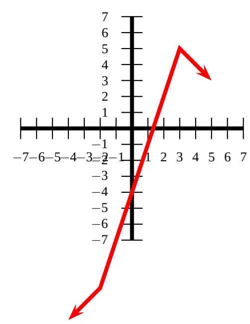


Figure 2.25: Example 154.

Homework

155 Problem Draw the graphs of the following equations.

❶ $y = |x+2|$

❷ $y = 3 - |x+2|$

❸ $y = 2|x+2|$

❹ $y = |x-1| + |x+1|$

❺ $y = |x-1| - |x+1|$

❻ $y = |x+1| - |x-1|$

❼ $y = |x-1| + |x| + |x+1|$

❽ $y = |x-1| - |x| + |x+1|$

❾ $y = |x-1| + x + |x+1|$

❿ $y = |x+3| + 2|x-1| - |x-4|$

2.8 Distance of a Point to a Line

156 Theorem (Distance between a point and a line) Let $L: y = mx + k$ be a fixed line on the plane and let $P = (x_0, y_0)$ be a point not on L. Then the distance $\mathbf{d}\langle L, P\rangle$ is given by

$$\frac{|x_0 m + k - y_0|}{\sqrt{1+m^2}} \tag{2.11}$$

Proof: *If the line has infinite gradient, then the line is of the form $x = c$ for a constant c, and clearly*

$$\mathbf{d}\langle L, P\rangle = |x_0 - c|.$$

If $m = 0$, then L is parallel to the x-axis and clearly

$$\mathbf{d}\langle L, P\rangle = |y_0 - k|,$$

which agrees with the theorem. Suppose now that $m \neq 0$. The line L has gradient m and any line perpendicular to it has gradient $-\frac{1}{m}$. The distance from P to L is the length of the line segment joining P to the intersection point (x_1, y_1) of the line L_1 perpendicular to L and passing through P. Now, it is easy to see that L_1 has equation

$$L_1 : y = -\frac{1}{m}x + y_0 + \frac{x_0}{m},$$

and so, L and L_1 intersect at

$$x_1 = \frac{y_0 m + x_0 - bm}{1+m^2}, \quad y_1 = \frac{y_0 m^2 + x_0 m + k}{1+m^2}.$$

Therefore

$$\begin{aligned}
\mathbf{d}\langle L, P\rangle &= \mathbf{d}\langle (x_0, y_0), (x_1, y_1)\rangle \\
&= \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} \\
&= \sqrt{\left(x_0 - \frac{y_0 m + x_0 - km}{1+m^2}\right)^2 + \left(y_0 - \frac{y_0 m^2 + x_0 m + k}{1+m^2}\right)^2} \\
&= \frac{\sqrt{(x_0 m^2 - y_0 m + km)^2 + (y_0 - x_0 m - k)^2}}{1+m^2} \\
&= \frac{\sqrt{(m^2+1)(x_0 m - y_0 + k)^2}}{1+m^2} \\
&= \frac{|x_0 m - y_0 + k|}{\sqrt{1+m^2}},
\end{aligned}$$

proving the theorem. ❑

157 Example Find the distance between the line $L: 2x - 3y = 1$ and the point $(-1, 1)$.

Solution: The equation of the line L can be rewritten in the form $L: y = \frac{2}{3}x - \frac{1}{3}$. Using Theorem 156, we have

$$\mathbf{d}\langle L, P\rangle = \frac{|-\frac{2}{3} - 1 - \frac{1}{3}|}{\sqrt{1 + (\frac{2}{3})^2}} = \frac{6\sqrt{13}}{13}.$$

Homework

158 Problem Find the distance from the point $(1,1)$ to the line $y = -x$.

159 Problem Let $a \in \mathbb{R}$. Find the distance from the point $(a,0)$ to the line $L : y = ax + 1$.

160 Problem Find the equation of the circle with centre at $(3,4)$ and tangent to the line $x - 2y + 3 = 0$.

2.9 Parabolas

161 Definition A *parabola* is the collection of all the points on the plane whose distance from a fixed point F (called the *focus* of the parabola) is equal to the distance to a fixed line L (called the *directrix* of the parabola). See figure 2.26, where $FD = DP$.

We can draw a parabola as follows. Cut a piece of thread as long as the trunk of T-square (see figure 2.27). Tie one end to the end of the trunk of the T-square and tie the other end to the focus, say, using a peg. Slide the crosspiece of the T-square along the directrix, while maintaining the thread tight against the ruler with a pencil.

162 Theorem Let $d > 0$ be a real number. The equation of a parabola with focus at $(0,d)$ and directrix $y = -d$ is $y = \dfrac{x^2}{4d}$.

Proof: *Let (x,y) be an arbitrary point on the parabola. Then the distance of (x,y) to the line $y = -d$ is $|y+d|$. The distance of (x,y) to the point $(0,d)$ is $\sqrt{x^2 + (y-d)^2}$. We have*

$$\begin{aligned} |y+d| = \sqrt{x^2+(y-d)^2} &\implies (|y+d|)^2 = x^2 + (y-d)^2 \\ &\implies y^2 + 2yd + d^2 = x^2 + y^2 - 2yd + d^2 \\ &\implies 4dy = x^2 \\ &\implies y = \frac{x^2}{4d}, \end{aligned}$$

as wanted. ❑

Observe that the midpoint of the perpendicular line segment from the focus to the directrix is on the parabola. We call this point the vertex. *For the parabola $y = \dfrac{x^2}{4d}$ of Theorem 162, the vertex is clearly $(0,0)$.*

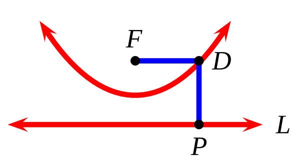


Figure 2.26: Definition of a parabola.

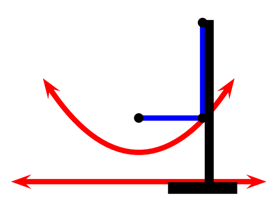


Figure 2.27: Drawing a parabola.

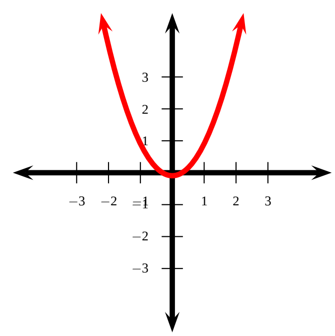


Figure 2.28: Example 163.

163 Example Draw the parabola $y = x^2$.

Solution: From Theorem 162, we want $\dfrac{1}{4d} = 1$, that is, $d = \dfrac{1}{4}$. Following Theorem 162, we locate the focus at $(0, \frac{1}{4})$ and the directrix at $y = -\dfrac{1}{4}$ and use a T-square with these references. The vertex of the parabola is at $(0,0)$. The graph is in figure 2.28.

Homework

164 Problem Let $d > 0$ be a real number. Prove that the equation of a parabola with focus at $(d, 0)$ and directrix $x = -d$ is $x = \frac{y^2}{4d}$.

165 Problem Find the focus and the directrix of the parabola $x = y^2$.

166 Problem Find the equation of the parabola with directrix $y = -x$ and vertex at $(1, 1)$.

2.10 Hyperbolas

167 Definition A *hyperbola* is the collection of all the points on the plane whose absolute value of the difference of the distances from two distinct fixed points F_1 and F_2 (called the *foci*[1] of the hyperbola) is a positive constant. See figure 2.29, where $|F_1D - F_2D| = |F_1D' - F_2D'|$.

We can draw a hyperbola as follows. Put tacks on F_1 and F_2 and measure the distance F_1F_2. Attach piece of thread to one end of the ruler, and the other to F_2, while letting the other end of the ruler to pivot around F_1. The lengths of the ruler and the thread must satisfy

$$\text{length of the ruler} - \text{length of the thread} < F_1F_2.$$

Hold the pencil against the side of the rule and tighten the thread, as in figure 2.30.

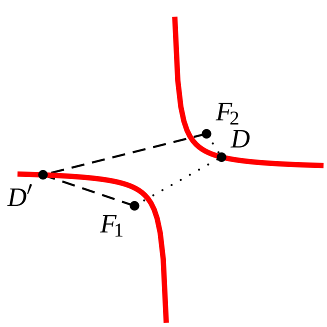


Figure 2.29: Definition of a hyperbola.

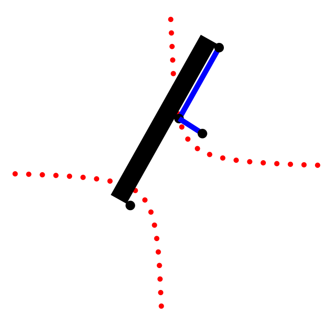


Figure 2.30: Drawing a hyperbola.

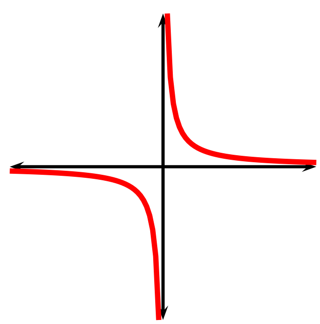


Figure 2.31: The hyperbola $y = \frac{1}{x}$.

168 Theorem Let $c > 0$ be a real number. The hyperbola with foci at $F_1 = (-c, -c)$ and $F_2 = (c, c)$, and whose absolute value of the difference of the distances from its points to the foci is $2c$ has equation $xy = \frac{c^2}{2}$.

[1]*Foci* is the plural of *focus*.

Proof: *Let (x,y) be an arbitrary point on the hyperbola. Then*

$$\begin{aligned}
&|\mathbf{d}\langle(x,y),(-c,-c)\rangle - \mathbf{d}\langle(x,y),(c,c)\rangle| = 2c \\
\iff & \left|\sqrt{(x+c)^2+(y+c)^2} - \sqrt{(x-c)^2+(y-c)^2}\right| = 2c \\
\iff & (x+c)^2+(y+c)^2+(x-c)^2+(y-c)^2 - 2\sqrt{(x+c)^2+(y+c)^2}\cdot\sqrt{(x-c)^2+(y-c)^2} = 4c^2 \\
\iff & 2x^2+2y^2 = 2\sqrt{(x^2+y^2+2c^2)+(2xc+2yc)}\cdot\sqrt{(x^2+y^2+2c^2)-(2xc+2yc)} \\
\iff & 2x^2+2y^2 = 2\sqrt{(x^2+y^2+2c^2)^2-(2xc+2yc)^2} \\
\iff & (2x^2+2y^2)^2 = 4\left((x^2+y^2+2c^2)^2-(2xc+2yc)^2\right) \\
\iff & 4x^4+8x^2y^2+4y^4 = 4((x^4+y^4+4c^4+2x^2y^2+4y^2c^2+4x^2c^2)-(4x^2c^2+8xyc^2+4y^2c^2)) \\
\iff & xy = \frac{c^2}{2},
\end{aligned}$$

where we have used the identities

$$(A+B+C)^2 = A^2+B^2+C^2+2AB+2AC+2BC \qquad \text{and} \qquad \sqrt{A-B}\cdot\sqrt{A+B} = \sqrt{A^2-B^2}.$$

❑

Observe that the points $\left(-\frac{c}{\sqrt{2}}, -\frac{c}{\sqrt{2}}\right)$ and $\left(\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$ are on the hyperbola $xy = \frac{c^2}{2}$. We call these points the vertices[2] *of the hyperbola $xy = \frac{c^2}{2}$.*

169 Example To draw the hyperbola $y = \frac{1}{x}$ we proceed as follows. According to Theorem 168, its two foci are at $(-\sqrt{2}, -\sqrt{2})$ and $(\sqrt{2}, \sqrt{2})$. Put length of the ruler − length of the thread $= 2\sqrt{2}$. By alternately pivoting about these points using the procedure above, we get the picture in figure 2.31.

Answers

59 $\sqrt{3} - \sqrt{\sqrt{15}-2}$

60 For $x > \frac{1}{2}$, we have $|1-2x| = 2x-1$. Thus $|x-|1-2x|| = |x-(2x-1)| = |-x+1|$. If $x > 1$ then $|-x+1| = x-1$. In conclusion, for all $x > 1$ (and *a fortiori* $x > 2$, we have $|x-|1-2x|| = x-1$.

61 We have

$$\begin{aligned}
|5x-2| = |2x+1| \quad &\iff \quad (5x-2 = 2x+1) \text{ or } (5x-2 = -(2x+1)) \\
&\iff \quad (x = 1) \text{ or } (x = \frac{1}{7}) \\
&\iff \quad x \in \left\{\frac{1}{7}, 1\right\}
\end{aligned}$$

62 The first term vanishes when $x = 2$ and the second term vanishes when $x = 3$. We decompose $\mathbb{R}$ into (overlapping) intervals with endpoints at the places where each of the expressions in absolute values vanish. Thus we have

$$\mathbb{R} =]-\infty;2] \cup [2;3] \cup [3;+\infty[.$$

We examine the sign diagram

$x \in$	$]-\infty;2]$	$[2;3]$	$[3;+\infty[$
$\|x-2\| =$	$-x+2$	$x-2$	$x-2$
$\|x-3\| =$	$-x+3$	$-x+3$	$x-3$
$\|x-2\|+\|x-3\| =$	$-2x+5$	1	$2x-5$

Thus on $]-\infty;2]$ we need $-2x+5 = 1$ from where $x = 2$. On $[2;3]$ we obtain the identity $1 = 1$. This means that all the numbers on this interval are solutions to this equation. On $[3;+\infty[$ we need $2x-5 = 1$ from where $x = 3$. Upon assembling all this, the solution set is $\{x | x \in [2;3]\}$.

63 $\{-\frac{1}{2}, \frac{3}{2}\}$

64 $\{x | x \in [0;1]\}$

65 $\{-1\}$

66 $[1;+\infty[$

67 $]-\infty;-2]$

69 $\{\frac{3}{2} + \frac{\sqrt{17}}{2}, \frac{3}{2} - \frac{\sqrt{17}}{2}, 1, 2\}$

70 $\{-1, 1\}$

71 $\{-3, -2, 2, 3\}$

72 $\{-6, 1, 2, 3\}$

73 We have

$$|x+3| = \begin{cases} -x-3 & \text{if } x+3 < 0, \\ x+3 & \text{if } x+3 \geq 0. \end{cases} \qquad |x-4| = \begin{cases} -x+4 & \text{if } x-4 < 0, \\ x-4 & \text{if } x-4 \geq 0. \end{cases}$$

This means that when $x < -3$

$$|x+3| - |x-4| = (-x-3) - (-x+4) = -7,$$

a constant. Since at $x = -3$ we also obtain -7, the result holds true for the larger interval $x \leq -3$.

82 $2\sqrt{10}$

[2] *Vertices* is the plural of *vertex*.

83 $\sqrt{2}|b-a|$

84 $\sqrt{(a^2-b)^2+(b^2-a)^2}$

85 $\sqrt{4x^2+(a+b)^2t^2}$

86 The bug should travel along two line segments: first from $(-1,1)$ the origin, and then from the origin to $(2,1)$ avoiding quadrant II altogether. For, if $a>0, b>0$ then the line segment joining $(-b,0)$ and $(a,0)$ lies in quadrant II, it is $\sqrt{a^2+b^2}$ long, and the bug spends an amount of time equal to $\frac{\sqrt{a^2+b^2}}{2}$ on this line. But a path on the axes from $(-b,0)$ to $(a,0)$ is $a+b$ units long and the bug spends an amount of time equal to $a+b$ there. Thus as long as

$$a+b \le \frac{a^2+b^2}{2}$$

the bug should avoid quadrant II completely. But by the Arithmetic-Mean-Geometric-Mean Inequality we have

$$2ab \le a^2+b^2 \implies (a+b)^2 = a^2+2ab+b^2 \le 2a^2+2b^2 \implies a+b \le \sqrt{2}\sqrt{a^2+b^2},$$

which means that as long as the speed of the bug in quadrant II is $< \frac{1}{\sqrt{2}}$ then the bug will better avoid quadrant II. Since $\frac{1}{2} < \frac{1}{\sqrt{2}}$, this follows in our case.

87 $(0,-3/4)$

88 $(2b-a, 2a-b)$

89 $(5, -\frac{1}{2})$

90 $(11, -\frac{15}{2})$

91 Its x coordinate is

$$\frac{1}{2}-\frac{1}{8}+\frac{1}{32}-\cdots = \frac{\frac{1}{2}}{1-\frac{-1}{4}} = \frac{2}{5}.$$

Its y coordinate is

$$1-\frac{1}{4}+\frac{1}{16}-\cdots = \frac{1}{1-\frac{-1}{4}} = \frac{4}{5}.$$

Therefore, the fly ends up in

$$\left(\frac{2}{5}, \frac{4}{5}\right).$$

Here we have used the fact the sum of an infinite geometric progression with common ratio r, with $|r|<1$ and first term a is

$$a+ar+ar^2+ar^3+\cdots = \frac{a}{1-r}.$$

92 $(\frac{3a+b}{4}, \frac{3b+a}{4})$

93 (a,b) ; $(-a,-b)$; $(a,-b)$

94 It is enough to prove this in the case when a,b,c,d are all positive. To this end, put $O=(0,0)$, $L=(a,b)$ and $M=(a+c,b+d)$. By the triangle inequality $OM \le OL+LM$, where equality occurs if and only if the points are collinear. But then

$$\sqrt{(a+c)^2+(b+d)^2} = OM \le OL+LM = \sqrt{a^2+b^2}+\sqrt{c^2+d^2},$$

and equality occurs if and only if the points are collinear, that is $\frac{a}{b}=\frac{c}{d}$.

96 Use the above generalisation of Minkowski's Inequality and the fact that $17^2+144^2=145^2$. The desired value is S_{12}.

103 $(x-2)^2+(y-3)^2=2$

104 We must find the radius of this circle. Since the radius is the distance from the centre of the circle to any point on the circle, we see that the required radius is

$$\sqrt{(-1-1)^2+(1-2)^2} = \sqrt{5} \approx 2.236.$$

The equation sought is thus

$$(x+1)^2+(y-1)^2=5.$$

105 (1) $x^2+(y-1)^2=36, C=(0,1), R=6$. (2) $(x+2)^2+(y-1)^2=25, C=(-2,1), R=5$, (3) $(x+2)^2+(y-1)^2=10, C=(-2,1), R=\sqrt{10}$, (4) $(x-2)^2+y^2=12, C=(2,0), R=2\sqrt{3}$ (5) $(x+\frac{1}{2})^2+(y-\frac{3}{2})^2=\frac{5}{8}, C=(-\frac{1}{2},\frac{3}{2}), R=\sqrt{\frac{5}{8}}$, (6) $(x+\frac{1}{\sqrt{3}})^2+(y-\frac{\sqrt{3}}{3})^2=\frac{5}{3}, C=(-\frac{\sqrt{3}}{3},\sqrt{3}), R=\sqrt{\frac{5}{3}}$

107 $(x-1)^2+(y-3)^2=5$

108 $(x-\frac{9}{10})^2+(y-\frac{1}{10})^2=\frac{221}{50}$

124 $\frac{7}{9}$

125 $-\frac{b}{a}$

126 -6

127 $y=\frac{x}{4}+\frac{1}{4}$

128 $y=-x+b+a$

129 $y=(a+b)x-ab$

130 $m=2$

132 $x=3$

139 $y=4x-14$

140 $y=\frac{b}{a}x$

141 $y=-\frac{1}{4}x+\frac{29}{4}$

142 $y=-\frac{a}{b}x+b+\frac{a^2}{b}$

143 $y=\frac{3}{4}x-9$

144 $y=-\frac{4}{3}x+16$

145 Notice that there is a radius of the circle connecting $(0,0)$ and $(\frac{1}{2},\frac{\sqrt{3}}{2})$. The line passing through these two points is $y=\sqrt{3}x$. Hence, since the tangent line is perpendicular to the radius at the point of tangency, the line sought is of the form $y=-\frac{\sqrt{3}}{3}x+k$. To find k observe that $\frac{\sqrt{3}}{2}=-\frac{3}{4}+k \implies k=\frac{3\sqrt{3}}{4}$. Finally, the desired line is $y=-\frac{\sqrt{3}}{3}x+\frac{3\sqrt{3}}{4}$.

146 $L_1: y=(a+b)x+1-a-b, L_2: y=-\frac{x}{a+b}+\frac{a+b+1}{a+b}$

147 $l: y=\frac{5}{4}x-1.1, l': y=-\frac{4}{5}x+3$

148 (1) $t=4/3$, (2) $t=6/7$, (3) $t=1/2$, (4) $t=3$, (5) $t=-7$, (6) $t=7/9$, (7) $t=7/4$, (8) $(3,-1)$

149 We have

❶ If L_t passes through $(-2,3)$ then

$$(t-2)(-2)+(t+3)(3)+10t-5=0,$$

from where $t=-\frac{8}{11}$. In this case the line is

$$-\frac{30}{11}x+\frac{25}{11}y-\frac{135}{11}=0.$$

❷ L_t will be parallel to the x-axis if the x-term disappears, which necessitates $t-2=0$ or $t=2$. In this case the line is

$$y=-3.$$

❸ L_t will be parallel to the y-axis if the y-term disappears, which necessitates $t+3=0$ or $t=-3$. In this case the line is

$$x=-7.$$

❹ The line $x-2y-6=0$ has gradient $\frac{1}{2}$ and L_t has gradient $\frac{2-t}{t+3}$. The lines will be parallel when $\frac{2-t}{t+3}=\frac{1}{2}$ or $t=1/3$. In this case the line is

$$-\frac{5}{3}x+\frac{10}{3}y-\frac{5}{3}=0.$$

❺ The line $y=-\frac{1}{4}x-5$ has gradient $-\frac{1}{4}$ and L_t has gradient $\frac{2-t}{t+3}$. The lines will be perpendicular when $\frac{2-t}{t+3}=4$ or $t=-2$. In this case the line is

$$-4x+y-25=0.$$

❻ If such a point existed, it would pass through the horizontal and vertical lines found above, thus $(x_0,y_0)=(-7,-3)$ is a candidate for the point sought. That $(-7,-3)$ passes through every line L_t, no matter the choice of t is seen from

$$(t-2)(-7)+(t+3)(-3)+10t-5=-7t+14-3t-9+10t-5=0.$$

158 $\sqrt{2}$

159 $\sqrt{1+a^2}$

160 The radius of the circle is the distance from the centre to the tangent line. This radius is then

$$r=\left|\frac{3-2\cdot 4+3}{\sqrt{1^2+(-2)^2}}\right|=\frac{2}{\sqrt{5}}.$$

The desired equation is

$$(x-3)^2+(y-4)^2=\frac{4}{5}.$$

165 By the preceding exercise the focus is $(\frac{1}{4},0)$ and the directrix is $x=-\frac{1}{4}$.

166 If (x,y) is an arbitrary point on this parabola we must have

$$\frac{|-x-y|}{\sqrt{1+(-1)^2}}=\sqrt{(x-1)^2+(y-1)^2}$$

Squaring and rearranging, the desired equation is

$$x^2+y^2-2xy-4x-4y+4=0.$$

Chapter 3

Functions I: Assignment Rules

This chapter introduces the central concept of a function. We concentrate on real-valued functions whose domains are subsets of the real numbers. We will use the curves obtained in the last chapter as examples to see how various transformations affect the graph of a function.

3.1 Basic Definitions

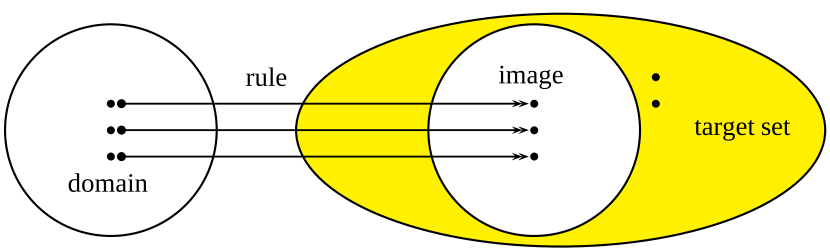


Figure 3.1: The main ingredients of a function.

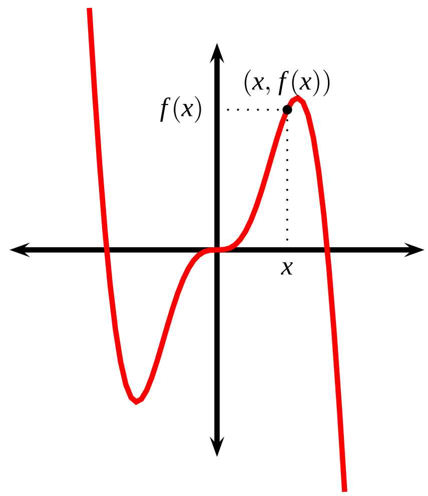


Figure 3.2: The graph of a function.

170 Definition By a *(real-valued) function* $f : \begin{array}{ccc}\mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x)\end{array}$ we mean the collection of the following ingredients:

❶ a *name* for the function. Usually we use the letter f.

❷ a set of real number inputs—usually an interval or a finite union of intervals—called the *domain* of the function. The domain of f is denoted by $\mathbf{Dom}(f)$.

❸ an *input parameter* , also called *independent variable* or *dummy variable*. We usually denote a typical input by the letter x.

❹ a set of possible real number outputs—usually an interval or a finite union of intervals—of the function, called the *target set* of the function. The target set of f is denoted by $\mathbf{Target}(f)$.

❺ an *assignment rule* or *formula*, assigning to **every input** a **unique** output. This assignment rule for f is usually denoted by $x \mapsto f(x)$. The output of x under f is also referred to as the *image of x under f*, and is denoted by $f(x)$.

See figure 3.1.

171 Definition The *image* of a function $f: \begin{array}{rcl} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$ is the set

$$\mathbf{Im}(f) = \{f(x) : x \in \mathbf{Dom}(f)\},$$

that is, the collection of all outputs of f.

We see that necessarily we have $\mathbf{Im}(f) \subseteq \mathbf{Target}(f)$, but we will see later on that these two sets may not be equal.

172 Definition The *graph* of a function $f: \begin{array}{rcl} \mathbf{Dom}(f) & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array}$ is the set $\{(x,y) \in \mathbb{R}^2 : y = f(x)\}$ on the plane. For ellipsis, we usually say *the graph of* f, or *the graph* $y = f(x)$ or the *the curve* $y = f(x)$. See figure 3.2.

From now on, unless otherwise stated, we will take $\mathbb{R}$ as the target set of all the functions below.

It must be emphasised that the uniqueness of the image of an element of the domain is crucial. For example, the diagram in figure 3.3 *does not* represent a function. The element 1 in the domain is assigned to more than one element of the target set. Also important in the definition of a function is the fact that *all the elements* of the domain must be operated on. For example, the diagram in 3.4 *does not* represent a function. The element 3 in the domain is not assigned to any element of the target set. Also, by the definition of the graph of a function, the x-axis contains the set of inputs and y-axis has the set of outputs. Therefore, if a vertical line crosses two or more points of a graph, the graph does not represent a function. See figures 3.5 and 3.6.

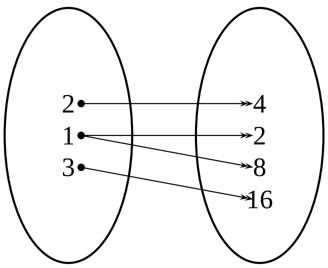


Figure 3.3: Not a function.

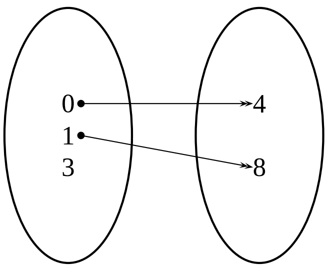


Figure 3.4: Not a function.

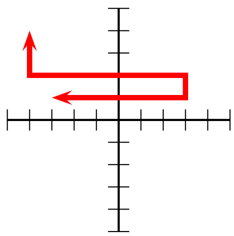


Figure 3.5: Not a function.

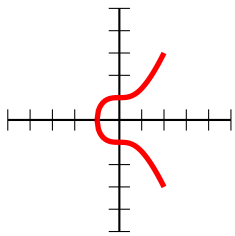


Figure 3.6: Not a function.

173 Example (The Identity Function) Consider the function

$$\mathbf{Id}: \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}.$$

This function assigns to every real its own value. Thus $\mathbf{Id}(-1) = -1$, $\mathbf{Id}(0) = 0$, $\mathbf{Id}(4) = 4$, etc. By Theorem 118, the graph of identity function is a straight line, and it is given in figure 3.7.

174 Example (Affine Function) Consider the function

$$\mathbf{Id} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & mx+b \end{array},$$

where m and b are real number constants. By Theorem 118, the graph of any affine function is a straight line.

175 Example (The Square Function) Consider the function

$$\mathbf{Id}^2 : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}.$$

This function assigns to every real its square. Thus $\mathbf{Id}^2(-1) = 1$, $\mathbf{Id}^2(0) = 0$, $\mathbf{Id}^2(2) = 4$, etc. By Theorem 162, the graph of the square function is given in figure 3.8.

For ellipsis, we usually refer to the identity function $\mathbf{Id} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}$ *as "the function* $\mathbf{Id}$ *" or "the function* $x \mapsto x$*." Similarly, in situations when the domain of a function is not in question, we will simply give the assignment rule or the name of the function. So we will speak of "the function* f*" or "the function* $x \mapsto f(x)$*," e.g., "the function* $\mathbf{Id}^2$*" or "the function* $x \mapsto x^2$*."*

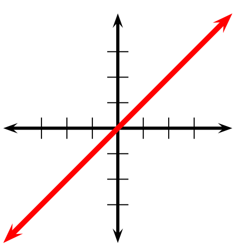


Figure 3.7: $\mathbf{Id}$

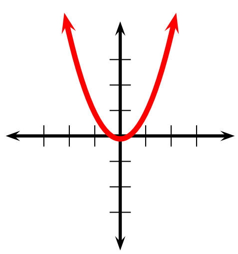


Figure 3.8: $\mathbf{Id}^2$

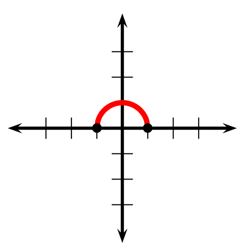


Figure 3.9: $x \mapsto \sqrt{1-x^2}$

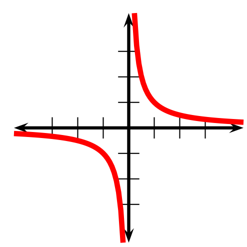


Figure 3.10: $x \mapsto \dfrac{1}{x}$

176 Example Consider the function[1]

$$f : \begin{array}{ccc} [-1;1] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{1-x^2} \end{array}.$$

Then $f(-1) = 0$, $f(0) = 1$, $f\left(\frac{1}{2}\right) = \dfrac{\sqrt{3}}{2} \approx .866$, etc. By Example 110, the graph of f is the upper unit semicircle, which is shewn in figure 3.9.

177 Example (The Reciprocal function) Consider the function[2]

$$g : \begin{array}{ccc} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array}.$$

Then $g(-1) = -1$, $g(1) = 1$, $g\left(\frac{1}{2}\right) = 2$, etc. By Example 169, the graph of g is the hyperbola shewn in figure 3.10.

[1] Since we are concentrating exclusively on real-valued functions, the formula for f only makes sense in the interval $[-1;1]$.
[2] g only makes sense when $x \neq 0$.

178 Example Find all functions with domain $\{a,b\}$ and target set $\{c,d\}$.

Solution: There are $2^2 = 4$ such functions, namely:

1. f_1 given by $f_1(a) = f_1(b) = c$. Observe that $\mathbf{Im}(f_1) = \{c\}$.
2. f_2 given by $f_2(a) = f_2(b) = d$. Observe that $\mathbf{Im}(f_2) = \{d\}$.
3. f_3 given by $f_3(a) = c$, $f_3(b) = d$. Observe that $\mathbf{Im}(f_1) = \{c,d\}$.
4. f_4 given by $f_4(a) = d$, $f_4(b) = c$. Observe that $\mathbf{Im}(f_1) = \{c,d\}$.

It is easy to see that if A has n elements and B has m elements, the number of functions from A to B is m^n.

179 Example Let

$$\gamma: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2-2 \end{array}.$$

Find

1. $\gamma(6)$
2. $\gamma(1)$
3. $\gamma(\gamma(x))$

Solution: We have

1. $\gamma(6) = 6^2 - 2 = 34$
2. $\gamma(1) = 1^2 - 2 = -1$
3. $\gamma(\gamma(x)) = (\gamma(x))^2 - 2 = (x^2-2)^2 - 2 = x^4 - 4x^2 + 2$

180 Example Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(2x+4) = x^2 - 2$. Find

1. $f(6)$
2. $f(1)$
3. $f(x)$
4. $f(f(x))$

Solution:

1. We need $2x+4 = 6 \implies x = 1$. Hence
$$f(6) = f(2(1)+4) = 1^2 - 2 = -1.$$
2. We need $2x+4 = 1 \implies x = -\frac{3}{2}$. Hence
$$f(1) = f\left(2\left(-\frac{3}{2}\right)+4\right) = \left(-\frac{3}{2}\right)^2 - 2 = \frac{1}{4}.$$
3. First rename the dummy variable: say $f(2u+4) = u^2 - 2$. We need $2u+4 = x \implies u = \frac{x-4}{2}$. Hence
$$f(x) = f\left(2\left(\frac{x-4}{2}\right)+4\right) = \left(\frac{x-4}{2}\right)^2 - 2 = \frac{x^2}{4} - 2x + 2.$$
4. Using the above part,
$$\begin{aligned} f(f(x)) &= \frac{(f(x))^2}{4} - 2f(x) + 2 \\ &= \frac{\left(\frac{x^2}{4} - 2x + 2\right)^2}{4} - 2\left(\frac{x^2}{4} - 2x + 2\right) + 2 \\ &= \frac{x^4}{64} - \frac{x^3}{4} + \frac{3x^2}{4} + 2x - 1 \end{aligned}$$

181 Example $f:\mathbb{R}\to\mathbb{R}$ is a function satisfying $f(3)=2$ and $f(x+3)=f(3)f(x)$. Find $f(-3)$.

Solution: Since we are interested in $f(-3)$, we first put $x=-3$ in the relation, obtaining

$$f(0)=f(3)f(-3).$$

Thus we must also know $f(0)$ in order to find $f(-3)$. Letting $x=0$ in the relation,

$$f(3)=f(3)f(-3) \implies f(-3)=1.$$

Notice that the fact that $f(3)=2$ is irrelevant, we only need $f(3)\neq 0$.

182 Example (Cauchy's Functional Equation) Suppose f satisfies $f(x+y)=f(x)+f(y)$. Prove that $\exists c\in\mathbb{R}$ such that $f(x)=cx,\ \forall x\in\mathbb{Q}$.

Solution: Letting $y=0$ we obtain $f(x)=f(x)+f(0)$, and so $f(0)=0$. If k is a positive integer we obtain

$$\begin{aligned} f(kx) &= f(x+(k-1)x) \\ &= f(x)+f((k-1)x) \\ &= f(x)+f(x)+f((k-2)x)=2f(x)+f((k-2)x) \\ &= 2f(x)+f(x)+f((k-3)x)=3f(x)+f((k-3)x) \\ &\vdots \\ &= \cdots = kf(x)+f(0)=kf(x). \end{aligned}$$

Letting $y=-x$ we obtain $0=f(0)=f(x)+f(-x)$ and so $f(-x)=-f(x)$. Hence $f(nx)=nf(x)$ for $n\in\mathbb{Z}$. Let $x\in\mathbb{Q}$, which means that $x=\dfrac{s}{t}$ for integers s,t with $t\neq 0$. This means that $tx=s\cdot 1$ and so $f(tx)=f(s\cdot 1)$ and by what was just proved for integers, $tf(x)=sf(1)$. Hence $f(x)=\dfrac{s}{t}f(1)=xf(1)$. Since $f(1)$ is a constant, we may put $c=f(1)$. Thus $f(x)=cx$ for rational numbers x.

Homework

183 Problem Draw the graph of the function $f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & |2x-1| \end{array}$ and find $f(-1), f(0), f(1)$.

184 Problem Find all functions from $\{0,1,2\}$ to $\{-1,1\}$.

185 Problem Find all functions from $\{-1,1\}$ to $\{0,1,2\}$.

186 Problem Let $f:\mathbf{Dom}(f)\to\mathbb{R}$ be a function. f is said to have a *fixed point* at $t\in\mathbf{Dom}(f)$ if $f(t)=t$. Let $s:[0;+\infty[\to\mathbb{R}$, $s(x)=x^5-2x^3+2x$. Find all fixed points of s.

187 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto x^2-x$. Find

$$\frac{f(x+h)-f(x-h)}{h}.$$

188 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto x^3-3x$. Find

$$\frac{f(x+h)-f(x-h)}{h}.$$

189 Problem Let $a:\mathbb{R}\to\mathbb{R}$, be given by $a(2-x)=x^2-5x$. Find $a(3)$, $a(x)$ and $a(a(x))$.

190 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $f(1-x)=x^2-2$. Find $f(-2)$, $f(x)$ and $f(f(x))$.

191 Problem Let $:\mathbb{R}\to\mathbb{R}$, $h(x+2)=1+x-x^2$. Express $h(x-1)$, $h(x)$, $h(x+1)$ as powers of x.

192 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $f(x+1)=x^2$. Find $f(x), f(x+2)$ and $f(x-2)$ as powers of x.

193 Problem Let $h:\mathbb{R}\to\mathbb{R}$ be given by $h(1-x)=2x$. Find $h(3x)$.

194 Problem Consider the polynomial

$$(1-x^2+x^4)^{2003} = a_0 + a_1x + a_2x^2 + \cdots + a_{8012}x^{8012}.$$

Find

❶ a_0

❷ $a_0 + a_1 + a_2 + \cdots + a_{8012}$

❸ $a_0 - a_1 + a_2 - a_3 + \cdots - a_{8011} + a_{8012}$

❹ $a_0 + a_2 + a_4 + \cdots + a_{8010} + a_{8012}$

❺ $a_1 + a_3 + \cdots + a_{8009} + a_{8011}$

195 Problem Let $f : \mathbb{R} \to \mathbb{R}$, be a function such that $\forall x \in]0; +\infty[$,

$$[f(x^3+1)]^{\sqrt{x}} = 5,$$

find the value of

$$\left[f\left(\frac{27+y^3}{y^3}\right)\right]^{\sqrt{\frac{27}{y}}}$$

for $y \in]0; +\infty[$.

196 Problem Find all functions g that satisfy $g(x+y)+g(x-y) = 2x^2+2y^2$.

197 Problem Find all the functions f that satisfy $f(xy) = yf(x)$.

198 Problem Find all functions f for which

$$f(x)+2f(\frac{1}{x}) = x.$$

199 Problem Let f satisfy $f(n+1) = (-1)^{n+1}n - 2f(n), n \geq 1$ If $f(1) = f(1001)$ find $f(1)+f(2)+f(3)+\cdots+f(1000)$.

200 Problem If $f(a)f(b) = f(a+b)\ \forall\, a,b \in \mathbb{R}$ and $f(x) > 0\ \forall\, x \in \mathbb{R}$, find $f(0)$. Also, find $f(-a)$ and $f(2a)$ in terms of $f(a)$.

201 Problem Find all functions $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$ such that

$$(f(x))^2 \cdot f\left(\frac{1-x}{1+x}\right) = 64x.$$

3.2 Piecewise Functions

Sometimes the assignment rule of a function varies from interval to interval. We call any such function a *piecewise function*.

202 Example Write $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |2x-1|$ as a piecewise function, and draw its graph.

Solution: We have $f(x) = 2x-1$ for $2x-1 \geq 0$ and $f(x) = -(2x-1)$ for $2x-1 < 0$. This gives

$$f(x) = \begin{cases} 2x-1 & \text{if } x \leq \frac{1}{2} \\ 1-2x & \text{if } x > \frac{1}{2} \end{cases}$$

The graph can be observed in figure 2.22.

203 Example A function f is only defined for $x \in [-4;4]$, and it is made of straight lines, as in figure 3.11. Find a piecewise formula for f.

Solution: The first line segment $\mathscr{L}_1$ has slope

$$\text{slope } \mathscr{L}_1 = \frac{1-(-3)}{-1-(-4)} = \frac{4}{3},$$

and so the equation of the line containing this line segment is of the form $y = \frac{4}{3}x + k_1$. Since $(-1,1)$ is on the line, $1 = -\frac{4}{3} + k_1 \implies k_1 = \frac{7}{3}$, so this line segment is contained in the line $y = \frac{4}{3}x + \frac{7}{3}$. The second line segment $\mathscr{L}_2$ has slope

$$\text{slope } \mathscr{L}_2 = \frac{1-1}{2-(-1)} = 0,$$

and so this line segment is contained in the line $y = 1$. Finally, the third line segment $\mathscr{L}_3$ has slope

$$\text{slope } \mathscr{L}_3 = \frac{-5-1}{4-2} = -3,$$

and so this line segment is part of the line of the form $y = -3x + k_2$. Since $(1,2)$ is on the line, we have $2 = -3 + k_2 \implies k_2 = 5$, and so the line segment is contained on the line $y = -3x+5$. Upon assembling all this we see that the piecewise function required is

$$f(x) = \begin{cases} \frac{4}{3}x + \frac{7}{3} & \text{if } x \in [-4;-1] \\ 1 & \text{if } x \in [-1;2] \\ -3x+5 & \text{if } x \in [2;4] \end{cases}$$

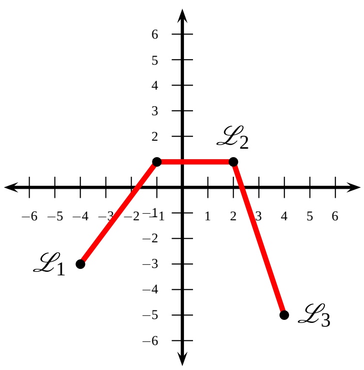


Figure 3.11: Example 203.

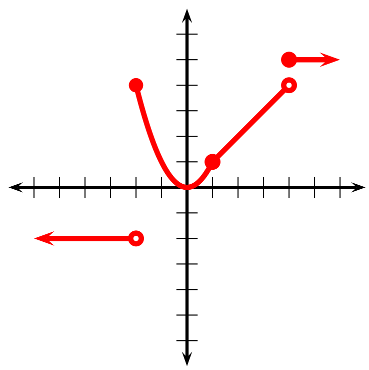


Figure 3.12: Example 204.

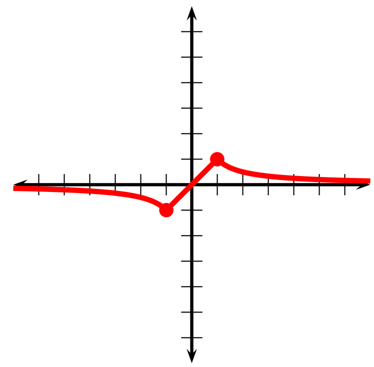


Figure 3.13: Example 205.

Sometimes the pieces in a piecewise function do not connect at a particular point, let us say at $x = a$. Then we write $f(a-)$ for the value that $f(x)$ would have if we used the assignment rule for values near a but smaller than a, and $f(a+)$ for the value that $f(x)$ would have if we used the assignment rule for values near a but larger than a.

204 Example The function $f : \mathbb{R} \to \mathbb{R}$ is piecewise defined by

$$f(x) = \begin{cases} -2 & \text{if } x \in]-\infty;-2[\\ x^2 & \text{if } x \in [-2;1] \\ x & \text{if } x \in]1;4[\\ 5 & \text{if } x \in [4;+\infty[\end{cases}$$

Its graph appears in figure 3.12. We have, for example,

1. $f(-3) = -2$
2. $f(-2-) = -2$
3. $f(-2) = (-2)^2 = 4$
4. $f(-2+) = (-2)^2 = 4$
5. $f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$
6. $f(1-) = (1)^2 = 1$
7. $f(1) = (1)^2 = 1$
8. $f(1+) = 1$
9. $f(2) = 2$
10. $f(4-) = 4$
11. $f(4) = 5$
12. $f(4+) = 5$

205 Example The function $g : \mathbb{R} \to \mathbb{R}$ is piecewise defined by

$$g(x) = \begin{cases} \frac{1}{x} & \text{if } x \in]-\infty;-1[\\ x & \text{if } x \in [-1;1] \\ \frac{1}{x} & \text{if } x \in]1;+\infty[\end{cases}$$

Its graph appears in figure 3.13. We have, for example,

1. $g(-\infty) = 0$, using 1.9
2. $g(-1-) = \frac{1}{-1} = -1$
3. $g(-1) = -1$
4. $g(-1+) = -1$
5. $g(0) = 0$
6. $g(1) = 1$
7. $g(1+) = \frac{1}{1} = 1$
8. $g(+\infty) = 0$, using 1.9

206 Definition The *floor function* $\lfloor \cdot \rfloor : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lfloor x \rfloor \end{array}$ is defined as follows. $\lfloor x \rfloor$ is the unique integer satisfying the inequalities

$$x - 1 < \lfloor x \rfloor \leq x.$$

The *ceiling function* $\lceil \cdot \rceil : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lceil x \rceil \end{array}$ is defined as follows. $\lceil x \rceil$ is the unique integer satisfying the inequalities

$$x \leq \lceil x \rceil < x + 1.$$

If x is an integer, then $\lfloor x \rfloor = x = \lceil x \rceil$. If x is not an integer then $\lceil x \rceil$ is the integer just to the right of x and $\lfloor x \rfloor$ is the integer just to the left of x.

A portion of the graph of the floor function appears in figure 3.14 and a portion of the graph of the ceiling function appears in figure 3.15.

207 Example We have

$$\lfloor \pi \rfloor = 3, \quad \lfloor -\pi \rfloor = -4, \quad \lceil \pi \rceil = 4, \quad \lceil -\pi \rceil = -3, \quad \lfloor -8 \rfloor = \lceil -8 \rceil = -8.$$

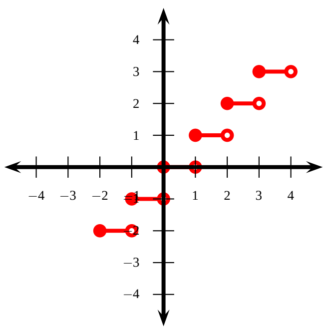


Figure 3.14: The floor function $x \mapsto \lfloor x \rfloor$.

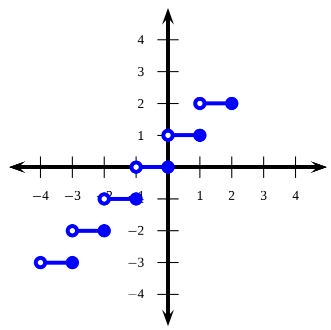


Figure 3.15: The ceiling function $x \mapsto \lceil x \rceil$.

Homework

208 Problem Write $f:\mathbb{R}\to\mathbb{R}$, $f(x)=|x-1|+|x+2|$ as a piecewise function. Graph this function.

209 Problem Write $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x|x|$ as a piecewise function.

210 Problem Write $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x+|x|$ as a piecewise function and draw its graph.

211 Problem Graph $x\mapsto\lfloor 2x\rfloor$.

212 Problem Graph $x\mapsto\lfloor x^2\rfloor$.

213 Problem Let $g:\mathbb{R}\to\mathbb{R}$ be piecewise defined by

$$g(x)=\begin{cases}\dfrac{1}{x} & \text{if } x\in]-\infty;-1[\\ |x| & \text{if } x\in[-1;1]\\ 1+2x & \text{if } x\in]1;+\infty[\end{cases}$$

Graph it and determine

1. $g(-\infty)$
2. $g(-1-)$
3. $g(-1)$
4. $g(-1+)$
5. $g(0)$
6. $g(1-)$
7. $g(1)$
8. $g(1+)$
9. $g(+\infty)$

214 Problem Solve the equation $\lfloor\dfrac{x}{5}\rfloor=10$.

3.3 Translations

In this section we study how several rigid transformations affect both the graph of a function and its assignment rule.

215 Theorem Let f be a function and let v and h be real numbers. If (x_0,y_0) is on the graph of f, then (x_0,y_0+v) is on the graph of g, where $g(x)=f(x)+v$, and if (x_1,y_1) is on the graph of f, then (x_1-h,y_1) is on the graph of j, where $j(x)=f(x+h)$.

Proof: *Let $\Gamma_f,\Gamma_g,\Gamma_j$ denote the graphs of f,g,j respectively.*

$$(x_0,y_0)\in\Gamma_f \iff y_0=f(x_0) \iff y_0+v=f(x_0)+v \iff y_0+v=g(x_0) \iff (x_0,y_0+v)\in\Gamma_g.$$

Similarly,

$$(x_1,y_1)\in\Gamma_f \iff y_1=f(x_1) \iff y_1=f(x_1-h+h) \iff y_1=j(x_1-h) \iff (x_1-h,y_1)\in\Gamma_j.$$

❑

216 Definition Let f be a function and let v and h be real numbers. We say that the curve $y=f(x)+v$ is a *vertical translation* of the curve $y=f(x)$. If $v>0$ the translation is v up, and if $v<0$, it is v units down. Similarly, we say that the curve $y=f(x+h)$ is a *horizontal translation* of the curve $y=f(x)$. If $h>0$, the translation is h units left, and if $h<0$, then the translation is h units right.

217 Example If $f(x)=x^2$, then figures 3.16, 3.17 and 3.18 shew vertical translations 3 units up and 3 units down, respectively. Figures 3.19, 3.20, and 3.21, respectively shew a horizontal translation 3 units right, 3 units left, and a simultaneous translation 3 units left and down. The corresponding assignment rules are

$$a(x)=f(x)+3=x^2+3, \quad b(x)=f(x)-3=x^2-3; \quad c(x)=f(x+3)=(x-3)^2; \quad d(x)=(x+3)^2; \quad g(x)=(x+3)^2-3$$

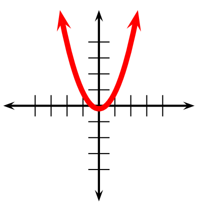


Figure 3.16: $y=f(x)=x^2$

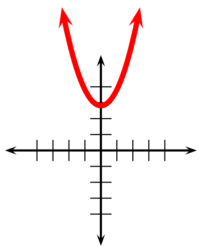


Figure 3.17: $y=a(x)=x^2+3$

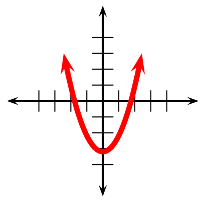


Figure 3.18: $y=b(x)=x^2-3$

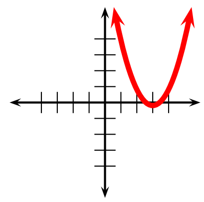


Figure 3.19: $y=c(x)=(x-3)^2$

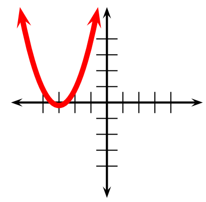


Figure 3.20: $y=d(x)=(x+3)^2$

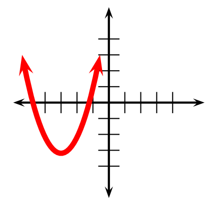


Figure 3.21: $y=g(x)=(x+3)^2-3$

218 Example If $g(x)=x$ (figure 3.22), then figures , 3.23 and 3.24 shew vertical translations 3 units up and 3 units down, respectively. Notice than in this case $g(x+t)=x+t=g(x)+t$, so a vertical translation by t units has exactly the same graph as a horizontal translation t units.

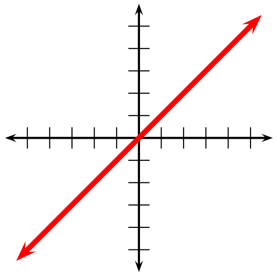


Figure 3.22: $y=g(x)=x$

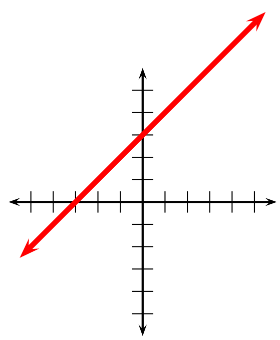


Figure 3.23: $y=g(x)+3=x+3$

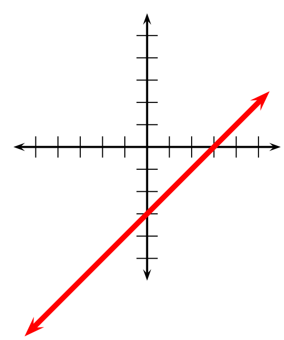


Figure 3.24: $y=g(x)-3=x-3$

219 Definition Given a function f we write $f(-\infty)$ for the value that f may eventually approach for large (in absolute value) and negative inputs and $f(+\infty)$ for the value that f may eventually approach for large (in absolute value) and positive input. The line $y=b$ is a (horizontal) *asymptote* for the function f if either

$$f(-\infty)=b \qquad \text{or} \qquad f(+\infty)=b.$$

220 Definition Let $k>0$ be an integer. A function f has a *pole of order* k at the point $x=a$ if $(x-a)^{k-1}f(x)\longrightarrow\pm\infty$ as $x\longrightarrow a$, but $(x-a)^k f(x)$ as $x\longrightarrow a$ is finite. Some authors prefer to use the term *vertical asymptote*, rather than pole.

221 Example Since $xf(x)=1$, $f(0-)=-\infty$, $f(0+)=+\infty$ for $f:\begin{array}{ccc}\mathbb{R}\setminus\{0\} & \to & \mathbb{R}\\ x & \mapsto & \dfrac{1}{x}\end{array}$, f has a pole of order 1 at $x=0$.

222 Example Figures 3.25 through 3.27 exhibit various transformations of $y = a(x) = \frac{1}{x}$. Notice how the poles and the asymptotes move with the various transformations.

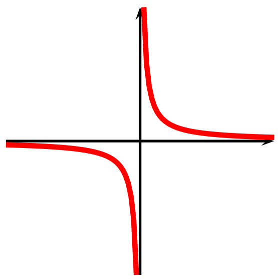


Figure 3.25: $x \mapsto \frac{1}{x}$

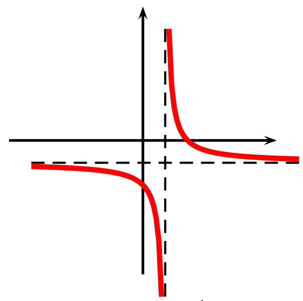


Figure 3.26: $x \mapsto \frac{1}{x-1} - 1$

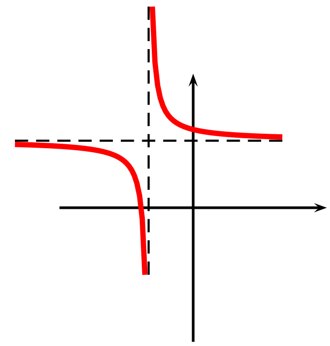


Figure 3.27: $x \mapsto \frac{1}{x+2} + 3$

Homework

223 Problem Graph the following curves:

❶ $y = |x-2| + 3$

❷ $y = (x-2)^2 + 3$

❸ $y = \frac{1}{x-2} + 3$

❹ $y = \sqrt{1-(x-2)^2} + 3$

❺ $y = \sqrt{4-x^2} + 1$

224 Problem What is the equation of the curve $y = f(x) = x^3 - \frac{1}{x}$ after a successive translation one unit down and two units right?

225 Problem Suppose the curve $y = f(x)$ is translated a units vertically and b units horizontally, in this order. Would that have the same effect as translating the curve b units horizontally first, and then a units vertically?

3.4 Distortions

226 Theorem Let f be a function and let $V \neq 0$ and $H \neq 0$ be real numbers. If (x_0, y_0) is on the graph of f, then (x_0, Vy_0) is on the graph of g, where $g(x) = Vf(x)$, and if (x_1, y_1) is on the graph of f, then $\left(\frac{x_1}{H}, y_1\right)$ is on the graph of j, where $j(x) = f(Hx)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff Vy_0 = Vf(x_0) \iff Vy_0 = g(x_0) \iff (x_0, Vy_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f\left(\frac{x_1}{H} \cdot H\right) \iff y_1 = j\left(\frac{x_1}{H}\right) \iff \left(\frac{x_1}{H}, y_1\right) \in \Gamma_j.$$

❑

227 Definition Let $V > 0$, $H > 0$, and let f be a function. The curve $y = Vf(x)$ is called a *vertical distortion* of the curve $y = f(x)$. The graph of $y = Vf(x)$ is a *vertical dilatation* of the graph of $y = f(x)$ if $V > 1$ and a *vertical contraction* if $0 < V < 1$. The curve $y = f(Hx)$ is called a *horizontal distortion* of the curve $y = f(x)$ The graph of $y = f(Hx)$ is a *horizontal dilatation* of the graph of $y = f(x)$ if $0 < H < 1$ and a *horizontal contraction* if $H > 1$.

228 Example Let $a(x) = \sqrt{4-x^2}$. If $y = \sqrt{4-x^2}$, then $x^2 + y^2 = 4$, which is a circle with centre at $(0,0)$ and radius 2 by virtue of 2.10. Hence

$$y = a(x) = \sqrt{4-x^2}$$

is the upper semicircle of this circle. Figures 3.28 through 3.33 shew various transformations of this curve.

229 Example Draw the graph of the curve $y = 2x^2 - 4x + 1$.

Solution: We complete squares.

$$\begin{aligned} y = 2x^2 - 4x + 1 \quad &\iff \quad \frac{y}{2} = x^2 - 2x + \frac{1}{2} \\ &\iff \quad \frac{y}{2} + 1 = x^2 - 2x + 1 + \frac{1}{2} \\ &\iff \quad \frac{y}{2} + 1 = (x-1)^2 + \frac{1}{2} \\ &\iff \quad \frac{y}{2} = (x-1)^2 - \frac{1}{2} \\ &\iff \quad y = 2(x-1)^2 - 1, \end{aligned}$$

whence to obtain the graph of $y = 2x^2 - 4x + 1$ we (i) translate $y = x^2$ one unit right, (ii) dilate the above graph by factor of two, (iii) translate the above graph one unit down. This succession is seen in figures 3.34 through 3.36.

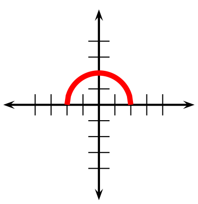


Figure 3.28: $y = a(x) = \sqrt{4-x^2}$

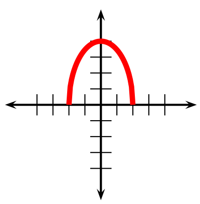


Figure 3.29: $y = 2a(x) = 2\sqrt{4-x^2}$

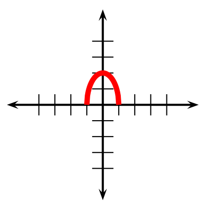


Figure 3.30: $y = a(2x) = \sqrt{4-4x^2}$

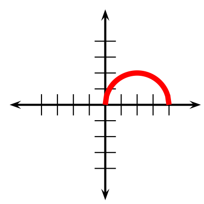


Figure 3.31: $y = a(x-2) = \sqrt{-x^2+4x}$

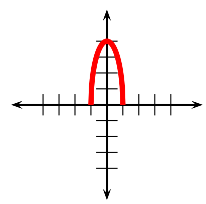


Figure 3.32: $y = 2a(2x) = 2\sqrt{4-4x^2}$

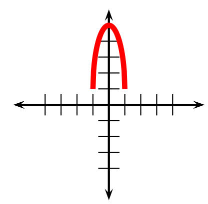


Figure 3.33: $y = 2a(2x) + 1 = 2\sqrt{4-4x^2} + 1$

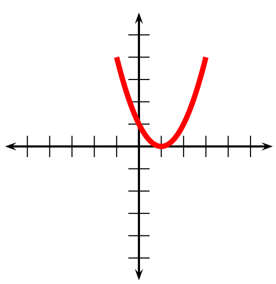


Figure 3.34: $y = (x-1)^2$

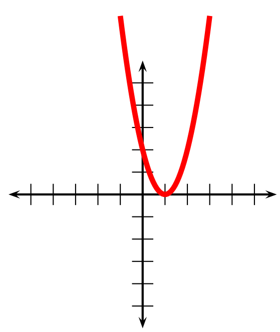


Figure 3.35: $y = 2(x-1)^2$

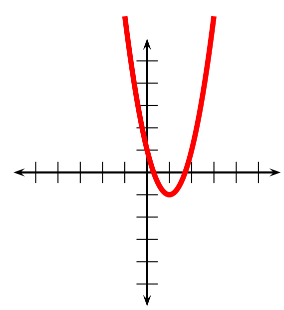


Figure 3.36: $y = 2(x-1)^2 - 1$

230 Example The curve $y = x^2 + \frac{1}{x}$ experiences the following successive transformations: (i) a translation one unit up, (ii) a horizontal shrinkage by a factor of 2, (iii) a translation one unit left. Find its resulting equation.

Solution: After a translation one unit up, the curve becomes

$$y = f(x) + 1 = x^2 + \frac{1}{x} + 1 = a(x).$$

After a horizontal shrinkage by a factor of 2 the curve becomes

$$y = a(2x) = 4x^2 + \frac{1}{2x} + 1 = b(x).$$

After a translation one unit left the curve becomes

$$y = b(x+1) = 4(x+1)^2 + \frac{1}{2x+2} + 1.$$

The required equation is thus

$$y = 4(x+1)^2 + \frac{1}{2x+2} + 1 = 4x^2 + 8x + 5 + \frac{1}{2x+2}.$$

Homework

231 Problem Draw the graphs of the following curves:

❶ $y = \frac{x^2}{2}$

❷ $y = \frac{x^2}{2} - 1$

❸ $y = 2|x| + 1$

❹ $y = \frac{2}{x}$

❺ $y = x^2 + 4x + 5$

❻ $y = 2x^2 + 8x$

232 Problem The curve $y = \frac{1}{x}$ experiences the following successive transformations: (i) a translation one unit left, (ii) a vertical dilatation by a factor of 2, (iii) a translation one unit down. Find its resulting equation and make a rough sketch of the resulting curve.

3.5 Reflexions

233 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, -y_0)$ is on the graph of g, where $g(x) = -f(x)$, and if (x_1, y_1) is on the graph of f, then $(-x_1, y_1)$ is on the graph of j, where $j(x) = f(-x)$.

Proof: *Let $\Gamma_f, \Gamma_g, \Gamma_j$ denote the graphs of f, g, j respectively.*

$$(x_0, y_0) \in \Gamma_f \iff y_0 = f(x_0) \iff -y_0 = -f(x_0) \iff -y_0 = g(x_0) \iff (x_0, -y_0) \in \Gamma_g.$$

Similarly,

$$(x_1, y_1) \in \Gamma_f \iff y_1 = f(x_1) \iff y_1 = f(-(-x_1)) \iff y_1 = j(-x_1) \iff (-x_1, y_1) \in \Gamma_j.$$

❑

234 Definition Let f be a function. The curve $y = -f(x)$ is said to be the *reflexion of f about the x-axis* and the curve $y = f(-x)$ is said to be the *reflexion of f about the y-axis*.

235 Example Figures 3.37 through 3.40 shew various reflexions about the axes.

236 Example Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with

$$f(x) = x + \frac{2}{x} - 1.$$

The curve $y = f(x)$ experiences the following successive transformations:

❶ A reflexion about the x-axis.

❷ A translation 3 units left.

❸ A reflexion about the y-axis.

❹ A vertical dilatation by a factor of 2.

Find the equation of the resulting curve. Note also how the domain of the function is affected by these transformations.

Solution:

❶ A reflexion about the x-axis gives the curve

$$y = -f(x) = 1 - \frac{2}{x} - x = a(x),$$

say, with $\textbf{Dom}(a) = \mathbb{R} \setminus \{0\}$.

❷ A translation 3 units left gives the curve

$$y = a(x+3) = 1 - \frac{2}{x+3} - (x+3) = -2 - \frac{2}{x+3} - x = b(x),$$

say, with $\mathbf{Dom}\,(b) = \mathbb{R} \setminus \{-3\}$.

❸ A reflexion about the y-axis gives the curve

$$y = b(-x) = -2 - \frac{2}{-x+3} + x = c(x),$$

say, with $\mathbf{Dom}\,(c) = \mathbb{R} \setminus \{3\}$.

❹ A vertical dilatation by a factor of 2 gives the curve

$$y = 2c(x) = -4 + \frac{4}{x-3} + 2x = d(x),$$

say, with $\mathbf{Dom}\,(d) = \mathbb{R} \setminus \{3\}$. Notice that the resulting curve is

$$y = d(x) = 2c(x) = 2b(-x) = 2a(-x+3) = -2f(-x+3).$$

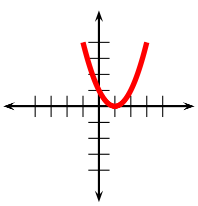


Figure 3.37: $y = d(x) = (x-1)^2$

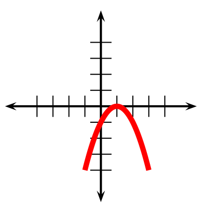


Figure 3.38: $y = -d(x) = -(x-1)^2$

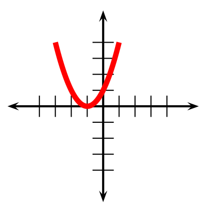


Figure 3.39: $y = d(-x) = (-x-1)^2$

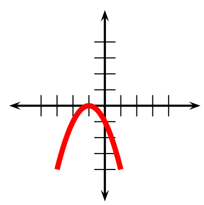


Figure 3.40: $y = -d(-x) = -(-x-1)^2$

Homework

237 Problem Draw the following curves in succession:

❶ $y = \dfrac{1}{x}$

❷ $y = \dfrac{1}{x+1}$

❸ $y = \dfrac{1}{-x+1}$

❹ $y = \dfrac{1}{-x+1} + 2$

238 Problem Let $f : \mathbb{R} \to \mathbb{R}$ with

$$f(x) = 2 - |x|.$$

The curve $y = f(x)$ experiences the following successive transformations:

❶ A reflexion about the x-axis.

❷ A translation 3 units up.

❸ A horizontal stretch by a factor of $\frac{3}{4}$.

Find the equation of the resulting curve.

239 Problem The graphs of the following curves suffer the following successive, rigid transformations:

1. a vertical translation of 2 units down,
2. a reflexion about the y-axis, and finally,
3. a horizontal translation of 1 unit to the left.

Find the resulting equations after all the transformations have been exerted.

❶ $y = x(1-x)$

❷ $y = 2x - 3$

❸ $y = |x+2| + 1$

3.6 Symmetry

240 Definition A function f is *even* if for all x it is verified that $f(x) = f(-x)$, that is, if the portion of the graph for $x < 0$ is a mirror reflexion of the part of the graph for $x > 0$. This means that the graph of f is symmetric about the y-axis. A function g is *odd* if for all x it is verified that $g(-x) = -g(x)$, in other words, g is odd if it is symmetric about the origin. This implies that the portion of the graph appearing in quadrant I is a 180° rotation of the portion of the graph appearing in quadrant III, and the portion of the graph appearing in quadrant II is a 180° rotation of the portion of the graph appearing in quadrant IV.

241 Example The curve in figure 3.41 is even. The curve in figure 3.42 is odd.

242 Theorem Let $\varepsilon_1, \varepsilon_2$ be even functions, and let ω_1, ω_2 be odd functions, all sharing the same common domain. Then

❶ $\varepsilon_1 \pm \varepsilon_2$ is an even function.

❷ $\omega_1 \pm \omega_2$ is an odd function.

❸ $\varepsilon_1 \cdot \varepsilon_2$ is an even function.

❹ $\omega_1 \cdot \omega_2$ is an even function.

❺ $\varepsilon_1 \cdot \omega_1$ is an odd function.

Proof: *We have*

❶ $(\varepsilon_1 \pm \varepsilon_2)(-x) = \varepsilon_1(-x) \pm \varepsilon_2(-x) = \varepsilon_1(x) \pm \varepsilon_2(x)$.
❷ $(\omega_1 \pm \omega_2)(-x) = \omega_1(-x) \pm \omega_2(-x) = -\omega_1(x) \mp \omega_2(x) = -(\omega_1 \pm \omega_2)(x)$
❸ $(\varepsilon_1\varepsilon_2)(-x) = \varepsilon_1(-x)\varepsilon_2(-x) = \varepsilon_1(x)\varepsilon_2(x)$
❹ $(\omega_1\omega_2)(-x) = \omega_1(-x)\omega_2(-x) = (-\omega_1(x))(-\omega_2(x)) = \omega_1(x)\omega_2(x))$
❺ $(\varepsilon_1\omega_1)(-x) = \varepsilon_1(-x)\omega_1(-x) = -\varepsilon_1(x)\omega_1(x)$

❑

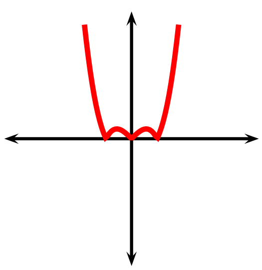


Figure 3.41: Example 241. The graph of an even function.

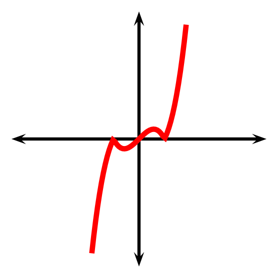


Figure 3.42: Example 241. The graph of an odd function.

243 Corollary Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with real coefficients. Then the function

$$p: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & p(x) \end{array}$$

is an even function if and only if each of its terms has even degree.

Proof: *Assume p is even. Then $p(x) = p(-x)$ and so*

$$\begin{aligned} p(x) &= \frac{p(x)+p(-x)}{2} \\ &= \frac{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n}{2} \\ &\quad + \frac{a_0 - a_1x + a_2x^2 - a_3x^3 + \cdots + (-1)^{n-1}a_{n-1}x^{n-1} + (-1)^na_nx^n}{2} \\ &= a_0 + a_2x^2 + a_4x^4 + \cdots + \end{aligned}$$

and so the polynomial has only terms of even degree. The converse of this statement is trivial. ❑

244 Example Prove that in the product

$$(1-x+x^2-x^3+\cdots-x^{99}+x^{100})(1+x+x^2+x^3+\cdots+x^{99}+x^{100})$$

after multiplying and collecting terms, there does not appear a term in x of odd degree.

Solution: Let $f: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & f(x) \end{array}$ with

$$f(x) = (1-x+x^2-x^3+\cdots-x^{99}+x^{100})(1+x+x^2+x^3+\cdots+x^{99}+x^{100})$$

Then

$$f(-x) = (1+x+x^2+x^3+\cdots+x^{99}+x^{100})(1-x+x^2-x^3+\cdots-x^{99}+x^{100}) = f(x),$$

which means that f is an even function. Since f is a polynomial, this means that f does not have a term of odd degree.

Analogous to Corollary 243, we may establish the following.

245 Corollary Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ be a polynomial with real coefficients. Then the function

$$p: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & p(x) \end{array}$$

is an odd function if and only if each of its terms has odd degree.

246 Theorem Let $f: \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Then f can be written as the sum of an even function and an odd function.

Proof: *Given $x \in \mathbb{R}$, put $E(x) = f(x) + f(-x)$, and $O(x) = f(x) - f(-x)$. We claim that E is an even function and that O is an odd function. First notice that*

$$E(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = E(x),$$

which proves that E is even. Also,

$$O(-x) = f(-x) - f(-(-x)) = -(f(x) - f(-x)) = -O(x),$$

which proves that O is an odd function. Clearly

$$f(x) = \frac{1}{2}E(x) + \frac{1}{2}O(x),$$

which proves the theorem. ❑

247 Example Investigate which of the following functions are even, odd, or neither.

❶ $a: \mathbb{R} \to \mathbb{R}$, $a(x) = \dfrac{x^3}{x^2+1}$.

❷ $b: \mathbb{R} \to \mathbb{R}$, $b(x) = \dfrac{|x|}{x^2+1}$.

❸ $c: \mathbb{R} \to \mathbb{R}$, $c(x) = |x| + 2$.

❹ $d: \mathbb{R} \to \mathbb{R}$, $d(x) = |x+2|$.

❺ $f: [-4;5] \to \mathbb{R}$, $f(x) = |x| + 2$.

Solution:

❶
$$a(-x) = \frac{(-x)^3}{(-x)^2+1} = -\frac{x^3}{x^2+1} = -a(x),$$
whence a is odd, since its domain is also symmetric.

❷
$$b(-x) = \frac{|-x|}{(-x)^2+1} = \frac{|x|}{x^2+1} = b(x),$$
whence b is even, since its domain is also symmetric.

❸
$$c(-x) = |-x|+2 = |x|+2 = c(x),$$
whence c is even, since its domain is also symmetric.

❹ $d(-1) = |-1+2| = 1$, but $d(1) = 3$. This function is neither even nor odd.

❺ The domain of f is not symmetric, so f is neither even nor odd.

248 Theorem Let f be a function. Then both $x \mapsto f(|x|)$ and $x \mapsto f(-|x|)$ are even functions.

> **Proof:** *Put $a(x) = f(|x|)$. Then $a(-x) = f(|-x|) = f(|x|) = a(x)$, whence $x \mapsto a(x)$ is even. Similarly, if $b(x) = f(-|x|)$, then $b(-x) = f(-|-x|) = f(-|x|) = b(x)$ proving that $x \mapsto b(x)$ is even.* ❑

Notice that $f(x) = f(|x|)$ for $x > 0$. Since $x \mapsto f(|x|)$ is even, the graph of $x \mapsto f(|x|)$ is thus obtained by erasing the portion of the graph of $x \mapsto f(x)$ for $x < 0$ and reflecting the part for $x > 0$. Similarly, since $f(x) = f(-|x|)$ for $x < 0$, the graph of $x \mapsto f(-|x|)$ is obtained by erasing the portion of the graph of $x \mapsto f(x)$ for $x > 0$ and reflecting the part for $x < 0$.

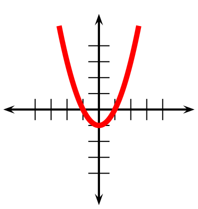


Figure 3.43: $y = g(x) = x^2 - 1$

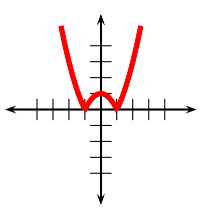


Figure 3.44: $y = |g(x)| = |x^2 - 1|$

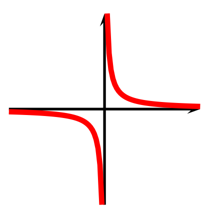


Figure 3.45: $x \mapsto \frac{1}{x}$

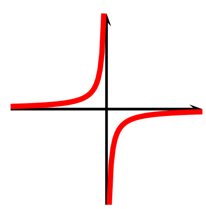


Figure 3.46: $x \mapsto -\frac{1}{x}$

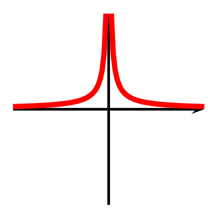


Figure 3.47: $x \mapsto \left|\frac{1}{x}\right|$

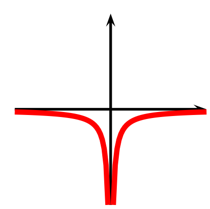


Figure 3.48: $x \mapsto -\left|\frac{1}{x}\right|$

249 Theorem Let f be a function If (x_0, y_0) is on the graph of f, then $(x_0, |y_0|)$ is on the graph of g, where $g(x) = |f(x)|$.

> **Proof:** *Let Γ_f, Γ_g denote the graphs of f, g, respectively.*
> $$(x_0, y_0) \in \Gamma_f \implies y_0 = f(x_0) \implies |y_0| = |f(x_0)| \implies |y_0| = g(x_0) \implies (x_0, |y_0|) \in \Gamma_g.$$
> ❑

250 Example Figures 3.43 and 3.44 shew $y = x^2 - 1$ and $y = |x^2 - 1|$ respectively. Figures 3.45 through 3.48 exhibit various transformations of $x \mapsto \frac{1}{x}$. Figures 3.49 through 3.52 exhibit various transformations of $x \mapsto (x-1)^2 - 3$.

251 Example Figures 6.7 through 6.9 shew a few transformations of $x \mapsto \frac{1}{x-1} - 1$.

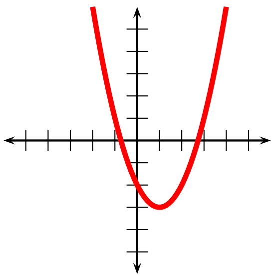


Figure 3.49: $y = f(x) = (x-1)^2 - 3$

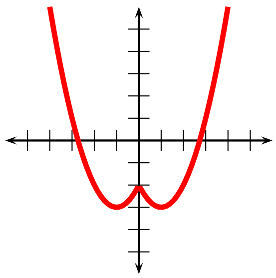


Figure 3.50: $y = f(|x|) = (|x|-1)^2 - 3$

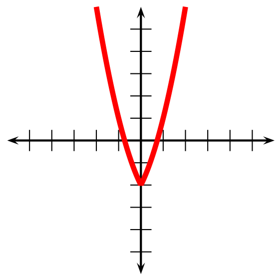


Figure 3.51: $y = f(-|x|) = (-|x|-1)^2 - 3$

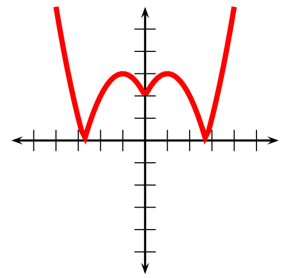


Figure 3.52: $y = |f(|x|)| = |(|x|-1)^2 - 3|$

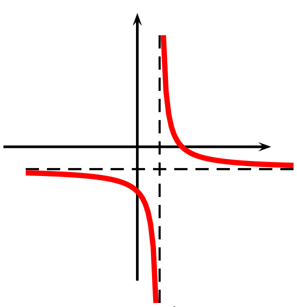


Figure 3.53: $x \mapsto \frac{1}{x-1} - 1$

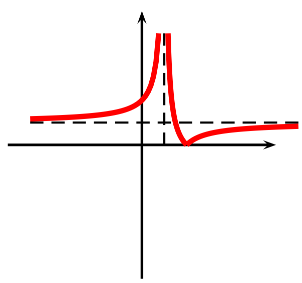


Figure 3.54: $x \mapsto \left|\frac{1}{x-1} - 1\right|$

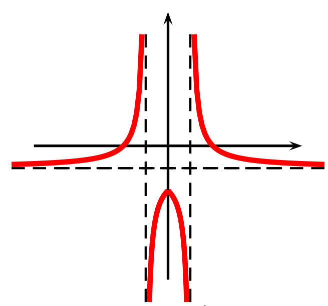


Figure 3.55: $x \mapsto \frac{1}{|x|-1} - 1$

Homework

252 Problem Draw the graph of $f : \mathbb{R} \to \mathbb{R}$, with assignment rule $f(x) = x|x|$.

253 Problem Draw the following curves in succession:

❶ $y = x^2$
❷ $y = (x-1)^2$
❸ $y = (|x|-1)^2$

254 Problem Draw the following curves in succession:

❶ $y = x^2$
❷ $y = x^2 - 1$
❸ $y = |x^2 - 1|$

255 Problem Let f be an odd function and assume that f is defined at $x = 0$. Prove that $f(0) = 0$.

256 Problem Draw the following curves in succession:

❶ $y = x^2 + 2x + 3$
❷ $y = x^2 + 2|x| + 3$
❸ $y = |x^2 + 2x + 3|$
❹ $y = |x^2 + 2|x| + 3|$

257 Problem Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ with $f(x) = \frac{1}{x}$. Draw the following curves in succession and state the changes in the domain of each new curve. Are any of these curves identical?

❶ $y = f(x-2)$
❷ $y = |f(x-2)|$
❸ $y = f(|x|-2)$
❹ $y = f(|x-2|)$

258 Problem Can a curve be even and odd simultaneously? What would the graph of such a curve look like?

259 Problem Draw the following curves in succession:

❶ $y = 1 - x$
❷ $y = |1 - x|$
❸ $y = 1 - |1 - x|$
❹ $y = |1 - |1 - x||$
❺ $y = 1 - |1 - |1 - x||$
❻ $y = |1 - |1 - |1 - x|||$
❼ $y = 1 - |1 - |1 - |1 - x|||$

❽ $y = |1 - |1 - |1 - |1 - x||||$

260 Problem Put $f_1(x) = x$; $f_2(x) = |1 - f_1(x)|$; $f_3(x) = |1 - f_2(x)|$; $\ldots$ $f_n(x) = |1 - f_{n-1}(x)|$. Prove that the solutions of the equation $f_n(x) = 0$ are $\{\pm 1, \pm 3, \ldots, \pm(n-3), (n-1)\}$ if n is even and $\{0, \pm 2, \ldots, \pm(n-3), (n-1)\}$ if n is odd.

261 Problem Use f in figure 3.56 to draw each of the curves below.

❶ $y = 2f(x)$

❷ $y = f(2x)$

❸ $y = f(-x)$

❹ $y = -f(x)$

❺ $y = -f(-x)$

❻ $y = f(|x|)$

❼ $y = |f(x)|$

❽ $y = f(-|x|)$

❾ $y = |f(|x|)|$

❿ $y = |f(-|x|)|$

262 Problem Given in figures 3.57 and 3.58 are the graphs of two curves, $y = f(x)$ and $y = f(ax)$ for some real constant $a < 0$.

❶ Determine the value of the constant a.

❷ Determine the value of C.

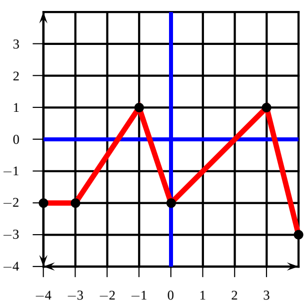


Figure 3.56: $y = f(x)$

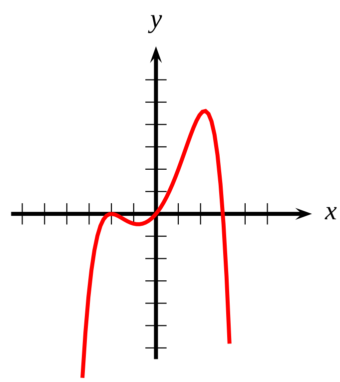


Figure 3.57: Problem 262. $y = f(x)$

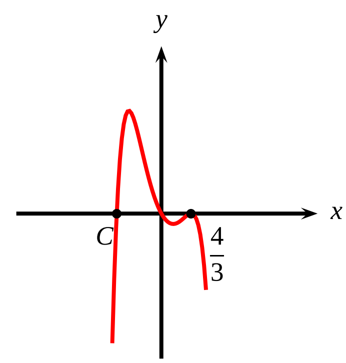


Figure 3.58: Problem 262. $y = f(ax)$

3.7 Behaviour of the Graphs of Functions

So far we have limited our study of functions to those families of functions whose graphs are known to us: lines, parabolas, hyperbolas, or semicircles. Through some arguments involving symmetry we have been able to extend this collection to compositions of the above listed functions with the absolute value function. We would now like to increase our repertoire of functions that we can graph. For that we need the machinery of Calculus, which will be studied in subsequent courses. We will content ourselves with introducing various terms useful when describing curves and with proving that these properties hold for familiar curves.

263 Definition A function f is said to be *continuous* at the point $x = a$ if $f(a-) = f(a) = f(a+)$. It is continuous on the interval I if it is continuous on every point of I.

Heuristically speaking, a continuous function is one whose graph has no "breaks."

264 Example Given that

$$f(x) = \begin{cases} 6 + x & \text{if } x \in]-\infty; -2] \\ 3x^2 + xa & \text{if } x \in]-2; +\infty[\end{cases}$$

is continuous, find a.

Solution: Since $f(-2-) = f(-2) = 6 - 2 = 4$ and $f(-2+) = 3(-2)^2 - 2a = 12 - 2a$ we need

$$f(-2-) = f(-2+) \implies 4 = 12 - 2a \implies a = 4.$$

The graph of f is given in figure 3.59.

265 Example Given that

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

is continuous, find a.

Solution: For $x \neq 1$ we have $f(x) = \dfrac{x^2-1}{x-1} = x+1$. Since $f(1-) = 2$ and $f(1+) = 2$ we need $a = f(1) = 2$. The graph of f is given in figure 3.60.

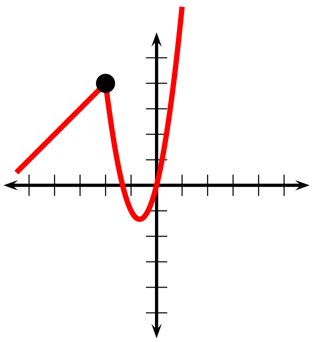


Figure 3.59: Example 264.

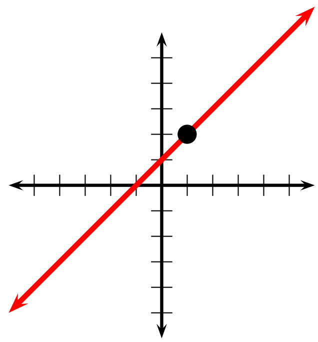


Figure 3.60: Example 265.

We will accept the following results without proof.

266 Theorem (Bolzano's Intermediate Value Theorem) If f is continuous on the interval $[a;b]$ and f and there are two different values in this interval for which f changes sign, then f is vanishes somewhere in this interval, that is, there is $c \in [a;b]$ such that $f(c) = 0$.

267 Corollary If f is continuous on the interval $[a;b]$ with $f(a) \neq f(b)$ then f assumes every value between $f(a)$ and $f(b)$, that is, for d with $\min(f(a), f(b)) \leq d \leq \max(f(a), f(b))$ there is $c \in [a;b]$ such that $f(c) = d$.

268 Theorem (Weierstrass's Theorem) A continuous function on a finite closed interval $[a;b]$ assumes a maximum value and a minimum value.

269 Definition A function f is said to be *increasing* (respectively, *strictly increasing*) if $a < b \implies f(a) \leq f(b)$ (respectively, $a < b \implies f(a) < f(b)$). A function g is said to be *decreasing* (respectively, *strictly decreasing*) if $a < b \implies g(a) \leq g(b)$ (respectively, $a < b \implies g(a) < g(b)$). A function is *monotonic* if it is either (strictly) increasing or decreasing. By the *intervals of monotonicity of a function* we mean the intervals where the function might be (strictly) increasing or decreasing.

If the function f is (strictly) increasing, its opposite $-f$ is (strictly) decreasing, and viceversa.

The following theorem is immediate.

270 Theorem A function f is (strictly) increasing if for all $a < b$ for which it is defined

$$\frac{f(b)-f(a)}{b-a} \geq 0 \qquad \text{(respectively, } \frac{f(b)-f(a)}{b-a} > 0\text{)}.$$

Similarly, a function g is (strictly) decreasing if for all $a < b$ for which it is defined

$$\frac{g(b)-g(a)}{b-a} \leq 0 \qquad \text{(respectively, } \frac{g(b)-g(a)}{b-a} < 0\text{)}.$$

271 Lemma Let $(a,b) \in \mathbb{R}^2, a < b$. Then every number of the form $\lambda a + (1-\lambda)b$, $\lambda \in [0;1]$ belongs to the interval $[a;b]$. Conversely, if $x \in [a;b]$ then we can find a $\lambda \in [0;1]$ such that $x = \lambda a + (1-\lambda)b$.

Proof: *Clearly* $\lambda a + (1-\lambda)b = b + \lambda(a-b)$ *and since* $a - b < 0$,

$$b = b + 0(a-b) \geq b + \lambda(a-b) \geq b + 1(a-b) = a,$$

whence the first assertion follows.

Assume now that $x \in [a;b]$. *Solve the equation* $x = \lambda a + (1-\lambda)b$ *for* λ *obtaining* $\lambda = \frac{x-b}{b-a}$. *All what remains to prove is that* $0 \leq \lambda \leq 1$, *but this is evident, as* $0 \leq x - b \leq b - a$. *This concludes the proof.* ❑

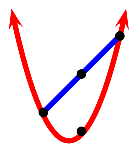


Figure 3.61: A convex curve

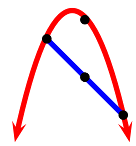


Figure 3.62: A concave curve.

272 Definition A function $f : A \to B$ is *convex* in A if $\forall (a,b,\lambda) \in A^2 \times [0;1]$,

$$f(\lambda a + (1-\lambda)b) \leq f(a)\lambda + (1-\lambda)f(b).$$

Similarly, a function $g : A \to B$ is *concave* in A if $\forall (a,b,\lambda) \in A^2 \times [0;1]$,

$$g(\lambda a + (1-\lambda)b) \geq g(a)\lambda + (1-\lambda)g(b).$$

By the *intervals of convexity (concavity) of a function* we mean the intervals where the function is convex (concave). An *inflexion point* is a point where a graph changes convexity.

By Lemma 271, $\lambda a + (1-\lambda)b$ lies in the interval $[a;b]$ for $0 \leq \lambda \leq 1$. Hence, geometrically speaking, a convex function is one such that if two distinct points on its graph are taken and the straight line joining these two points drawn, then the midpoint of that straight line is above the graph. In other words, the graph of the function bends upwards. Notice that if f is convex, then its opposite $-f$ is concave.

273 Theorem (Graph of the Identity Function) The graph of the identity function

$$\mathbf{Id} : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array}$$

is a straight line that bisects the first and third quadrant. **Id** is an increasing odd function and $\mathbf{Im}(\mathbf{Id}) = \mathbb{R}$.

Proof: *Let the points* $O(0,0)$, $B(1,1)$ *and* $C(x,x)$ *be on the graph of* **Id**. *Consider the projections* $B'(1,0)$, $C'(x,0)$ *of* B *and* C *respectively onto the x-axis.* $\triangle OBB'$ *is rectangle at* $\angle B'$ *and* $OB = \sqrt{2}$. *Also,* $\triangle OCC'$ *is rectangle at angle* C' *and* $OC = \sqrt{2}|x|$. *Thus* $\triangle OBB'$ *and* OCC' *are similar, and hence* $\angle B'OB = \angle C'OC$. *Thus* O, B, *and* C *are collinear. It is clear that the line passing through* $(0,0)$ *and* $(1,1)$ *splits the first and third quadrant halfway. As* $\mathbf{Id}(-x) = -x = -\mathbf{Id}(x)$, *the identity function is an odd function. Since for* $a < b$

$$\frac{\mathbf{Id}(b) - \mathbf{Id}(a)}{b-a} = \frac{b-a}{b-a} = 1 > 0,$$

Id *is a strictly increasing function. Also given any* $x \in \mathbb{R}$ *we have* $\mathbf{Id}(x) = x$ *and so every real number is an image of* **Id** *meaning that* $\mathbf{Im}(\mathbf{Id}) = \mathbb{R}$. ❑

274 Theorem (Graph of the Square Function) The graph of the square function

$$\mathbf{Square}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^2 \end{array}$$

is a convex curve which is strictly decreasing for $x < 0$ and strictly increasing for $x > 0$. Moreover, $x \mapsto x^2$ is an even function and $\mathbf{Im}(\mathbf{Square}) = [0; +\infty[$.

Proof:
As $\mathbf{Square}(-x) = (-x)^2 = x^2 = \mathbf{Square}(x)$, *the square function is an even function. Now, for* $a < b$

$$\frac{\mathbf{Square}(b) - \mathbf{Square}(a)}{b-a} = \frac{b^2 - a^2}{b-a} = b + a.$$

If $a < b < 0$ *the sum* $a+b$ *is negative and* $x \mapsto x^2$ *is a strictly decreasing function. If* $0 < a < b$ *the sum* $a+b$ *is positive and* $x \mapsto x^2$ *is a strictly increasing function. To prove that* $x \mapsto x^2$ *is convex we observe that*

$$\mathbf{Square}(\lambda a + (1-\lambda)b) \leq \lambda \mathbf{Square}(a) + (1-\lambda)\mathbf{Square}(b)$$

$$\begin{aligned}
&\iff \lambda^2 a^2 + 2\lambda(1-\lambda)ab + (1-\lambda)^2 b^2 \leq \lambda a^2 + (1-\lambda)b^2 \\
&\iff 0 \leq \lambda(1-\lambda)a^2 - 2\lambda(1-\lambda)ab + ((1-\lambda) - (1-\lambda)^2)b^2 \\
&\iff 0 \leq \lambda(1-\lambda)a^2 - 2\lambda(1-\lambda)ab + \lambda(1-\lambda)b^2 \\
&\iff 0 \leq \lambda(1-\lambda)(a^2 - 2ab + b^2) \\
&\iff 0 \leq \lambda(1-\lambda)(a-b)^2.
\end{aligned}$$

This last inequality is clearly true for $\lambda \in [0;1]$, *establishing the claim. Also suppose that* $y \in \mathbf{Im}(\mathbf{Square})$. *Thus there is* $x \in \mathbb{R}$ *such that* $x^2 = x \mapsto x^2(x) = y$. *But the equation* $y = x^2$ *is solvable only for* $y \geq 0$ *and so only positive numbers appear as the image of* $x \mapsto x^2$. *Hence* $\mathbf{Im}(\mathbf{Square}) = [0 : +\infty[$. *The graph of the* $x \mapsto x^2$ *is called a* parabola. *We summarise this information by means of the following diagram.*

x	$-\infty$	0	$+\infty$
$f(x) = x^2$	$\searrow$	0	$\nearrow$

❑

275 Theorem (Graph of the Absolute Value Function) The graph of the absolute value function

$$\mathbf{AbsVal}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & |x| \end{array}$$

is convex. $x \mapsto |x|$ is an even function, decreasing for $x < 0$ and increasing for $x > 0$. Moreover, $\mathbf{Im}(\mathbf{AbsVal}) = [0; +\infty[$.

Proof: *To prove that* $x \mapsto |x|$ *is convex, we use the triangle inequality theorem 57 and the fact that* $|\lambda| = \lambda$, $|1-\lambda| = 1-\lambda$ *for* $\lambda \in [0;1]$. *We have*

$$\begin{aligned}
\mathbf{AbsVal}(\lambda a + (1-\lambda)b) &= |\lambda a + (1-\lambda)b| \\
&\leq |\lambda a| + |(1-\lambda)b| \\
&= \lambda|a| + (1-\lambda)|b| \\
&= \lambda \mathbf{AbsVal}(a) + (1-\lambda)\mathbf{AbsVal}(b),
\end{aligned}$$

whence $x \mapsto |x|$ is convex. As $\mathbf{AbsVal}(-x) = |-x| = |x| = \mathbf{AbsVal}(x)$*, the absolute value function is an even function. For* $a < b < 0$,

$$\frac{\mathbf{AbsVal}(b) - \mathbf{AbsVal}(a)}{b-a} = \frac{|b|-|a|}{b-a} = \frac{-b-(-a)}{b-a} = -1 < 0,$$

$x \mapsto |x|$ is a strictly decreasing function for $x < 0$. Similarly, for $0 < a < b$

$$\frac{\mathbf{AbsVal}(b) - \mathbf{AbsVal}(a)}{b-a} = \frac{|b|-|a|}{b-a} = \frac{b-a}{b-a} = 1 > 0,$$

and so $x \mapsto |x|$ is a strictly increasing function for $x > 0$. Also, assume that $y \in$ $\mathbf{Im}(\mathbf{AbsVal})$*. Then $\exists x \in \mathbb{R}$ with $y =$* $\mathbf{AbsVal}(x) = |x|$*, which means that $y \geq 0$ and so* $\mathbf{Im}(\mathbf{AbsVal}) = [0; +\infty[$.

To obtain the graph of $x \mapsto |x|$ we graph the line $y = -x$ for $x < 0$ and the line $y = x$ for $x \geq 0$. ❑

276 Theorem (Graph of the Reciprocal Function) The graph of the reciprocal function

$$\mathbf{Rec}: \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{1}{x} \end{array}$$

is concave for $x < 0$ and convex for $x > 0$. $x \mapsto \dfrac{1}{x}$ is decreasing for $x < 0$ and $x > 0$. $x \mapsto \dfrac{1}{x}$ is an odd function and $\mathbf{Im}(\mathbf{Rec}) = \mathbb{R}\setminus\{0\}$.

Proof: *Assume first that $0 < a < b$ and that $\lambda \in [0;1]$. By the Arithmetic-Mean-Geometric-Mean Inequality, Theorem 16, we deduce that*

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

Hence the product

$$\begin{aligned}(\lambda a + (1-\lambda)b)\left(\frac{\lambda}{a} + \frac{1-\lambda}{b}\right) &= \lambda^2 + (1-\lambda)^2 + \lambda(1-\lambda)\left(\frac{a}{b} + \frac{b}{a}\right) \\ &\geq \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) \\ &= (\lambda + 1 - \lambda)^2 \\ &= 1.\end{aligned}$$

Thus for $0 < a < b$ we have

$$\frac{1}{\lambda a + (1-\lambda)b} \leq \left(\frac{\lambda}{a} + \frac{1-\lambda}{b}\right) \implies \mathbf{Rec}(\lambda a + (1-\lambda)b) \leq \lambda\mathbf{Rec}(a) + (1-\lambda)\mathbf{Rec}(b),$$

from where $x \mapsto \dfrac{1}{x}$ is convex for $x > 0$. If we replace a, b by $-a, -b$ then the inequality above is reversed and we obtain that $x \mapsto \dfrac{1}{x}$ is concave for $x < 0$.

As $\mathbf{Rec}(-x) = \dfrac{1}{-x} = -\dfrac{1}{x} = -\mathbf{Rec}(x)$*, the reciprocal function is an odd function. Assume $a < b$ are non-zero and have the same sign. Then*

$$\frac{\mathbf{Rec}(b) - \mathbf{Rec}(a)}{b-a} = \frac{\frac{1}{b} - \frac{1}{a}}{b-a} = -\frac{1}{ab} < 0,$$

since we are assuming that a, b have the same sign, whence $x \mapsto \dfrac{1}{x}$ is a strictly decreasing function whenever arguments have the same sign. Also given any $y \in$ $\mathbf{Im}(\mathbf{Rec})$ *we have $y =$* $\mathbf{Rec}(x) = \dfrac{1}{x}$*, but this equation is solvable only if $y \neq 0$. and so every real number is an image of* **Id** *meaning that* $\mathbf{Im}(\mathbf{Rec}) = \mathbb{R}\setminus\{0\}$.

❑

277 Definition Let f be a function. If f is defined at $x = 0$, then $(0, f(0))$ is its *y-intercept*. The points $(x, 0)$ on the x-axis for which $f(x) = 0$, if any, are the *x-intercepts* of f.

278 Definition A *zero* or *root* of a function f is a solution to the equation $f(x) = 0$.

279 Definition Let $\gamma : \begin{array}{ccc} \mathbf{Dom}(\gamma) & \to & \mathbf{Im}(\gamma) \\ x & \mapsto & \gamma(x) \end{array}$ be a function. We say that γ is periodic of period P if $\forall x \in \mathbf{Dom}(\gamma)$, we have $x + P \in \mathbf{Dom}(\gamma)$ and $\gamma(x + P) = \gamma(x)$.

Observe that if P is the period of γ then $\forall n \in \mathbb{Z}, \gamma(x + nP) = \gamma(x)$. Thus the graph of a periodic curve consists of horizontal translations of copies of a single period.

280 Example The graph in figure 3.63 represents the periodic function $f : \mathbb{R} \to [0;1]$ with $f(x) = \{x\} = x - \lfloor x \rfloor$, that is, the decimal part of the real number x.

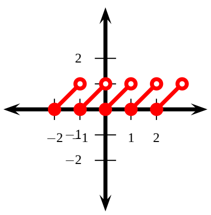


Figure 3.63: Example 280

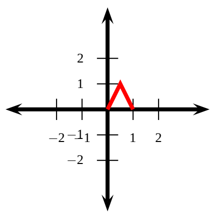


Figure 3.64: Example 281.

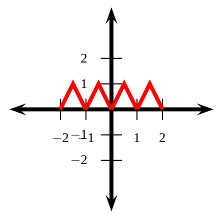


Figure 3.65: Example 281.

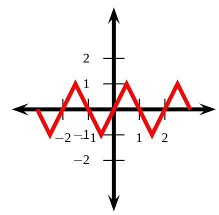


Figure 3.66: Example 281.

281 Example The graph in figure 3.64 shews the portion of a curve. Extend the graph of the curve if it is to be (i) periodic and even, (ii) periodic and odd.

Solution: The required graphs are shewn in figure 3.65 and 3.66. Notice that the period of (i) is 1 and the period of (ii) is 2.

Homework

282 Problem Give an example of a function which is discontinuous on the set $\{-1, 0, 1\}$ but continuous everywhere else.

283 Problem Give an example of a function discontinuous at the points $\pm\sqrt[3]{1}, \pm\sqrt[3]{2}, \pm\sqrt[3]{3}, \pm\sqrt[3]{4}, \pm\sqrt[3]{5}, \ldots$.

284 Problem Give an example of a function r discontinuous at the reciprocal of every non-zero integer.

285 Problem Give an example of a function discontinuous at the odd integers.

286 Problem Give an example of a function discontinuous at the square of every integer.

287 Problem Given that

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x \leq 1 \\ 2x + 3a & \text{if } x > 1 \end{cases}$$

is continuous, find a.

288 Problem Let n be a strictly positive integer. Given that

$$f(x) = \begin{cases} \dfrac{x^n - 1}{x - 1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

is continuous, find a.

289 Problem Let $||x|| = \min_{n \in \mathbb{Z}} |x - n|$. Prove that $x \mapsto ||x||$ is periodic and find its period. Notice that this function measures the distance of a real number to its nearest integer.

290 Problem Prove that $f : \mathbb{R} \to \mathbb{R},\ f(t+1) = \dfrac{1}{2} + \sqrt{f(t) - (f(t))^2}$ has period 2.

Answers

183 The graph is in figure **??**. $f(-1)=3$, $f(0)=1$, $f(1)=1$.

184 There are $2^3=8$ such functions:

1. f_1 given by $f_1(0)=f_1(1)=f_1(2)=-1$
2. f_2 given by $f_2(0)=1, f_2(1)=f_2(2)=-1$
3. f_3 given by $f_3(0)=f_3(1)=-1, f_3(2)=1$
4. f_4 given by $f_4(0)=-1, f_4(1)=1, f_4(2)=-1$
5. f_5 given by $f_5(0)=f_5(1)=f_5(2)=1$
6. f_6 given by $f_6(0)=-1, f_6(1)=f_6(2)=1$
7. f_7 given by $f_7(0)=f_7(1)=1, f_7(2)=-1$
8. f_8 given by $f_8(0)=1, f_8(1)=-1, f_8(2)=1$

185 There are $3^2=9$ such functions:

1. f_1 given by $f_1(-1)=f_1(1)=0$
2. f_2 given by $f_2(-1)=f_2(1)=1$
3. f_3 given by $f_3(-1)=f_3(1)=2$
4. f_4 given by $f_4(-1)=0, f_4(1)=1$
5. f_5 given by $f_5(-1)=0, f_5(1)=2$
6. f_6 given by $f_6(-1)=1, f_6(1)=2$
7. f_7 given by $f_7(-1)=1, f_7(1)=0$
8. f_8 given by $f_8(-1)=2, f_8(1)=0$
9. f_9 given by $f_9(-1)=2, f_9(1)=1$

186 We must look for all $x\in \mathbf{Dom}(f)$ such that $s(x)=x$. Thus

$$\begin{array}{lll} s(x)=x & \Longrightarrow & x^5-2x^3+2x=x \\ & \Longrightarrow & x^5-2x^3+x=0 \\ & \Longrightarrow & x(x^4-2x^2+1)=0 \\ & \Longrightarrow & x(x^2-1)^2=0 \\ & \Longrightarrow & x(x+1)^2(x-1)^2=0. \end{array}$$

The solutions to this last equation are $\{-1,0,1\}$. Since $-1\notin \mathbf{Dom}(s)$, the only fixed points of s are $x=0$ and $x=1$.

187 $4x-2$

188 $6x^2+2h^2-6$

189 $a(3)=6$; x^2+x-6; $24-11x-10x^2+2x^3+x^4$

190 $7, x^2-2x-1, x^4-4x^3+8x+2$

191 $h(x-1)=-11+7x-x^2;\; h(x)=-5+5x-x^2;\; h(x+1)=-1+3x-x^2$

192 $f(x)=x^2-2x+1;\; f(x+2)=x^2+2x+1;\; f(x-2)=x^2-6x+9$

193 Rename the independent variable, say $h(1-s)=2s$. Now, if $1-s=3x$ then $s=1-3x$. Hence

$$h(3x)=h(1-s)=2s=2(1-3x)=2-6x.$$

194 Put

$$p(x)=(1-x^2+x^4)^{2003}=a_0+a_1x+a_2x^2+\cdots+a_{8012}x^{8012}.$$

Then

1. $a_0=p(0)=(1-0^2+0^4)^{2003}=1.$
2. $a_0+a_1+a_2+\cdots+a_{8012}=p(1)=(1-1^2+1^4)^{2003}=1.$
3. $$\begin{array}{lll} a_0-a_1+a_2-a_3+\cdots-a_{8011}+a_{8012} & = & p(-1) \\ & = & (1-(-1)^2+(-1)^4)^{2003} \\ & = & 1. \end{array}$$
4. The required sum is $\dfrac{p(1)+p(-1)}{2}=1.$
5. The required sum is $\dfrac{p(1)-p(-1)}{2}=0.$

195 We have

$$\begin{array}{lll} \left[f\left(\frac{27+y^3}{y^3}\right)\right]^{\sqrt{\frac{27}{y}}} & = & \left[f\left(\left(\frac{3}{y}\right)^3+1\right)\right]^{3\sqrt{\frac{3}{y}}} \\ & = & \left(\left[f\left(\left(\frac{3}{y}\right)^3+1\right)\right]^{\sqrt{\frac{3}{y}}}\right)^3 \\ & = & 5^3 \\ & = & 125. \end{array}$$

196 Let $y=0$. Then $2g(x)=2x^2$, that is, $g(x)=x^2$. Let us check that $g(x)=x^2$ works. We have

$$g(x+y)+g(x-y)=(x+y)^2+(x-y)^2=x^2+2xy+y^2+x^2-2xy+y^2=2x^2+2y^2,$$

which is the functional equation given. Our choice of g works.

197 Let $x=1$. Then $f(y)=yf(1)$. Since $f(1)$ is a constant, we may let $k=f(1)$. So all the functions satisfying the above equation satisfy $f(y)=ky$.

198 From $f(x)+2f(\frac{1}{x})=x$ we obtain $f(\frac{1}{x})=\frac{x}{2}-\frac{1}{2}f(x)$. Also, substituting $1/x$ for x on the original equation we get

$$f(1/x)+2f(x)=1/x.$$

Hence

$$f(x)=\frac{1}{2x}-\frac{1}{2}f(1/x)=\frac{1}{2x}-\frac{1}{2}\left(\frac{x}{2}-\frac{1}{2}f(x)\right),$$

which yields $f(x)=\dfrac{2}{3x}-\dfrac{x}{3}$.

199 We have

$$\begin{array}{lllll} f(2) & = & (-1)^2 1-2f(1) & = & 1-2f(1) \\ f(3) & = & (-1)^3 2-2f(2) & = & -2-2f(2) \\ f(4) & = & (-1)^4 3-2f(3) & = & 3-2f(3) \\ f(5) & = & (-1)^5 4-2f(4) & = & -4-2f(4) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f(999) & = & (-1)^{999}998-2f(998) & = & -998-2f(998) \\ f(1000) & = & (-1)^{1000}999-2f(999) & = & 999-2f(999) \\ f(1001) & = & (-1)^{1001}1000-2f(1000) & = & -1000-2f(1000) \end{array}$$

Adding columnwise,

$$f(2)+f(3)+\cdots+f(1001)=1-2+3-\cdots+999-1000-2(f(1)+f(2)+\cdot+f(1000)).$$

This gives

$$2f(1)+3(f(2)+f(3)+\cdots+f(1000))+f(1001)=-500.$$

Since $f(1)=f(1001)$ we have $2f(1)+f(1001)=3f(1)$. Therefore

$$f(1)+f(2)+\cdots+f(1000)=-\frac{500}{3}.$$

200 Set $a=b=0$. Then $(f(0))^2=f(0)f(0)=f(0+0)=f(0)$. This gives $f(0)^2=f(0)$. Since $f(0)>0$ we can divide both sides of this equality to get $f(0)=1$.

Further, set $b=-a$. Then $f(a)f(-a)=f(a-a)=f(0)=1$. Since $f(a)\neq 0$, may divide by $f(a)$ to obtain $f(-a)=\dfrac{1}{f(a)}$.

Finally taking $a=b$ we obtain $(f(a))^2=f(a)f(a)=f(a+a)=f(2a)$. Hence $f(2a)=(f(a))^2$

201 We have

$$(f(x))^2\cdot f\left(\frac{1-x}{1+x}\right)=64x,$$

whence

$$(f(x))^4\cdot\left(f\left(\frac{1-x}{1+x}\right)\right)^2=64^2x^2 \qquad (I)$$

Substitute x by $\frac{1-x}{1+x}$. Then

$$f\left(\frac{1-x}{1+x}\right)^2 f(x)=64\left(\frac{1-x}{1+x}\right). \qquad (II)$$

Divide (I) by (II),

$$f(x)^3=64x^2\left(\frac{1+x}{1-x}\right),$$

from where the result follows.

208 We have

$$f(x)=\begin{cases} -2x-1 & \text{if } x\in]-\infty;-2] \\ 3 & \text{if } x\in[-2;1] \\ 2x+1 & \text{if } x\in[1;+\infty[\end{cases}$$

Notice that in this case we allow the intervals to overlap.

209 We have

$$f(x)=\begin{cases} -x^2 & \text{if } x\in]-\infty;0[\\ x^2 & \text{if } x\in[0;+\infty[\end{cases}$$

210 We have

$$f(x) = \begin{cases} 0 & \text{if } x \in]-\infty; 0[\\ 2x & \text{if } x \in [0; +\infty[\end{cases}$$

213 $0; -1; 1; 1; 0; 1; 1; 3; +\infty$

214 $x \in [50; 55[$

224 $y = f(x-2) - 1 = (x-2)^2 - \dfrac{1}{x-2} - 1$

225 Yes.

232 The required equation is $y = \dfrac{1}{2x+2} - 1$.

238 Proceeding successively:

❶ A reflexion about the x-axis gives the curve

$$y = -f(x) = |x| - 2 = a(x),$$

say.

❷ A translation 3 units up gives the curve

$$y = a(x) + 3 = |x| + 1 = b(x),$$

say.

❸ A horizontal stretch by a factor of $\frac{3}{4}$ gives the curve

$$y = b\left(\frac{4}{3}x\right) = \left|\frac{4x}{3}\right| + 1 = \frac{4}{3}|x| + 1 = c(x),$$

say. Observe that the resulting curve is

$$y = c(x) = b\left(\frac{4}{3}x\right) = a\left(\frac{4}{3}x\right) + 3 = -f\left(\frac{4}{3}x\right) + 3.$$

239 (1) $y = -(x+1)(x+2) - 2$ (2) $y = -2x - 7$ (3) $y = |1-x| - 1$

255 Since f is odd, $f(-0) = -f(0)$. But $f(-0) = f(0)$, giving $f(0) = -f(0)$, that is, $2f(0) = 0$ which implies that $f(0) = 0$.

258 The constant function $\mathbb{R} \to \{0\}$ with assignment rule $f : x \mapsto 0$ is both even and odd. It is the only such function, for if g were both even and odd and $g(x) = a \neq 0$ for some real number x, then we would have $a = g(x) = g(-x) = -g(x) = -a$, implying that $a = 0$.

262 Notice that the graph of $y = f(ax)$ is a horizontal shrinking of the graph of $y = f(x)$. Put $g(x) = f(ax)$. Since $g(4/3) = 0$ we must have $4a/3 = -2 \implies a = -3/2$, so the point $(-2, 0)$ on the original graph was mapped to the point $(4/3, 0)$ on the new graph. Hence the point $(3, 0)$ in the old graph gets mapped to $(-2, 0)$ and so $C = -2$.

282 Take, among many possible examples, the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \dfrac{1}{x^3 - x}$ for $x \notin \{-1, 0, 1\}$ and $f(-1) = f(0) = f(1) = 0$.

283 Examine the assignment rule $x \mapsto \lfloor x^3 \rfloor$.

284 Examine the assignment rule $r(x) = \lfloor \frac{1}{x} \rfloor, x \neq 0$.

285 Examine the assignment rule $x \mapsto \lfloor \frac{x-1}{2} \rfloor$.

286 Examine the assignment rule $x \mapsto \lfloor \sqrt{|x|} \rfloor$.

287 We have $f(1-) = 0$ and $f(1+) = 2 + 3a$. We need then $0 = 2 + 3a$ or $a = -\dfrac{2}{3}$.

288 For $x \neq 1$ we have $f(x) = \dfrac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1$. Since $f(1-) = n$ and $f(1+) = n$ we need $a = f(1) = n$.

Chapter 4

Functions II: Domains and Images

4.1 Natural Domain of an Assignment Rule

291 Definition The *natural domain of an assignment rule* is the largest set of real number inputs that will give a real number output of a given assignment rule.

STOP *For the algebraic combinations that we are dealing with, we must then worry about having non-vanishing denominators and taking even-indexed roots of positive real numbers.*

292 Example Find the natural domain of the rule $x \mapsto \dfrac{1}{x^2 - x - 6}$.

Solution: In order for the output to be a real number, the denominator must not vanish. We must have $x^2 - x - 6 = (x+2)(x-3) \neq 0$, and so $x \neq -2$ nor $x \neq 3$. Thus the natural domain of this rule is $\mathbb{R} \setminus \{-2, 3\}$.

293 Example Find the natural domain of $x \mapsto \dfrac{1}{x^4 - 16}$.

Solution: Since $x^4 - 16 = (x^2-4)(x^2+4) = (x+2)(x-2)(x^2+4)$, the rule is undefined when $x = -2$ or $x = 2$. The natural domain is thus $\mathbb{R} \setminus \{-2, +2\}$.

294 Example Find the natural domain for the rule $f(x) = \dfrac{2}{4 - |x|}$.

Solution: The denominator must not vanish, hence $x \neq \pm 4$. The natural domain of this rule is thus $\mathbb{R} \setminus \{-4, 4\}$.

295 Example Find the natural domain of the rule $f(x) = \sqrt{x+3}$

Solution: In order for the output to be a real number, the quantity under the square root must be positive, hence $x + 3 \geq 0 \implies x \geq -3$ and the natural domain is the interval $[-3; +\infty[$.

296 Example Find the natural domain of the rule $g(x) = \dfrac{2}{\sqrt{x+3}}$

Solution: The denominator must not vanish, and hence the quantity under the square root must be positive, therefore $x > -3$ and the natural domain is the interval $]-3+;\infty[$.

297 Example Find the natural domain of the rule $x \mapsto \sqrt[4]{x^2}$.

Solution: Since for all real numbers $x^2 \geq 0$, the natural domain of this rule is $\mathbb{R}$.

298 Example Find the natural domain of the rule $x \mapsto \sqrt[4]{-x^2}$.

Solution: Since for all real numbers $-x^2 \leq 0$, the quantity under the square root is a real number only when $x = 0$, whence the natural domain of this rule is $\{0\}$.

299 Example Find the natural domain of the rule $x \mapsto \dfrac{1}{\sqrt{x^2}}$.

Solution: The denominator vanishes when $x = 0$. Otherwise for all real numbers, $x \neq 0$, we have $x^2 > 0$. The natural domain of this rule is thus $\mathbb{R} \setminus \{0\}$.

300 Example Find the natural domain of the rule $x \mapsto \dfrac{1}{\sqrt{-x^2}}$.

Solution: The denominator vanishes when $x = 0$. Otherwise for all real numbers, $x \neq 0$, we have $-x^2 < 0$. Thus $\sqrt{-x^2}$ is only a real number when $x = 0$, and in that case, the denominator vanishes. The natural domain of this rule is thus the empty set $\varnothing$.

301 Example Find the natural domain of the assignment rule

$$x \mapsto \sqrt{1-x} + \frac{1}{\sqrt{1+x}}.$$

Solution: We need simultaneously $1 - x \geq 0$ (which implies that $x \leq 1$) and $1 + x > 0$ (which implies that $x > -1$), so $x \in]-1;1]$.

302 Example Find the largest subset of real numbers where the assignment rule $x \mapsto \sqrt{x^2 - x - 6}$ gives real number outputs.

Solution: The quantity $x^2 - x - 6 = (x+2)(x-3)$ under the square root must be positive. Studying the sign diagram

$x \in$	$]-\infty;-2]$	$[-2;3]$	$[3;+\infty[$
$\text{signum}\,(x+2) =$	$-$	$+$	$+$
$\text{signum}\,(x-3) =$	$-$	$-$	$+$
$\text{signum}\,((x+2)(x-2)) =$	$+$	$-$	$+$

we conclude that the natural domain of this formula is the set $]-\infty;-2] \cup [3;+\infty[$.

303 Example Find the natural domain for the rule $f(x) = \dfrac{1}{\sqrt{x^2 - x - 6}}$.

The denominator must not vanish, so the quantity under the square root must be positive. By the preceding problem this happens when $x \in]-\infty;-2[\cup]3;+\infty[$.

304 Example Find the natural domain of the rule $x \mapsto \sqrt{x^2 + 1}$.

Solution: Since $\forall x \in \mathbb{R}$ we have $x^2 + 1 \geq 1$, the square root is a real number for all real x. Hence the natural domain is $\mathbb{R}$.

305 Example Find the natural domain of the rule $x \mapsto \sqrt{x^2 + x + 1}$.

Solution: The discriminant of $x^2 + x + 1 = 0$ is $1^2 - 4(1)(1) < 0$. Since the coefficient of x^2 is $1 > 0$, the expression $x^2 + x + 1$ is always positive, meaning that the required natural domain is all of $\mathbb{R}$.

Aliter: Observe that

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\geq\frac{3}{4},$$

the square root is a real number for all real x. Hence the natural domain is $\mathbb{R}$.

Homework

306 Problem Below are given some assignment rules. Verify that the accompanying set is the natural domain of the assignment rule.

Assignment Rule	Natural Domain
$x\mapsto\sqrt{(1-x)(x+3)}$	$x\in[-3;1]$.
$x\mapsto\sqrt{\frac{1-x}{x+3}}$	$x\in]-3;1]$
$x\mapsto\sqrt{\frac{x+3}{1-x}}$	$x\in[-3;1[$
$x\mapsto\sqrt{\frac{1}{(x+3)(1-x)}}$	$x\in]-3;1[$

307 Problem Find the natural domain for the given assignment rules.

❶ $x\mapsto\frac{1}{\sqrt{1+|x|}}$

❷ $x\mapsto\sqrt[4]{5-|x|}$

❸ $x\mapsto\sqrt[3]{5-|x|}$

❹ $x\mapsto\frac{1}{x^2+2x+2}$

❺ $x\mapsto\frac{1}{\sqrt{x^2-2x-2}}$

❻ $x\mapsto\frac{1}{|x-1|+|x+1|}$

❼ $x\mapsto\frac{\sqrt{-x}}{x^2-1}$

❽ $x\mapsto\frac{\sqrt{1-x^2}}{1-|x|}$

❾ $x\mapsto\sqrt{x}+\sqrt{-x}$

308 Problem Below are given some assignment rules. Verify that the accompanying set is the natural domain of the assignment rule.

Assignment Rule	Natural Domain
$x\mapsto\sqrt{\frac{x}{x^2-9}}$	$x\in]-3;0]\bigcup]3;+\infty[$
$x\mapsto\sqrt{-\|x\|}$	$x=0$
$x\mapsto\sqrt{-\|\|x\|-2\|}$	$x\in\{-2,2\}$
$x\mapsto\sqrt{\frac{1}{x}}$	$x\in]0;+\infty[$
$x\mapsto\sqrt{\frac{1}{x^2}}$	$x\in\mathbb{R}\setminus\{0\}$
$x\mapsto\sqrt{\frac{1}{-x}}$	$x\in]-\infty;0[$
$x\mapsto\sqrt{\frac{1}{-\|x\|}}$	$\varnothing$ (the empty set)
$x\mapsto\frac{1}{x\sqrt{x+1}}$	$x\in]-1;0[\bigcup]0;+\infty[$
$x\mapsto\sqrt{1+x}+\sqrt{1-x}$	$[-1;1]$

309 Problem Find the natural domain for the rule $f(x)=\sqrt{x^3-12x}$.

310 Problem Find the natural domain of the rule $x\mapsto\frac{1}{\sqrt{x^2-2x-2}}$.

311 Problem Find the natural domain for the following rules.

❶ $x\mapsto\sqrt{-(x+1)^2}$,

❷ $x\mapsto\frac{1}{\sqrt{-(x+1)^2}}$

❸ $f(x)=\frac{x^{1/2}}{\sqrt{x^4-13x^2+36}}$

❹ $g(x)=\frac{\sqrt[4]{3-x}}{\sqrt{x^4-13x^2+36}}$

❺ $h(x)=\frac{1}{\sqrt{x^6-13x^4+36x^2}}$

❻ $j(x)=\frac{1}{\sqrt{x^5-13x^3+36x}}$

❼ $k(x)=\frac{1}{\sqrt{|x^4-13x^2+36|}}$

4.2 Algebra of Functions

312 Definition Let $f:\textbf{Dom}(f)\to\mathbb{R}$ and $g:\textbf{Dom}(g)\to\mathbb{R}$. Then $\textbf{Dom}(f\pm g)=\textbf{Dom}(f)\cap\textbf{Dom}(g)$ and the sum (respectively, difference) function $f+g$ (respectively, $f-g$) is given by

$$f\pm g:\begin{array}{ccc}\textbf{Dom}(f)\cap\textbf{Dom}(g) & \to & \mathbb{R}\\ x & \mapsto & f(x)\pm g(x)\end{array}.$$

In other words, if x belongs both to the domain of f and g, then

$$(f\pm g)(x)=f(x)\pm g(x).$$

313 Definition Let $f : \mathbf{Dom}(f) \to \mathbb{R}$ and $g : \mathbf{Dom}(g) \to \mathbb{R}$. Then $\mathbf{Dom}(fg) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g)$ and the product function fg is given by

$$fg : \begin{array}{ccc} \mathbf{Dom}(f) \cap \mathbf{Dom}(g) & \to & \mathbb{R} \\ x & \mapsto & f(x) \cdot g(x) \end{array}.$$

In other words, if x belongs both to the domain of f and g, then

$$(fg)(x) = f(x) \cdot g(x).$$

314 Example Let

$$f : \begin{array}{ccc} [-1;1] & \to & \mathbb{R} \\ x & \mapsto & x^2 + 2x \end{array}, \quad g : \begin{array}{ccc} [0;2] & \to & \mathbb{R} \\ x & \mapsto & 3x + 2 \end{array}.$$

Find

1. $\mathbf{Dom}(f \pm g)$
2. $\mathbf{Dom}(fg)$
3. $(f+g)(-1)$
4. $(f+g)(1)$
5. $(fg)(1)$
6. $(f-g)(0)$
7. $(f+g)(2)$

Solution: We have

1. $\mathbf{Dom}(f \pm g) = \mathbf{Dom}(f) \cap \mathbf{Dom}(g) = [-1;1] \cap [0;2] = [0;1]$.
2. $\mathbf{Dom}(fg)$ is also $\mathbf{Dom}(f) \cap \mathbf{Dom}(g) = [0;1]$.
3. Since $-1 \notin [0;1]$, $(f+g)(-1)$ is undefined.
4. $(f+g)(1) = f(1) + g(1) = 3 + 5 = 8$.
5. $(fg)(1) = f(1)g(1) = (3)(5) = 15$.
6. $(f-g)(0) = f(0) - g(0) = 0 - 2 = -2$.
7. Since $2 \notin [0;1]$, $(f+g)(2)$ is undefined.

315 Definition Let $g : \mathbf{Dom}(g) \to \mathbb{R}$ be a function. The *support* of g, denoted by $\mathbf{supp}(g)$ is the set of elements in $\mathbf{Dom}(g)$ where g does not vanish, that is

$$\mathbf{supp}(g) = \{x \in \mathbf{Dom}(g) : g(x) \neq 0\}.$$

316 Example Let

$$g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 - 2x \end{array}.$$

Then $x^3 - 2x = x(x - \sqrt{2})(x + \sqrt{2})$. Thus

$$\mathbf{supp}(g) = \mathbb{R} \setminus \{-\sqrt{2}, 0\sqrt{2}\}.$$

317 Example Let

$$g : \begin{array}{ccc} [0;1] & \to & \mathbb{R} \\ x & \mapsto & x^3 - 2x \end{array}.$$

Then $x^3 - 2x = x(x - \sqrt{2})(x + \sqrt{2})$. Thus

$$\mathbf{supp}(g) = [0;1] \setminus \{-\sqrt{2}, 0\sqrt{2}\} =]0;1].$$

318 Definition Let $f : \mathbf{Dom}(f) \to \mathbb{R}$ and $g : \mathbf{Dom}(g) \to \mathbb{R}$. Then $\mathbf{Dom}\left(\frac{f}{g}\right) = \mathbf{Dom}(f) \cap \mathbf{supp}(g)$ and the quotient function $\frac{f}{g}$ is given by

$$\frac{f}{g} : \begin{array}{ccc} \mathbf{Dom}(f) \cap \mathbf{supp}(g) & \to & \mathbb{R} \\ x & \mapsto & \dfrac{f(x)}{g(x)} \end{array}.$$

In other words, if x belongs both to the domain of f and g and $g(x) \neq 0$, then $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

319 Example Let

$$f : \begin{array}{ccc} [-2;3] & \to & \mathbb{R} \\ x & \mapsto & x^3 - x \end{array}, \quad g : \begin{array}{ccc} [0;5] & \to & \mathbb{R} \\ x & \mapsto & x^3 - 2x^2 \end{array}.$$

Find

❶ $\mathbf{supp}(f)$

❷ $\mathbf{supp}(g)$

❸ $\mathbf{Dom}\left(\frac{f}{g}\right)$

❹ $\mathbf{Dom}\left(\frac{g}{f}\right)$

❺ $\frac{f}{g}(2)$

❻ $\frac{g}{f}(2)$

❼ $\frac{f}{g}(1/3)$

❽ $\frac{g}{f}(1/3)$

Solution:

❶ As $x^3 - x = x(x-1)(x+1)$, $\mathbf{supp}(f) = [-2;-1[\cup]-1;0[\cup]0;3]$

❷ As $x^3 - 2x^2 = x^2(x-2)$, $\mathbf{supp}(g) =]0;2[\cup]2;5]$.

❸ $\mathbf{Dom}\left(\frac{f}{g}\right) = \mathbf{Dom}(f) \cap \mathbf{supp}(g) = [-2;3] \cap (]0;2[\cup]2;5]) =]0;2[\cup]2;3]$

❹

$$\mathbf{Dom}\left(\frac{g}{f}\right) = \mathbf{Dom}(g) \cap \mathbf{supp}(f) = [0;5] \cap ([-2;-1[\cup]-1;0[\cup]0;3]) =]0;3]$$

❺ $\frac{f}{g}(2)$ is undefined, as $2 \notin]0;2[\cup]2;3]$.

❻ $\frac{g}{f}(2) = \frac{g(2)}{f(2)} = \frac{0}{6} = 0$.

❼ $\frac{f}{g}(1/3) = \frac{-\frac{8}{27}}{-\frac{5}{27}} = \frac{8}{5}$

❽ $\frac{g}{f}(1/3) = \frac{-\frac{5}{27}}{-\frac{8}{27}} = \frac{5}{8}$

We are now going to consider “functions of functions.”

320 Definition Let $f : \textbf{Dom}(f) \to \mathbb{R}$, $g : \textbf{Dom}(g) \to \mathbb{R}$ and let $U = \{x \in \textbf{Dom}(g) : g(x) \in \textbf{Dom}(f)\}$. We define the *composition* function of f and g as

$$f \circ g : \begin{array}{ccc} U & \to & \mathbb{R} \\ x & \mapsto & f(g(x)) \end{array}. \tag{4.1}$$

We read $f \circ g$ as "f *composed with* g."

STOP *We have* $\textbf{Dom}(f \circ g) = \{x \in \textbf{Dom}(g) : g(x) \in \textbf{Dom}(f)\}$. *Thus to find* $\textbf{Dom}(f \circ g)$ *we find those elements of* $\textbf{Dom}(g)$ *whose images are in* $\textbf{Dom}(f) \cap \textbf{Im}(g)$

321 Example Let

$$f : \begin{array}{ccc} \{-2,-1,0,1,2\} & \to & \mathbb{R} \\ x & \mapsto & 2x+1 \end{array}, \quad g : \begin{array}{ccc} \{0,1,2,3\} & \to & \mathbb{R} \\ x & \mapsto & x^2-4 \end{array}.$$

❶ Find $\textbf{Im}(f)$.

❷ Find $\textbf{Im}(g)$.

❸ Find $\textbf{Dom}(f \circ g)$.

❹ Find $\textbf{Dom}(g \circ f)$.

❺ Find $(f \circ g)(0)$.

❻ Find $(g \circ f)(0)$.

❼ Find $(f \circ g)(2)$.

❽ Find $(g \circ f)(2)$.

Solution:

❶ We have $f(-2) = -3$, $f(-1) = -1$, $f(0) = 1$, $f(1) = 3$, $f(2) = 5$. Hence $\textbf{Im}(f) = \{-3,-1,1,3,5\}$.

❷ We have $g(0) = -4$, $g(1) = -3$, $g(2) = 0$, $g(3) = 5$. Hence $\textbf{Im}(g) = \{-4,-3,0,5\}$.

❸ $\textbf{Dom}(f \circ g) = \{x \in \textbf{Dom}(g) : g(x) \in \textbf{Dom}(f)\} = \{2\}$.

❹ $\textbf{Dom}(g \circ f) = \{x \in \textbf{Dom}(f) : f(x) \in \textbf{Dom}(g)\} = \{0,1\}$.

❺ $(f \circ g)(0) = f(g(0)) = f(-4)$, but this last is undefined.

❻ $(g \circ f)(0) = g(f(0)) = g(1) = -3$.

❼ $(f \circ g)(2) = f(g(2)) = f(0) = 1$.

❽ $(g \circ f)(2) = g(f(2)) = g(5)$, but this last is undefined.

322 Example Let

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & 2x-3 \end{array}, \quad g : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & 5x+1 \end{array}.$$

❶ Demonstrate that $\textbf{Im}(f) = \mathbb{R}$.

❷ Demonstrate that $\textbf{Im}(g) = \mathbb{R}$.

❸ Find $(f \circ g)(x)$.

❹ Find $(g \circ f)(x)$.

❺ Is it ever true that $(f \circ g)(x) = (g \circ f)(x)$?

Solution:

❶ Take $b \in \mathbb{R}$. We must shew that $\exists x \in \mathbb{R}$ such that $f(x) = b$. But

$$f(x) = b \implies 2x - 3 = b \implies x = \frac{b+3}{2}.$$

Since $\frac{b+3}{2}$ is a real number satisfying $f\left(\frac{b+3}{2}\right) = b$, we have shewn that $\mathbf{Im}(f) = \mathbb{R}$.

❷ Take $b \in \mathbb{R}$. We must shew that $\exists x \in \mathbb{R}$ such that $g(x) = b$. But

$$g(x) = b \implies 5x + 1 = b \implies x = \frac{b-1}{5}.$$

Since $\frac{b-1}{5}$ is a real number satisfying $g\left(\frac{b-1}{5}\right) = b$, we have shewn that $\mathbf{Im}(g) = \mathbb{R}$.

❸ We have

$$(f \circ g)(x) = f(g(x)) = f(5x+1) = 2(5x+1) - 3 = 10x - 1$$

❹ We have

$$(g \circ f)(x) = g(f(x)) = g(2x-3) = 5(2x-3) + 1 = 10x - 14.$$

$(g \circ f)(x)$.

❺ If

$$(f \circ g)(x) = (g \circ f)(x)$$

then we would have

$$10x - 1 = 10x - 14$$

which entails that $-1 = -14$, absolute nonsense!

Composition of functions need not be commutative.

323 Example Consider

$$f: \begin{array}{ccc} [-\sqrt{3};\sqrt{3}] & \to & \mathbb{R} \\ x & \mapsto & \sqrt{3-x^2} \end{array}, \quad g: \begin{array}{ccc} [-2;+\infty[& \to & \mathbb{R} \\ x & \mapsto & -\sqrt{x+2} \end{array}.$$

❶ Find $\mathbf{Im}(f)$.

❷ Find $\mathbf{Im}(g)$.

❸ Find $\mathbf{Dom}(f \circ g)$.

❹ Find $f \circ g$.

❺ Find $\mathbf{Dom}(g \circ f)$.

❻ Find $g \circ f$.

Solution:

❶ Assume $y = \sqrt{3-x^2}$. Then $y \geq 0$. Moreover $x = \pm\sqrt{3-y^2}$. This makes sense only if $-\sqrt{3} \leq y \leq \sqrt{3}$. Hence $\mathbf{Im}(f) = [0;\sqrt{3}]$.

❷ Assume $y = -\sqrt{x+2}$. Then $y \leq 0$, and since $x \mapsto -\sqrt{x+2}$ decreases steadily, we have $\mathbf{Im}(g) =]-\infty;0]$.

❸

$$\begin{aligned} \mathbf{Dom}(f \circ g) &= \{x \in \mathbf{Dom}(g) : g(x) \in \mathbf{Dom}(f)\} \\ &= \{x \in [-2;+\infty[: -\sqrt{3} \leq -\sqrt{x+2} \leq \sqrt{3}\} \\ &= \{x \in [-2;+\infty[: -\sqrt{3} \leq -\sqrt{x+2} \leq 0\} \\ &= \{x \in [-2;+\infty[: x \leq 1\} \\ &= [-2;1] \end{aligned}$$

❹ $(f\circ g)(x) = f(g(x)) = f(-\sqrt{x+2}) = \sqrt{1-x}$.

❺

$$\begin{aligned}\mathbf{Dom}(g\circ f) &= \{x\in\mathbf{Dom}(f) : f(x)\in\mathbf{Dom}(g)\}\\ &= \{x\in[-\sqrt{3};\sqrt{3}] : \sqrt{3-x^2}\geq -2\}\\ &= \{x\in[-\sqrt{3};\sqrt{3}] : \sqrt{3-x^2}\geq 0\}\\ &= [-\sqrt{3};\sqrt{3}]\end{aligned}$$

❻ $(g\circ f)(x) = g(f(x)) = g(\sqrt{3-x^2}) = -\sqrt{\sqrt{3-x^2}+2}$.

STOP *Notice that* $\mathbf{Dom}(f\circ g) = [-2;1]$, *although the domain of definition of* $x\mapsto\sqrt{1-x}$ *is* $]-\infty;1]$.

324 Example Let

$$f:\begin{array}{ccc}\mathbb{R}\setminus\{1\} & \to & \mathbb{R}\\ x & \mapsto & \dfrac{2x}{x-1}\end{array},\quad g:\begin{array}{ccc}]-\infty;2] & \to & \mathbb{R}\\ x & \mapsto & \sqrt{2-x}\end{array}$$

❶ Find $\mathbf{Im}(f)$.

❷ Find $\mathbf{Im}(g)$.

❸ Find $\mathbf{Dom}(f\circ g)$.

❹ Find $f\circ g$.

❺ Find $\mathbf{Dom}(g\circ f)$.

❻ Find $g\circ f$.

Solution:

❶ Assume $y = \dfrac{2x}{x-1}$, $x\in\mathbf{Dom}(f)$ is solvable. Then

$$y(x-1) = 2x \implies yx-2x = +y \implies x = \frac{y}{y-2}.$$

Thus the equation is solvable only when $y\neq 2$. Thus $\mathbf{Im}(f) = \mathbb{R}\setminus\{2\}$.

❷ Assume that $y=\sqrt{2-x}$, $x\in\mathbf{Dom}(g)$ is solvable. Then $y\geq 0$ since $y=\sqrt{2-x}$ is the square root of a (positive) real number. All $y\geq 0$ will render $x = 2-y^2$ in the appropriate range, and so $\mathbf{Im}(g) = [0;+\infty[$.

❸

$$\begin{aligned}\mathbf{Dom}(f\circ g) &= \{x\in\mathbf{Dom}(g) : g(x)\in\mathbf{Dom}(f)\}\\ &= \{x\in]-\infty;2] : \sqrt{2-x}\neq 1\}\\ &=]-\infty;1[\cup]1;2]\end{aligned}$$

❹ $(f\circ g)(x) = f(g(x)) = f(\sqrt{2-x}) = \dfrac{1}{\sqrt{2-x}-1}$.

❺

$$\begin{aligned}\mathbf{Dom}(g\circ f) &= \{x\in\mathbf{Dom}(f) : f(x)\in\mathbf{Dom}(g)\}\\ &= \{x\in\mathbb{R}\setminus\{1\} : \frac{2x}{x-1}\leq 2\}\\ &= \{x\in\mathbb{R}\setminus\{1\} : \frac{2}{x-1}\leq 0\}\\ &=]-\infty;1[\end{aligned}$$

❻
$$(g\circ f)(x) = g(f(x)) = g\left(\frac{2x}{x-1}\right) = \sqrt{2-\frac{2x}{x-1}} = \sqrt{\frac{2}{1-x}}$$

325 Example Figure 4.1 shews two functions $x\mapsto f(x)=x+1$ and $x\mapsto g(x)=x-1$. Figure 4.2 shews their sum $x\mapsto 2x$, a line, figure 4.3 shews the difference $x\mapsto (f-g)(x)=2$, a horizontal line, and figure 4.4 shews their product $x\mapsto x^2-1$, a parabola. We also have $x\mapsto \left(\frac{g}{f}\right)(x) = \dfrac{x-1}{x+1} = 1-\dfrac{2}{x+1}$, a hyperbola with pole at $x=-1$ and asymptote at $y=1$ (figure 4.5); $x\mapsto \left(\frac{f}{g}\right)(x) = \dfrac{x+1}{x-1} = 1+\dfrac{2}{x-1}$, a hyperbola with pole at $x=1$ and asymptote at $y=1$ (figure 4.6); $(f\circ g) = \mathbf{Id}$ (figure 4.7); and $x\mapsto (f\circ f)(x)=x+2$ (figure 4.8).

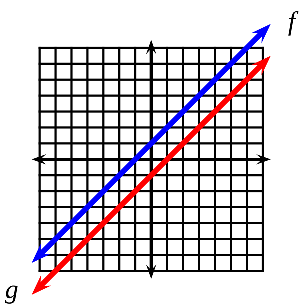


Figure 4.1: $f(x)=x+1$ and $g(x)=x-1$

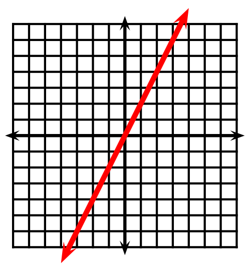


Figure 4.2: $x\mapsto (f+g)(x)=2x$

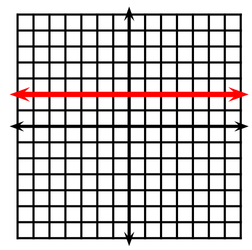


Figure 4.3: $x\mapsto (f-g)(x)=2$

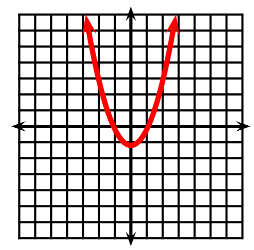


Figure 4.4: $x\mapsto (fg)(x)=x^2-1$

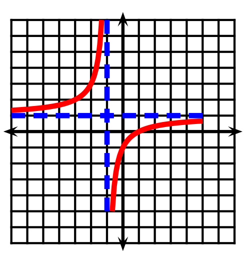


Figure 4.5: $\left(\frac{g}{f}\right)(x) = 1-\frac{2}{x+1}$

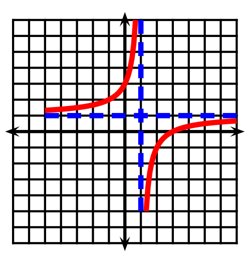


Figure 4.6: $\left(\frac{f}{g}\right)(x) = 1+\frac{2}{x-1}$

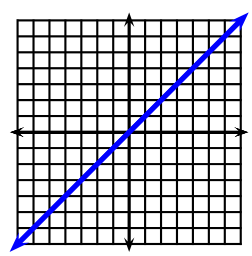


Figure 4.7: $(f\circ g)(x)=x$

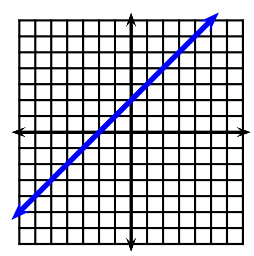


Figure 4.8: $(f\circ f)(x)=x+2$

Homework

326 Problem Let

$$f: \begin{array}{lcl}[-5;3] & \to & \mathbb{R}\\ x & \mapsto & x^4-16\end{array},\quad g: \begin{array}{lcl}[-4;2] & \to & \mathbb{R}\\ x & \mapsto & |x|-4\end{array}.$$

Find

❶ **Dom** $(f+g)$
❷ **Dom** (fg)
❸ **Dom** $\left(\dfrac{f}{g}\right)$
❹ **Dom** $\left(\dfrac{g}{f}\right)$
❺ $(f+g)(2)$
❻ $(fg)(2)$
❼ $\dfrac{f}{g}(2)$
❽ $\dfrac{g}{f}(2)$
❾ $\dfrac{f}{g}(1)$
❿ $\dfrac{g}{f}(1)$

327 Problem Let

$$f: \begin{array}{lcl}\{-2,-1,0,1,2\} & \to & \mathbb{Z}\\ x & \mapsto & 2x\end{array},\quad g: \begin{array}{lcl}\{0,1,2\} & \to & \mathbb{Z}\\ x & \mapsto & x^2\end{array}.$$

❶ Find $\mathbf{Im}(f)$.
❷ Find $\mathbf{Im}(g)$.
❸ Find $\mathbf{Dom}(f\circ g)$.
❹ Find $\mathbf{Dom}(g\circ f)$.

328 Problem Let $f,g,h:\{1,2,3,4\}\to\{1,2,10,1993\}$ be given by

$$f(1)=1, f(2)=2, f(3)=10, f(4)=1993,$$
$$g(1)=g(2)=2, g(3)=g(4)-1=1,$$
$$h(1)=h(2)=h(3)=h(4)+1=2.$$

❶ Compute $(f+g+h)(3)$
❷ Compute $(fg+gh+hf)(4)$.
❸ Compute $f(1+h(3))$.
❹ Compute $(f\circ f\circ f\circ f\circ f)(2)+f(g(2)+2)$.

329 Problem If $a,b,c:\mathbb{R}\to\mathbb{R}$ are functions with $a(t)=t-2, b(t)=t^3, c(t)=5$ demonstrate that

$$\begin{aligned}(a\circ b)(t) &= t^3-2\\ (b\circ a)(t) &= (t-2)^3\\ (b\circ c)(t) &= 125\\ (c\circ b)(t) &= 5\\ (c\circ a)(t) &= 5\\ (a\circ b\circ c)(t) &= 123\\ (c\circ b\circ a)(t) &= 5\\ (a\circ c\circ b)(t) &= 3\end{aligned}$$

330 Problem Let

$$f:\begin{array}{ccc}[2;+\infty[& \to & \mathbb{R}\\ x & \mapsto & \sqrt{x-2}\end{array},\quad g:\begin{array}{ccc}[-2;2] & \to & \mathbb{R}\\ x & \mapsto & \sqrt{4-x^2}\end{array}.$$

❶ Find $\mathbf{Im}(f)$.
❷ Find $\mathbf{Im}(g)$.
❸ Find $\mathbf{Dom}(f\circ g)$.
❹ Find $\mathbf{Dom}(g\circ f)$.
❺ Find $(f\circ g)(x)$.
❻ Find $(g\circ f)(x)$.

331 Problem Let

$$f:\begin{array}{ccc}[-\sqrt{2};+\sqrt{2}[& \to & \mathbb{R}\\ x & \mapsto & \sqrt{2-x^2}\end{array},\quad g:\begin{array}{ccc}]-\infty;0] & \to & \mathbb{R}\\ x & \mapsto & -\sqrt{-x}\end{array}.$$

❶ Find $\mathbf{Im}(f)$.
❷ Find $\mathbf{Im}(g)$.
❸ Find $\mathbf{Dom}(f\circ g)$.
❹ Find $\mathbf{Dom}(g\circ f)$.
❺ Find $(f\circ g)(x)$.
❻ Find $(g\circ f)(x)$.

332 Problem Let $f,g,h:\mathbb{R}\to\mathbb{R}$ be functions. Prove that their composition is associative

$$f\circ(g\circ h)=(f\circ g)\circ h$$

whenever the given expressions make sense.

333 Problem Let $f:\mathbb{R}\to\mathbb{R}$ be the function defined by $f(x)=ax^2-\sqrt{2}$ for some positive a. If $(f\circ f)(\sqrt{2})=-\sqrt{2}$ find the value of a.

334 Problem Let $f:]0:+\infty[\to]0:+\infty[$, such $f(2x)=\dfrac{2}{2+x}$. Find $2f(x)$.

335 Problem Let $f,g:\mathbb{R}\setminus\{1\}\to\mathbb{R}$, with $f(x)=\dfrac{4}{x-1}, g(x)=2x$, find all x for which $(g\circ f)(x)=(f\circ g)(x)$.

336 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $f(1-x)=x^2$. Find $(f\circ f)(x)$.

337 Problem Let $f^{[1]}(x)=f(x)=x+1, f^{[n+1]}=f\circ f^{[n]}, n\geq 1$. Find a closed formula for $f^{[n]}$

338 Problem Let $f^{[1]}(x)=f(x)=2x, f^{[n+1]}=f\circ f^{[n]}, n\geq 1$. Find a closed formula for $f^{[n]}$

339 Problem Let $f^{[1]}=f$ be given by $f(x)=\dfrac{1}{1-x}$. Find
(i) $f^{[2]}=f\circ f$,
(ii) $f^{[3]}=f\circ f\circ f$, and
(iii) $f^{[69]}=\underbrace{f\circ f\circ\cdots f\circ f}_{\text{69 compositions with itself}}$.

340 Problem Let $f:\mathbb{R}\setminus\{-\frac{3}{2}\}\to\mathbb{R}\setminus\{\frac{c}{2}\}, x\mapsto\dfrac{cx}{2x+3}$ be such that $(f\circ f)(x)=x$. Find the value of c.

4.3 Injections and Surjections

341 Definition A function $f:\begin{array}{ccc}\mathbf{Dom}(f) & \to & \mathbf{Target}(f)\\ a & \mapsto & f(a)\end{array}$ is said to be *injective* or *one-to-one* if $(a_1,a_2)\in(\mathbf{Dom}(f))^2$,

$$a_1\neq a_2 \implies f(a_1)\neq f(a_2).$$

That is,

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

f is said to be *surjective* or *onto* if **Target** $(f) =$ **Im** (f). That is, if $(\forall b \in B)$ $(\exists a \in A)$ such that $f(a) = b$. f is *bijective* if it is both injective and surjective. The number a is said to the the *pre-image* of b.

342 Example The function α in the diagram 4.9 is an injective function. The function represented by the diagram 4.10, however is not injective, since $\beta(3) = \beta(1) = 4$, but $3 \neq 1$. The function γ represented by diagram 4.11 is surjective. The function δ represented by diagram 4.12 is not surjective as 8 does not have a preimage.

It is easy to see that a graphical criterion for a function to be injective is that every horizontal line crossing the function must meet it at most one point and one for a function to be surjective is that every horizontal line passing through a point of the target set (a subset of the y-axis) of the function must also meet the curve.

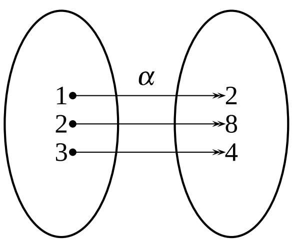


Figure 4.9: An injection.

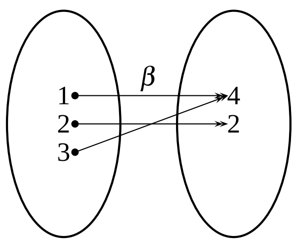


Figure 4.10: Not an injection

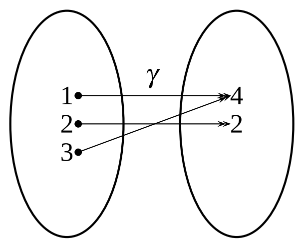


Figure 4.11: A surjection

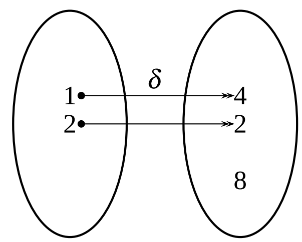


Figure 4.12: Not a surjection

343 Example Prove that

$$g: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ s & \mapsto & 2s+1 \end{array}$$

is a bijection.

Solution: Assume $g(s_1) = g(s_2)$. Then

$$\begin{array}{rcrcl} g(s_1) = g(s_2) & \implies & 2s_1 + 1 & = & 2s_2 + 1 \\ & \implies & 2s_1 & = & 2s_2 \\ & \implies & s_1 & = & s_2 \end{array}$$

We have shewn that $g(s_1) = g(s_2) \implies s_1 = s_2$, and the function is thus injective.

To prove that g is surjective, we must prove that $(\forall\ b \in \mathbb{R})$ $(\exists a)$ such that $g(a) = b$. We choose a so that $a = \dfrac{b-1}{2}$. Then

$$g(a) = g\left(\frac{b-1}{2}\right) = 2\left(\frac{b-1}{2}\right) + 1 = b - 1 + 1 = b.$$

Our choice of a works and hence the function is surjective.

344 Example Prove that

$$t: \begin{array}{ccc} \mathbb{R} \setminus \{1\} & \to & \mathbb{R} \setminus \{1\} \\ x & \mapsto & \dfrac{x+1}{x-1} \end{array}$$

is a bijection.

Solution: Assume $t(a) = t(b)$. Then

$$\begin{array}{rcrcl} t(a) = t(b) & \implies & \dfrac{a+1}{a-1} & = & \dfrac{b+1}{b-1} \\ & \implies & (a+1)(b-1) & = & (b+1)(a-1) \\ & \implies & ab - a + b - 1 & = & ab - b + a - 1 \\ & \implies & 2a & = & 2b \\ & \implies & a & = & b \end{array}$$

We have proved that $t(a) = t(b) \implies a = b$, which shews that t is injective.

To prove that t is surjective, we must be able to find a solution a with $t(a) = y$ for $y \neq 1$. But

$$t(a) = y \iff \frac{a+1}{a-1} = y \iff ya - y = a + 1 \iff ya - a = 1 + y \iff a = \frac{1+y}{1-y}.$$

This last expression makes sense for $y \neq 1$ and thus t is surjective.

345 Example Prove that

$$h : \begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is a bijection.

Solution: Assume $h(b) = h(a)$. Then

$$\begin{array}{rcc} h(a) = h(b) & \implies & a^3 = b^3 \\ & \implies & a^3 - b^3 = 0 \\ & \implies & (a-b)(a^2 + ab + b^2) = 0 \end{array}$$

Now,

$$b^2 + ab + a^2 = \left(b + \frac{a}{2}\right)^2 + \frac{3a^2}{4}.$$

This shews that $b^2 + ab + a^2$ is positive unless both a and b are zero. Hence $b - a = 0$ in all cases. We have shewn that $h(b) = h(a) \implies b = a$, and the function is thus injective.

To prove that h is surjective, we must prove that $(\forall\, b \in \mathbb{R})\ (\exists a)$ such that $h(a) = b$. We choose a so that $a = b^{1/3}$. Then

$$h(a) = h(b^{1/3}) = (b^{1/3})^3 = b.$$

Our choice of a works and hence the function is surjective.

346 Example Observe that

$$h : \begin{array}{rcl} \mathbb{R} & \to & [0; +\infty[\\ w & \mapsto & w^2 \end{array}$$

is not injective. For $h(-2) = h(2) = 4$ but $-2 \neq 2$. The graph of h is a full parabola opening up and in fact, most vertical lines crossing h meet the parabola at two points.

347 Example If we however restrict h in example 346 to the positive reals, as

$$j : \begin{array}{rcl} [0; +\infty[& \to & [0; +\infty[\\ w & \mapsto & w^2 \end{array}$$

then j is injective. For, it is only the right half of a parabola opening up. We can prove this analytically as follows. Clearly if $j(s) = 0$ then $s = 0$. Assume now that $j(a) = j(b)$ with a, b positive. Then $a^2 = b^2$ which is to say $(a-b)(a+b) = 0$. Since a, b are positive real numbers, $a + b > 0$. Therefore $a - b = 0$ which is to say $a = b$.

348 Example Prove that $f: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{x^{1/3}}{x^{1/3}-1} \end{array}$ is injective but not surjective.

Solution: We have

$$\begin{array}{rcrcl} f(a)=f(b) & \implies & \dfrac{a^{1/3}}{a^{1/3}-1} & = & \dfrac{b^{1/3}}{b^{1/3}-1} \\ & \implies & a^{1/3}b^{1/3}-a^{1/3} & = & a^{1/3}b^{1/3}-b^{1/3} \\ & \implies & -a^{1/3} & = & -b^{1/3} \\ & \implies & a & = & b, \end{array}$$

whence f is injective. To prove that f is not surjective assume that $f(x)=b, b\in\mathbb{R}$. Then

$$f(x)=b \implies \frac{x^{1/3}}{x^{1/3}-1}=b \implies x=\frac{b^3}{(b-1)^3}.$$

The expression for x is not a real number when $b=1$, and so there is no real x such that $f(x)=1$.

Homework

349 Problem Prove that $h:\mathbb{R}\to\mathbb{R}$ given by $h(s)=3-s$ is a bijection.

350 Problem Prove that $g:\mathbb{R}\to\mathbb{R}$ given by $g(x)=x^{1/3}$ is a bijection.

351 Problem Prove that $f:\mathbb{R}\setminus\{-1\}\to\mathbb{R}\setminus\{1\}$ given by $f(x)=\dfrac{x-1}{x+1}$ is a bijection.

352 Problem Prove that $f: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R}\setminus\{2\} \\ x & \mapsto & \dfrac{2x}{x+1} \end{array}$ is surjective but that $g: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R} \\ x & \mapsto & \dfrac{2x}{x+1} \end{array}$ is not surjective.

353 Problem Classify each of the following as injective, surjective, bijective or neither.

❶ $f:\mathbb{R}\to\mathbb{R},\quad x\mapsto x^4$

❷ $f:\mathbb{R}\to\{1\},\quad x\mapsto 1$

❸ $f:\{1,2,3\}\to\{a,b\},\quad f(1)=f(2)=a, f(3)=b$

❹ $f:[0;+\infty[\to\mathbb{R},\quad x\mapsto x^3$

❺ $f:\mathbb{R}\to\mathbb{R},\quad x\mapsto |x|$

❻ $f:[0;+\infty[\to\mathbb{R},\quad x\mapsto -|x|$

❼ $f:\mathbb{R}\to[0;+\infty[,\quad x\mapsto |x|$

❽ $f:[0;+\infty[\to[0;+\infty[,\quad x\mapsto x^4$

354 Problem Let $f:E\to F, g:F\to G$ be two functions. Prove that if $g\circ f$ is surjective then g is surjective.

355 Problem Let $f:E\to F, g:F\to G$ be two functions. Prove that if $g\circ f$ is injective then f is injective.

4.4 Inversion

356 Definition Let $A\times B\subseteq\mathbb{R}^2$. A function $F:A\to B$ is said to be *invertible* if there exists a function F^{-1} (called the *inverse* of F) such that $F\circ F^{-1}=\mathbf{Id}_B$ and $F^{-1}\circ F=\mathbf{Id}_A$. Here $\mathbf{Id}_S$ is the identity on the set S function with rule $\mathbf{Id}_S(x)=x$.

Consider the functions $u:\{a,b,c\}\to\{x,y,z\}$ and $v:\{x,y,z\}\to\{a,b,c\}$ as given by diagram 4.13. It is clear the v undoes whatever u does. Furthermore, we observe that u and v are bijections and that the domain of u is the image of v and vice-versa. This example motivates the following theorem.

357 Theorem A function $f:A\to B$ is invertible if and only if it is a bijection.

Proof: *Assume first that f is invertible. Then there is a function $f^{-1}:B\to A$ such that*

$$f\circ f^{-1}=\mathbf{Id}_B \quad\text{and}\quad f^{-1}\circ f=\mathbf{Id}_A. \tag{4.2}$$

Let us prove that f is injective and surjective. Let s,t be in the domain of f and such that $f(s)=f(t)$. Applying f^{-1} to both sides of this equality we get $(f^{-1}\circ f)(s)=(f^{-1}\circ f)(t)$. By the definition of inverse function, $(f^{-1}\circ f)(s)=s$ and $(f^{-1}\circ f)(t)=t$. Thus $s=t$. Hence $f(s)=f(t) \implies s=t$ implying that f is injective. To prove that f is surjective we must shew that for every $b\in f(A)$ $\exists a\in A$ such that $f(a)=b$. We take $a=f^{-1}(b)$ (observe that $f^{-1}(b)\in A$). Then $f(a)=f(f^{-1}(b))=(f\circ f^{-1})(b)=b$ by definition of inverse function. This shews that f is surjective. We conclude that if f is invertible then it is also a bijection.

Assume now that f is a bijection. For every $b\in B$ there exists a unique a such that $f(a)=b$. This makes the rule $g:B\to A$ given by $g(b)=a$ a function. It is clear that $g\circ f=\mathbf{Id}_A$ and $f\circ g=\mathbf{Id}_B$. We may thus take $f^{-1}=g$. This concludes the proof. ❑

STOP *Since by Theorem 138, $(x,f(x))$ and $(f(x),x)$ are symmetric with respect to the line $y=x$, the graph of a function f is symmetric with its inverse with respect to the line $y=x$.*

Figure 4.13: A function and its inverse.

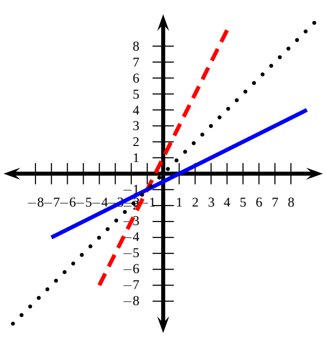


Figure 4.14: Example 358.

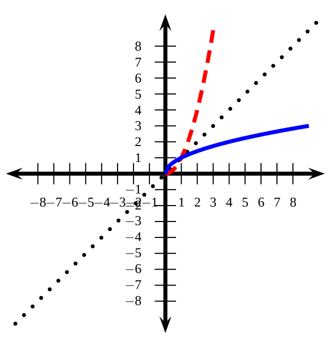


Figure 4.15: Example 359.

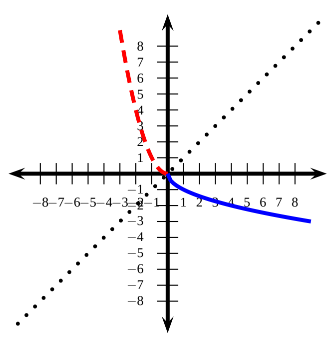


Figure 4.16: Example 360.

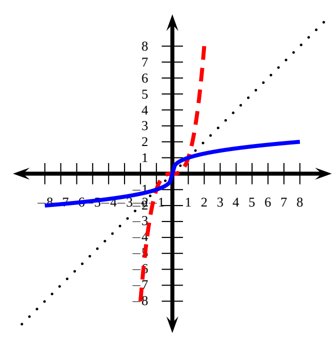


Figure 4.17: Example 361.

358 Example By Example 343,

$$g:\begin{array}{ccc}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & 2x+1\end{array}$$

is a bijection. Find the inverse of g.

Solution: The graph of a is a line non-parallel to any of the axes. Since $g(g^{-1}(x))=x$, we have $2g^{-1}(x)+1=x$. Solving for $g^{-1}(x)$ we obtain $g^{-1}(x)=\dfrac{x-1}{2}$. The inverse of g is thus

$$g^{-1}:\begin{array}{ccc}\mathbb{R} & \to & \mathbb{R}\\ x & \mapsto & \dfrac{x-1}{2}\end{array}.$$

The graphs of g (dashed), g^{-1} (solid) and the identity function (dotted) appear in figure 4.14.

359 Example

$$f: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$$

has inverse

$$f^{-1}: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & \sqrt{x} \end{array}.$$

The graph of f (dashed), f^{-1} (solid) and the line $y = x$ (dotted) appear in figure 4.15.

360 Example

$$g: \begin{array}{ccc}]-\infty;0] & \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$$

has inverse

$$g^{-1}: \begin{array}{ccc} [0;+\infty[& \to &]-\infty;0] \\ x & \mapsto & -\sqrt{x} \end{array}.$$

$g^{-1}(x) = -\sqrt{x}$. The graphs of g (dashed), g^{-1} (solid) and the line $y = x$ appear in figure 4.16.

361 Example Let

$$b: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

Find the inverse of b.

Solution: Since $b(b^{-1}(x)) = x$, we have $(b^{-1}(x))^3 = x$. Solving for $b^{-1}(x)$ we obtain $b^{-1}(x) = x^{1/3}$. The inverse of b is

$$b^{-1}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^{1/3} \end{array}.$$

We will see in Theorem 373 how to graph b. The graph of b and its inverse appear in figure 4.17.

362 Example Consider the rule

$$f(x) = \frac{1}{\sqrt{x^4 - 1}}.$$

❶ Find the natural domain of f.

❷ Find the inverse assignment rule f^{-1}.

❸ Find the image of the natural domain of f and find the natural domain of f^{-1}.

❹ Conclude.

Solution:

❶ The expression under the square root must be positive. Hence $x^4 - 1 > 0$ and the natural domain is $]-\infty;-1[\cup]1;+\infty[$. Since $f(-x) = f(x)$, f is not injective in this natural domain, so in order to find an inverse, we must restrict the domain to where f is injective.

❷ Put

$$y = \frac{1}{\sqrt{x^4 - 1}}.$$

Observe that $y > 0$. Now exchange x and y and solve for y:

$$x = \frac{1}{\sqrt{y^4 - 1}} \implies x^2(y^4 - 1) = 1 \implies y^4 = \frac{x^2+1}{x^2}.$$

Since $y > 0$ we choose the plus sign and so

$$y = \sqrt[4]{\frac{x^2+1}{x^2}}.$$

Hence

$$f^{-1}(x) = \sqrt[4]{\frac{x^2+1}{x^2}}.$$

❸ The expression for $f^{-1}(x)$ is undefined when $x = 0$ (notice that it is always positive). Hence the natural domain of f^{-1} is $\mathbb{R} \setminus \{0\}$. The image of the natural domain of f is, on the other hand, $]0; +\infty[$.

❹ The function

$$f : \begin{array}{ccc}]1; +\infty[& \to &]0; +\infty[\\ x & \mapsto & \dfrac{1}{\sqrt{x^4 - 1}} \end{array}$$

is a bijection with inverse

$$f^{-1} : \begin{array}{ccc}]0; +\infty[& \to &]1; +\infty[\\ x & \mapsto & \sqrt{\dfrac{x^2+1}{x^2}} \end{array}.$$

Homework

363 Problem Let

$$c : \begin{array}{ccc} \mathbb{R} \setminus \{-2\} & \to & \mathbb{R} \setminus \{1\} \\ x & \mapsto & \dfrac{x}{x+2} \end{array}.$$

Prove that c is bijective and find the inverse of c.

364 Problem Consider the rule

$$f(x) = \frac{1}{\sqrt[3]{x^5 - 1}}.$$

❶ Find the natural domain of f.

❷ Find the inverse assignment rule f^{-1}.

❸ Find the image of the natural domain of f and the natural domain of f^{-1}.

❹ Conclude.

365 Problem Verify that the functions below, with their domains and images, have the claimed inverses.

Assignment Rule	Natural Domain	Image	Inverse
$x \mapsto \sqrt{2-x}$	$]-\infty; 2]$	$[0; +\infty[$	$x \mapsto 2 - x^2$
$x \mapsto \dfrac{1}{\sqrt{2-x}}$	$]-\infty; 2[$	$]0; +\infty[$	$x \mapsto 2 - \dfrac{1}{x^2}$
$x \mapsto \dfrac{2+x^3}{2-x^3}$	$\mathbb{R} \setminus \{\sqrt[3]{2}\}$	$\mathbb{R} \setminus \{-1\}$	$x \mapsto \sqrt[3]{\dfrac{2x-2}{x+1}}$
$x \mapsto \dfrac{1}{x^3 - 1}$	$\mathbb{R} \setminus \{1\}$	$\mathbb{R} \setminus \{0\}$	$x \mapsto \sqrt[3]{1 + \dfrac{1}{x}}$

366 Problem Let $f, g, h : \{1, 2, 3, 4\} \to \{1, 2, 10, 1993\}$ be given by $f(1) = 1, f(2) = 2, f(3) = 10, f(4) = 1993, g(1) = g(2) = 2, g(3) = g(4) - 1 = 1, h(1) = h(2) = h(3) = h(4) + 1 = 2$.

❶ Is f invertible? Why? If so, what is $f^{-1}(f(h(4)))$?

❷ Is g one-to-one? Why?

367 Problem Given $g : \mathbb{R} \to \mathbb{R}$, $g(x) = 2x + 8$ and $f : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{0\}$, $f(x) = \dfrac{1}{x+2}$ find $(g \circ f^{-1})(-2)$.

368 Problem Let $f, g : A \to A$ be invertible. Shew that $f \circ g$ is invertible and that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

369 Problem Prove that $t : \begin{array}{ccc}]-\infty; 1] & \to & [0; +\infty[\\ x & \mapsto & \sqrt{1-x} \end{array}$ is a bijection and find t^{-1}.

370 Problem Let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=ax+b$. For which parameters a and b is $f=f^{-1}$?

371 Problem Prove that if $ab\neq -4$ and $f:\mathbb{R}\setminus\{2/b\}\to\mathbb{R}\setminus\{2/b\}$,
$$f(x)=\frac{2x+a}{bx-2}$$ then $f=f^{-1}$.

372 Problem Prove, without using a calculator, that
$$\sum_{k=1}^{9}\left(\left(\frac{k}{10}\right)^2+\sqrt{\frac{k}{10}}\right)<9.5$$

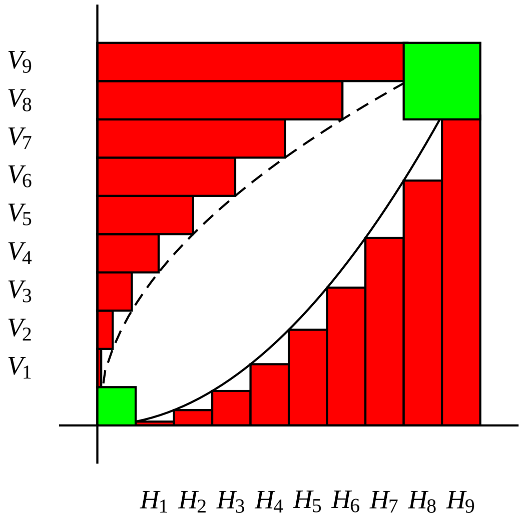


Figure 4.18: Problem 372.

Answers

307
❶ $\mathbb{R}$
❷ $[-5;5]$
❸ $\mathbb{R}$
❹ $\mathbb{R}$
❺ $]-\infty;1-\sqrt{3}[\cup]1+\sqrt{3};+\infty[$
❻ $\mathbb{R}$
❼ $]-\infty;-1[\cup]-1;0]$
❽ $]-1;1[$
❾ $\{0\}$

309 $[-2\sqrt{3};0]\cup[2\sqrt{3};+\infty[$.

310 $x\in]-\infty;1-\sqrt{3}[\cup]1+\sqrt{3};+\infty[$.

311 They are
❶ $\{-1\}$
❷ $\varnothing$
❸ $[0;2[\cup]3;+\infty[$
❹ $]3;+\infty[$
❺ $]-\infty;-3[\cup]-2;0[\cup]0;2[\cup]3;+\infty[$
❻ $]-3;-2[\cup]0;2[\cup]3;+\infty[$
❼ $\mathbb{R}\setminus\{-3,-2,2,3\}$

326
❶ $[-4;2]$
❷ $[-4;2]$
❸ $]-4;2]$
❹ $[-4;-2[\cup]-2;2[$
❺ -2
❻ 0
❼ 0
❽ undefined
❾ 5
❿ $\frac{1}{5}$

327 1)$\{-4,-2,0,2,4\}$ 2) $\{0,1,4\}$ 3) $\{0,1\}$. 4) $\{0,2\}$.

328 (1) 13, (2) 5981, (3) 10, (4) 1995

330 1) $[0;+\infty]$ 2) $[0;2]$ 3) $\{0\}$. 4) $[2;6]$. 5) $\sqrt{\sqrt{4-x^2}-2}$. 6) $\sqrt{6-x}$.

331 1) $[0;\sqrt{2}]$ 2) $]-\infty;0]$ 3) $[-2;0]$. 4) $\{-\sqrt{2},\sqrt{2}\}$. 5) $\sqrt{2+x}$. 6) $-\sqrt{-\sqrt{2-x^2}}$.

333 $\frac{\sqrt{2}}{2}$

334 $\frac{8}{4+x}$

335 $x=1/3$.

336 $(f\circ f)(x)=4x^2-4x^3+x^4$.

337 We have $f^{[2]}(x)=f(x+1)=(x+1)+1=x+2$, $f^{[3]}(x)=f(x+2)=(x+2)+1=x+3$ and so, recursively, $f^{[n]}(x)=x+n$.

338 We have $f^{[2]}(x)=f(2x)=2^2x$, $f^{[3]}(x)=f(2^2x)=2^3x$ and so, recursively, $f^{[n]}(x)=2^nx$.

339 We have (i) $f^{[2]}(x)=(f\circ f)(x)=f(f(x))=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x}$.
(ii) $f^{[3]}(x)=(f\circ f\circ f)(x)=f(f^{[2]}(x)))=f\left(\frac{x-1}{x}\right)=\frac{1}{1-\frac{x-1}{x}}=x$.
(iii) Notice that $f^{[4]}(x)=(f\circ f^{[3]})(x)=f(f^{[3]}(x))=f(x)=f^{[1]}(x)$. We see that f is cyclic of period 3, that is, $f^{[1]}=f^{[14]}=f^{[7]}=\ldots, f^{[2]}=f^{[5]}=f^{[8]}=\ldots, f^{[3]}=f^{[6]}=f^{[9]}=\ldots$. Hence $f^{[69]}(x)=f^{[3]}(x)=x$.

340 $c=-3$

352 We must shew that there is a solution x for the equation $f(x)=b, b\in\mathbb{R}\setminus\{2\}$. Now

$$f(x)=b \implies \frac{2x}{x+1}=b \implies x=\frac{b}{2-b}.$$

Thus as long as $b\neq 2$ there is $x\in\mathbb{R}$ with $f(x)=b$. Since there is no x such that $g(x)=2$ and $2\in\textbf{Target}(g)$, g is not surjective.

353

❶ neither, $f(-1)=f(1)$ so not injective. There is no a with $f(a)=-1$, so not surjective.

❷ surjective, $f(1)=f(-1)$ so not injective.

❸ surjective, not injective.

❹ injective, as proved in text, there is no a with $f(a)=-1$, so not surjective.

❺ neither, $|1|=|-1|$ so not injective, there is no a with $|a|=-1$, so not surjective.

❻ injective, non-surjective since, say, there is no a with $-|a|=1$.

❼ surjective, non-injective since, say, $|-1|=|1|$ but $-1\neq 1$.

❽ bijective.

363 Since $c(c^{-1}(x))=x$, we have $\dfrac{c^{-1}(x)}{c^{-1}(x)+2}=x$. Solving for $c^{-1}(x)$ we obtain $c^{-1}(x)=\dfrac{2x}{1-x}=-2+\dfrac{2}{1-x}$. The inverse of c is therefore

$$c^{-1}: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R}\setminus\{-2\} \\ x & \mapsto & -2+\dfrac{2}{1-x} \end{array}.$$

364 We have

❶ The expression under the cubic root must not be 0. Hence $x^5\neq 1$ and the natural domain is $\mathbb{R}\setminus\{1\}$.

❷ Put

$$y=\frac{1}{\sqrt[3]{x^5-1}}.$$

Now exchange x and y and solve for y:

$$x=\frac{1}{\sqrt[3]{y^5-1}} \implies x^3(y^5-1)=1 \implies y=\sqrt[5]{\frac{x^3+1}{x^3}}.$$

Hence

$$f^{-1}(x)=\sqrt[5]{\frac{x^3+1}{x^3}}.$$

❸ As x varies in $\mathbb{R}\setminus\{1\}$, the expression $\dfrac{1}{\sqrt[3]{x^5-1}}$ assumes all positive and negative values, but it is never 0. Thus $\textbf{Im}(f)=\mathbb{R}\setminus\{0\}$. The expression for $f^{-1}(x)$ is undefined when $x=0$. Hence the natural domain of f^{-1} is $\mathbb{R}\setminus\{0\}$.

❹ The function

$$f: \begin{array}{ccc} \mathbb{R}\setminus\{1\} & \to & \mathbb{R}\setminus\{0\} \\ x & \mapsto & \dfrac{1}{\sqrt[3]{x^5-1}} \end{array}$$

is a bijection with inverse

$$f^{-1}: \begin{array}{ccc} \mathbb{R}\setminus\{0\} & \to & \mathbb{R}\setminus\{1\} \\ x & \mapsto & \sqrt[5]{\dfrac{x^3+1}{x^3}} \end{array}.$$

366 1) Yes, f is a bijection. $f^{-1}(f(h(4)))=h(4)=1$, 2) No

367 3

369 $t^{-1}: \begin{array}{ccc} [0;+\infty[& \to &]-\infty;1] \\ x & \mapsto & 1-x^2 \end{array}$

370 Either $a=1, b=0$ or $a=-1$ and b arbitrary.

372 The inverse of

$$f: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & x^2 \end{array}$$

is

$$f^{-1}: \begin{array}{ccc} [0;+\infty[& \to & [0;+\infty[\\ x & \mapsto & \sqrt{x} \end{array}.$$

In diagram 4.18, each rectangle V_k has its lower left corner at $(0,\frac{k}{10})$, base $\sqrt{\frac{k}{10}}$ and height $\frac{1}{10}$. Each rectangle H_k has lower left corner at $(\frac{k}{10},0)$, base $\frac{1}{10}$ and height $(\frac{k}{10})^2$. The collective area of these rectangles is

$$\frac{1}{10}\left(\left(\frac{1}{10}\right)^2+\sqrt{\frac{1}{10}}+\left(\frac{2}{10}\right)^2+\sqrt{\frac{2}{10}}+\left(\frac{3}{10}\right)^2+\sqrt{\frac{3}{10}}+\cdots+\left(\frac{9}{10}\right)^2+\sqrt{\frac{9}{10}}\right)$$

Since these grey rectangles do not intersect with the green squares on the corners, their collective area is less than the area of the unit square minus these smaller squares: $1-\frac{1}{100}-\frac{4}{100}=\frac{95}{100}$. We thus conclude that

$$\frac{1}{10}\left(\left(\frac{1}{10}\right)^2+\sqrt{\frac{1}{10}}+\left(\frac{2}{10}\right)^2+\sqrt{\frac{2}{10}}+\left(\frac{3}{10}\right)^2+\sqrt{\frac{3}{10}}+\cdots+\left(\frac{9}{10}\right)^2+\sqrt{\frac{9}{10}}\right)<\frac{95}{100}.$$

Chapter 5

Polynomial Functions

5.1 Power Functions

By a *power function* we mean a function of the form $x \mapsto x^{\alpha}$, where $\alpha \in \mathbb{R}$. In this chapter we will only study the case when α is a positive integer.

If n is a positive integer, we are interested in how to graph $x \mapsto x^n$. We have already encountered a few instances of power functions. For $n = 0$, the function $x \mapsto 1$ is a constant function, whose graph is the straight line $y = 1$ parallel to the x-axis. For $n = 1$, the function $x \mapsto x$ is the identity function, whose graph is the straight line $y = x$, which bisects the first and third quadrant. For $n = 2$, we have the square function $x \mapsto x^2$ whose graph is the parabola $y = x^2$ encountered in example 163. We reproduce their graphs below in figures 5.1 through 5.3 for easy reference.

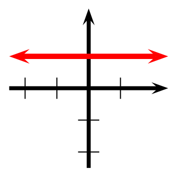


Figure 5.1: $x \mapsto 1$.

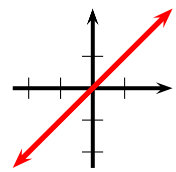


Figure 5.2: $x \mapsto x$.

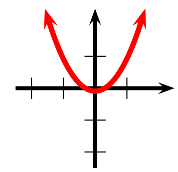


Figure 5.3: $x \mapsto x^2$.

We now deduce properties for the cube function.

373 Theorem (Graph of the Cubic Function) The graph of the cubic function

$$\mathbf{Cube}: \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x^3 \end{array}$$

is concave for $x < 0$ and convex for $x > 0$. $x \mapsto x^3$ is an increasing odd function and $\mathbf{Im}(\mathbf{Cube}) = \mathbb{R}$.

Proof: *Consider*

$$\mathbf{Cube}(\lambda a + (1-\lambda)b) - \lambda \mathbf{Cube}(a) - (1-\lambda)\mathbf{Cube}(b),$$

which is equivalent to

$$(\lambda a + (1-\lambda)b)^3 - \lambda a^3 - (1-\lambda)b^3,$$

which is equivalent to

$$(\lambda^3 - \lambda)a^3 + ((1-\lambda)^3 - (1-\lambda))b^3 + 3\lambda(1-\lambda)ab(\lambda a + (1-\lambda)b),$$

which is equivalent to

$$-(1-\lambda)(1+\lambda)\lambda a^3+(-\lambda^3+3\lambda^2-2\lambda)b^3+3\lambda(1-\lambda)ab(\lambda a+(1-\lambda)b),$$

which in turn is equivalent to

$$(1-\lambda)\lambda(-(1+\lambda)a^3+(\lambda-2)b^3+3ab(\lambda a+(1-\lambda)b)).$$

This last expression factorises as

$$-\lambda(1-\lambda)(a-b)^2(\lambda(a-b)+2b+a).$$

Since $\lambda(1-\lambda)(a-b)^2\geq 0$ *for* $\lambda\in[0;1]$,

$$\mathbf{Cube}(\lambda a+(1-\lambda)b)-\lambda\mathbf{Cube}(a)-(1-\lambda)\mathbf{Cube}(b)$$

has the same sign as

$$-(\lambda(a-b)+2b+a)=-(\lambda a+(1-\lambda)b+b+a).$$

If $(a,b)\in]0;+\infty[^2$ *then* $\lambda a+(1-\lambda)b\geq 0$ *by lemma 271 and so*

$$-(\lambda a+(1-\lambda)b+b+a)\leq 0$$

meaning that **Cube** *is convex for* $x\geq 0$. *Similarly, if* $(a,b)\in]-\infty;0[^2$ *then*

$$-(\lambda a+(1-\lambda)b+b+a)\geq 0$$

and so $x\mapsto x^3$ *is concave for* $x\geq 0$. *This proves the claim.*

As $\mathbf{Cube}(-x)=(-x)^3=-x^3=-\mathbf{Cube}(x)$, *the cubic function is an odd function. Since for* $a<b$

$$\frac{\mathbf{Cube}(b)-\mathbf{Cube}(a)}{b-a}=\frac{b^3-a^3}{b-a}=b^2+ab+b^2=\left(b+\frac{a}{2}\right)^2+\frac{3a^2}{4}>0,$$

Cube *is a strictly increasing function. Also if* $y\in\mathbf{Im}(\mathbf{Cube})$ *then there is* $x\in\mathbb{R}$ *such that* $x^3=\mathbf{Cube}(x)=y$. *The equation* $y=x^3$ *has a solution for every* $y\in\mathbb{R}$ *and so* $\mathbf{Im}(\mathbf{Cube})=\mathbb{R}$. *The graph of* $x\mapsto x^3$ *appears in figure 5.7.*
❑

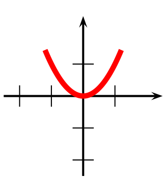


Figure 5.4: $y=x^2$.

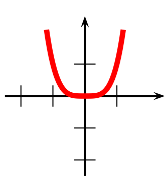


Figure 5.5: $y=x^4$.

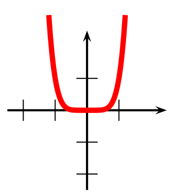


Figure 5.6: $y=x^6$.

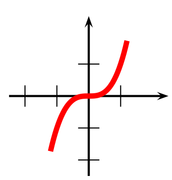


Figure 5.7: $y=x^3$.

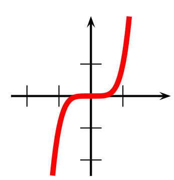


Figure 5.8: $y=x^5$.

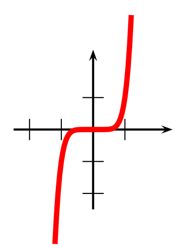


Figure 5.9: $y=x^7$.

374 Theorem Let $n\geq 2$ be an integer and $f(x)=x^n$. Then

- if n is even, f is convex, f is decreasing for $x<0$, and f is increasing for $x>0$. Also, $f(-\infty)=f(+\infty)=+\infty$.
- if n is odd, f is increasing, f is concave for $x<0$, and f is convex for $x>0$. Also, $f(-\infty)=-\infty$ and $f(+\infty)=+\infty$.

The graphs of $y = x^2$, $y = x^4$, $y = x^6$, etc., resemble one other. For $-1 \leq x \leq 1$, the higher the exponent, the flatter the graph (closer to the x-axis) will be, since

$$|x| < 1 \implies \cdots < x^6 < x^4 < x^2 < 1.$$

For $|x| \geq 1$, the higher the exponent, the steeper the graph will be since

$$|x| > 1 \implies \cdots > x^6 > x^4 > x^2 > 1.$$

Similarly for the graphs of $y = x^3$, $y = x^5$, $y = x^7$ etc. This information is summarised in the tables below.

x	$-\infty$	0	$+\infty$
$f(x) = x^n$	$\nearrow$	0	$\nearrow$

Table 5.1: $x \mapsto x^n$, with $n > 0$ integer and odd.

x	$-\infty$	0	$+\infty$
$f(x) = x^n$	$\searrow$	0	$\nearrow$

Table 5.2: $x \mapsto x^n$, with $n > 0$ integer and even.

375 Example Figures 5.10 through 5.12 shew a few transformations of the function $x \mapsto x^3$.

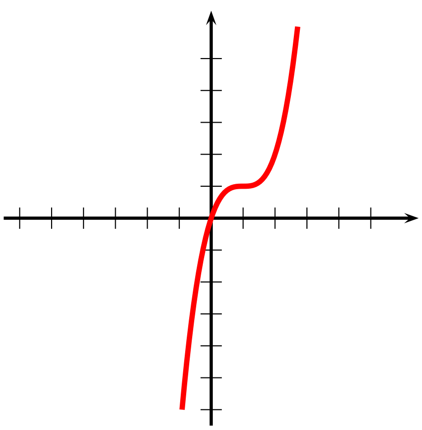


Figure 5.10: $y = (x-1)^3 + 1$

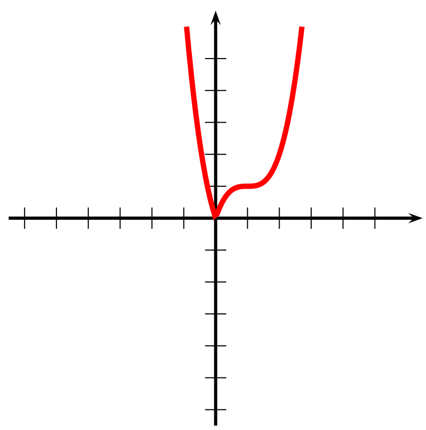


Figure 5.11: $y = |(x-1)^3 + 1|$

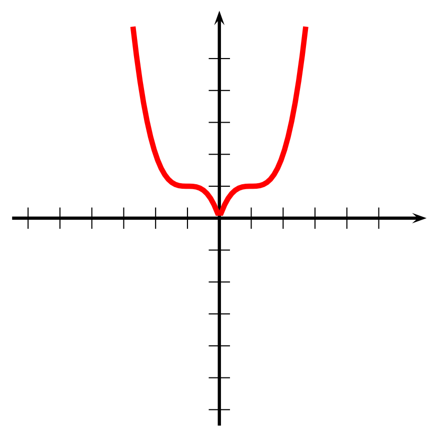


Figure 5.12: $y = (|x|-1)^3 + 1$

Homework

376 Problem Draw the following curves.

❶ $y = x^6 - 1$

❷ $y = 2 - x^5$

❸ $y = |2 - x^5|$

❹ $y = 2 - |x|^5$

5.2 Affine Functions

377 Definition Let m, k be real number constants. A function of the form $x \mapsto mx + k$ is called an *affine function*. In the particular case that $m = 0$, we call $x \mapsto k$ a *constant function*. If, however, $k = 0$, then we call the function $x \mapsto mx$ a *linear function*.

By virtue of Theorem 118, the graph of the function $x \mapsto mx+k$ is a straight line. We also know that $x \mapsto mx+k$ is strictly increasing if $m > 0$ and strictly decreasing if $m < 0$. If $m \neq 0$ then $mx+k=0 \implies x = -\frac{k}{m}$, meaning that $x \mapsto mx+k$ has a unique zero (crosses the x-axis) at $x = -\frac{k}{m}$. This information is summarised in the following tables.

x	$-\infty$	$-\frac{k}{m}$	$+\infty$
$f(x)=mx+k$	$\nearrow$	0	$\nearrow$

Table 5.3: $x \mapsto mx+k$, with $m > 0$.

x	$-\infty$	$-\frac{k}{m}$	$+\infty$
$f(x)=mx+k$	$\searrow$	0	$\searrow$

Table 5.4: $x \mapsto mx+k$, with $m < 0$.

5.3 Quadratic Functions

378 Definition Let a,b,c be real numbers, with $a \neq 0$. A function of the form

$$f: \begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R} \\ x & \mapsto & ax^2+bx+c \end{array}$$

is called a *quadratic function* with leading coefficient a.

379 Theorem Let $a \neq 0, b, c$ be real numbers and let $x \mapsto ax^2+bx+c$ be a quadratic function. Then its graph is a parabola. If $a > 0$ the parabola has a local minimum at $x = -\frac{b}{2a}$ and it is convex. If $a < 0$ the parabola has a local maximum at $x = -\frac{b}{2a}$ and it is concave.

Proof: *Put $f(x) = ax^2+bx+c$. Completing squares,*

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+2\frac{b}{2a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a} \\ &= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}, \end{aligned}$$

and hence this is a horizontal translation $-\frac{b}{2a}$ units and a vertical translation $\frac{4ac-b^2}{4a}$ units of the square function $x \mapsto x^2$ and so it follows from example 175 and Theorems 215 and 226, that the graph of f is a parabola.

Assume first that $a > 0$. Then f is convex, decreases if $x < -\frac{b}{2a}$ and increases if $x > -\frac{b}{2a}$, and so it has a minimum at $x = -\frac{b}{2a}$. The analysis of $-f$ yields the case for $a < 0$, and the Theorem is proved. ❑

The information of Theorem 379 is summarised in the following tables.

380 Definition The point $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$ lies on the parabola and it is called the *vertex* of the parabola $y = ax^2+bx+c$. The quantity b^2-4ac is called the *discriminant* of ax^2+bx+c. The equation

$$y = a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}$$

is called the *canonical equation of the parabola* $y = ax^2+bx+c$.

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$f(x) = ax^2+bx+c$	$\searrow$	0	$\nearrow$

Table 5.5: $x \mapsto ax^2+bx+c$, with $a > 0$.

x	$-\infty$	$-\frac{b}{2a}$	$+\infty$
$f(x) = ax^2+bx+c$	$\nearrow$	0	$\searrow$

Table 5.6: $x \mapsto ax^2+bx+c$, with $a < 0$.

The parabola $x \mapsto ax^2+bx+c$ is symmetric about the vertical line $x = -\dfrac{b}{2a}$ passing through its vertex.

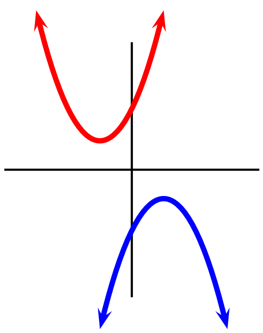


Figure 5.13: No real zeroes.

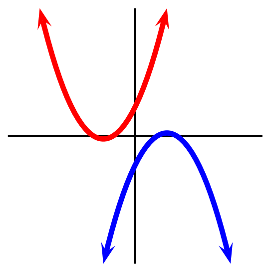


Figure 5.14: One real zero.

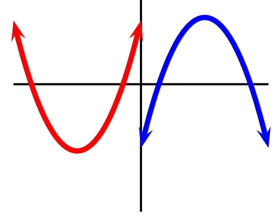


Figure 5.15: Two real zeros.

381 Corollary (Quadratic Formula) The roots of the equation $ax^2+bx+c = 0$ are given by the formula

$$ax^2+bx+c = 0 \iff x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{5.1}$$

If $a \neq 0, b, c$ are real numbers and $b^2-4ac = 0$, the parabola $x \mapsto ax^2+bx+c$ is tangent to the x-axis and has one (repeated) real root. If $b^2-4ac > 0$ then the parabola has two distinct real roots. Finally, if $b^2-4ac < 0$ the parabola has two complex roots.

Proof: *By Theorem 379 we have*

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a},$$

and so

$$\begin{aligned} ax^2+bx+c = 0 \quad &\iff \quad \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} \\ &\iff \quad x+\frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2|a|} \\ &\iff \quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \end{aligned}$$

where we have dropped the absolute values on the last line because the only effect of having $a < 0$ is to change from $\pm$ to $\mp$.

If $b^2-4ac = 0$ then the vertex of the parabola is at $\left(-\dfrac{b}{2a}, 0\right)$ on the x-axis, and so the parabola is tangent there. Also, $x = -\dfrac{b}{2a}$ would be the only root of this equation. This is illustrated in figure 5.14.

If $b^2-4ac>0$, then $\sqrt{b^2-4ac}$ is a real number $\neq 0$ and so $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ are distinct numbers. This is illustrated in figure 5.15.

If $b^2-4ac<0$, then $\sqrt{b^2-4ac}$ is a complex number $\neq 0$ and so $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ are distinct complex numbers. This is illustrated in figure 5.13. ❑

STOP *If a quadratic has real roots, then the vertex lies on a line crossing the midpoint between the roots.*

382 Example Consider the quadratic function $f(x)=x^2-5x+3$.

❶ Write this parabola in canonical form and hence find the vertex of f. Determine the intervals of monotonicity of f and its convexity.

❷ Find the x-intercepts and y-intercepts of f.

❸ Graph $y=f(x)$, $y=|f(x)|$, and $y=f(|x|)$.

❹ Determine the set of real numbers x for which $f(x)>0$.

Solution:

❶ Completing squares

$$y=x^2-5x+3=\left(x-\frac{5}{2}\right)^2-\frac{13}{4}.$$

From this the vertex is at $\left(\frac{5}{2},-\frac{13}{4}\right)$. Since the leading coefficient of f is positive, f will be increasing for $x>\frac{5}{2}$ and it will be decreasing for $x<\frac{5}{2}$ and f is concave for all real values of x.

❷ For $x=0$, $f(0)=0^2-5\cdot 0+3=3$, and hence $y=f(0)=3$ is the y-intercept. By the quadratic formula,

$$f(x)=0 \iff x^2-5x+3=0 \iff x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(3)}}{2(1)}=\frac{5\pm\sqrt{13}}{2}.$$

Observe that $\dfrac{5-\sqrt{13}}{2}\approx 0.697224362$ and $\dfrac{5+\sqrt{13}}{2}\approx 4.302775638$.

❸ The graphs appear in figures 5.16 through 5.18.

❹ From the graph in figure 5.16, $x^2-5x+3>0$ for values $x\in\left]-\infty\,;\frac{5-\sqrt{13}}{2}\right[$ or $x\in\left]\frac{5+\sqrt{13}}{2}\,;+\infty\right[$.

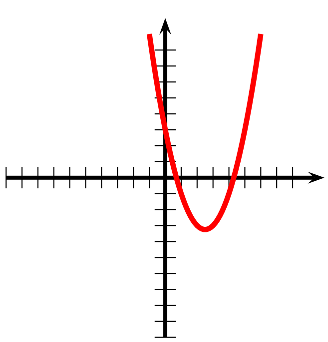


Figure 5.16: $y=x^2-5x+3$

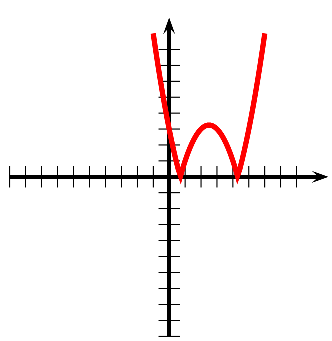


Figure 5.17: $y=|x^2-5x+3|$

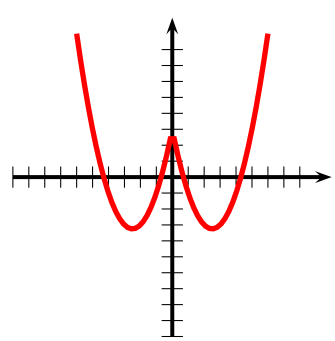


Figure 5.18: $y=|x|^2-5|x|+3$

383 Corollary If $a \neq 0, b, c$ are real numbers and if $b^2 - 4ac < 0$, then $ax^2 + bx + c$ has the same sign as a.

Proof: *Since*

$$ax^2 + bx + c = a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right),$$

and $4ac - b^2 > 0$, $\left(\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right) > 0$ *and so* $ax^2 + bx + c$ *has the same sign as* a. ❑

384 Example Prove that the quantity $q(x) = 2x^2 + x + 1$ is positive regardless of the value of x.

Solution: The discriminant is $1^2 - 4(2)(1) = -7 < 0$, hence the roots are complex. By Corollary 383, since its leading coefficient is $2 > 0$, $q(x) > 0$ regardless of the value of x. Another way of seeing this is to complete squares and notice the inequality

$$2x^2 + x + 1 = 2\left(x + \frac{1}{4}\right)^2 + \frac{7}{8} \geq \frac{7}{8},$$

since $\left(x + \frac{1}{4}\right)^2$ being the square of a real number, is ≥ 0.

By Corollary 381, if $a \neq 0, b, c$ are real numbers and if $b^2 - 4ac \neq 0$ then the numbers

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

are distinct solutions of the equation $ax^2 + bx + c = 0$. Since

$$r_1 + r_2 = -\frac{b}{a}, \qquad \text{and} \qquad r_1 r_2 = \frac{c}{a},$$

any quadratic can be written in the form

$$ax^2 + bx + c = a\left(x^2 + \frac{bx}{a} + \frac{c}{a}\right) = a\left(x^2 - (r_1 + r_2)x + r_1 r_2\right) = a(x - r_1)(x - r_2).$$

We call $a(x - r_1)(x - r_2)$ a *factorisation* of the quadratic $ax^2 + bx + c$.

385 Example A quadratic polynomial p has $1 \pm \sqrt{5}$ as roots and it satisfies $p(1) = 2$. Find its equation.

Solution: Observe that the sum of the roots is

$$r_1 + r_2 = 1 - \sqrt{5} + 1 + \sqrt{5} = 2$$

and the product of the roots is

$$r_1 r_2 = (1 - \sqrt{5})(1 + \sqrt{5}) = 1 - (\sqrt{5})^2 = 1 - 5 = -4.$$ [1]

Hence p has the form

$$p(x) = a\left(x^2 - (r_1 + r_2)x + r_1 r_2\right) = a(x^2 - 2x - 4).$$

Since

$$2 = p(1) \implies 2 = a(1^2 - 2(1) - 4) \implies a = -\frac{2}{5},$$

the polynomial sought is

$$p(x) = -\frac{2}{5}\left(x^2 - 2x - 4\right).$$

[1] As a shortcut for this multiplication you may wish to recall the *difference of squares identity*: $(a - b)(a + b) = a^2 - b^2$.

Homework

386 Problem Let

$$R_1 = \{(x,y) \in \mathbb{R}^2 | y \geq x^2 - 1\},$$
$$R_2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4\},$$
$$R_3 = \{(x,y) \in \mathbb{R}^2 | y \leq -x^2 + 4\}.$$

Sketch the following regions.

❶ $R_1 \setminus R_2$
❷ $R_1 \cap R_3$
❸ $R_2 \setminus R_1$
❹ $R_1 \cap R_2$

387 Problem Write the following parabolas in canonical form, determine their vertices and graph them: (i) $y = x^2 + 6x + 9$, (ii) $y = x^2 + 12x + 35$, (iii) $y = (x-3)(x+5)$, (iv) $y = x(1-x)$, (v) $y = 2x^2 - 12x + 23$, (vi) $y = 3x^2 - 2x + \frac{8}{9}$, (vii) $y = \frac{1}{5}x^2 + 2x + 13$

388 Problem Find the vertex of the parabola $y = (3x-9)^2 - 9$.

389 Problem Find the equation of the parabola whose axis of symmetry is parallel to the y-axis, with vertex at $(0,-1)$ and passing through $(3,17)$.

390 Problem Find the equation of the parabola having roots at $x = -3$ and $x = 4$ and passing through $(0,24)$.

391 Problem Let $0 \leq a,b,c \leq 1$. Prove that at least one of the products $a(1-b), b(1-c), c(1-a)$ is smaller than or equal to $\frac{1}{4}$.

392 Problem An apartment building has 30 units. If all the units are inhabited, the rent for each unit is \$700 per unit. For every empty unit, management increases the rent of the remaining tenants by \$25. What will be the profit $P(x)$ that management gains when x units are empty? What is the maximum profit?

393 Problem Find all real solutions to $|x^2 - 2x| = |x^2 + 1|$.

394 Problem Find all the real solutions to

$$(x^2 + 2x - 3)^2 = 2.$$

395 Problem Solve $x^3 - x^2 - 9x + 9 = 0$.

396 Problem Solve $x^3 - 2x^2 - 11x + 12 = 0$.

397 Problem Find all real solutions to $x^3 - 1 = 0$.

398 Problem Solve $9 + x^{-4} = 10x^{-2}$.

399 Problem Find all the real values of the parameter t for which the equation in x

$$t^2x - 3t = 81x - 27$$

has a solution.

400 Problem The sum of two positive numbers is 50. Find the largest value of their product.

401 Problem Of all rectangles having perimeter 20 shew that the square has the largest area.

402 Problem An orchard currently has 25 trees, which produce 600 fruits each. It is known that for each additional tree planted, the production of each tree diminishes by 15 fruits. Find:

❶ the current fruit production of the orchard,

❷ a formula for the production obtained from each tree upon planting x more trees,

❸ a formula $P(x)$ for the production obtained from the orchard upon planting x more trees.

❹ How many trees should be planted in order to yield maximum production?

403 Problem Points A , B, and C are on the parabola $y = \dfrac{x^2}{2}$ as shown in figure 5.19. If $\triangle ABC$ is equilateral, determine the x-coordinate of point B.

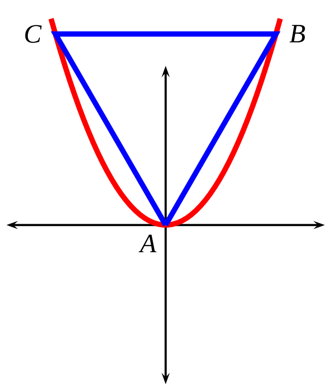


Figure 5.19: Problem 403.

5.4 Polynomials

5.4.1 Roots

In sections 5.2 and 5.3 we learned how to find the roots of equations (in the unknown x) of the type $ax+b=0$ and $ax^2+bx+c=0$, respectively. We would like to see what can be done for equations where the power of x is higher than 2. We recall that

404 Definition A *polynomial* $p(x)$ of degree $n \in \mathbb{N}$ is an expression of the form

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0, \qquad a_k \in \mathbb{R},$$

where the a_k are constants. If the a_k are all integers then we say that p has integer coefficients, and we write $p(x) \in \mathbb{Z}[x]$; if the a_k are real numbers then we say that p has real coefficients and we write $p(x) \in \mathbb{R}[x]$; etc. The degree of the polynomial p is denoted by $\deg p$. The coefficient a_n is called the *leading coefficient* of $p(x)$. A *root* of p is a solution to the equation $p(x) = 0$.

405 Example Here are a few examples of polynomials.

- $a(x) = 2x+1 \in \mathbb{Z}[x]$, is a polynomial of degree 1, and leading coefficient 2. It has $x = -\dfrac{1}{2}$ as its only root. A polynomial of degree 1 is also known as an *affine function*.
- $b(x) = \pi x^2 + x - \sqrt{3} \in \mathbb{R}[x]$, is a polynomial of degree 2 and leading coefficient π. By the quadratic formula b has the two roots
$$x = \frac{-1+\sqrt{1+4\pi\sqrt{3}}}{2\pi} \qquad \text{and} \qquad x = \frac{-1-\sqrt{1+4\pi\sqrt{3}}}{2\pi}.$$
A polynomial of degree 2 is also called a *quadratic polynomial* or *quadratic function*.
- $C(x) = 1 \equiv 1 \cdot x^0$, is a constant polynomial, of degree 0. It has no roots, since it is never zero.[2]

406 Theorem The degree of the product of two polynomials is the sum of their degrees. In symbols, if p, q are polynomials, $\deg pq = \deg p + \deg q$.

Proof: *If* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, *and* $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, *with* $a_n \neq 0$ *and* $b_m \neq 0$ *then upon multiplication,*

$$p(x)q(x) = (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)(b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0) = a_n b_m x^{m+n} + \cdots +,$$

with non-vanishing leading coefficient $a_n b_m$. ❑

407 Example The polynomial $p(x) = (1+2x+3x^3)^4(1-2x^2)^5$ has leading coefficient $3^4(-2)^5 = -2592$ and degree $3 \cdot 4 + 2 \cdot 5 = 22$.

408 Example What is the degree of the polynomial identically equal to 0? Put $p(x) \equiv 0$ and, say, $q(x) = x+1$. Then by Theorem 406 we must have $\deg pq = \deg p + \deg q = \deg p + 1$. But pq is identically 0, and hence $\deg pq = \deg p$. But if $\deg p$ were finite then

$$\deg p = \deg pq = \deg p + 1 \implies 0 = 1,$$

nonsense. Thus the 0-polynomial does not have any finite degree. We attach to it, by convention, degree $-\infty$.

409 Definition If all the roots of a polynomial are in $\mathbb{Z}$ (integer roots), then we say that the *polynomial splits or factors over* $\mathbb{Z}$. If all the roots of a polynomial are in $\mathbb{Q}$ (rational roots), then we say that the *polynomial splits or factors over* $\mathbb{Q}$. If all the roots of a polynomial are in $\mathbb{C}$ (complex roots), then we say that the *polynomial splits (factors) over* $\mathbb{C}$.

Since $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, *any polynomial splitting on a smaller set immediately splits over a larger set.*

[2] The symbol $\equiv$ is read "identically equal to" and it means that both expressions are always the same, regardless of the value of the input parameter.

410 Example The polynomial $l(x) = x^2 - 1 = (x-1)(x+1)$ splits over $\mathbb{Z}$. The polynomial $p(x) = 4x^2 - 1 = (2x-1)(2x+1)$ splits over $\mathbb{Q}$ but not over $\mathbb{Z}$. The polynomial $q(x) = x^2 - 2 = (x-\sqrt{2})(x+\sqrt{2})$ splits over $\mathbb{R}$ but not over $\mathbb{Q}$. The polynomial $r(x) = x^2 + 1 = (x-i)(x+i)$ splits over $\mathbb{C}$ but not over $\mathbb{R}$. Here $i = \sqrt{-1}$ is the imaginary unit.

5.4.2 Ruffini's Factor Theorem

411 Theorem (Division Algorithm) If the polynomial $p(x)$ is divided by $a(x)$ then there exist polynomials $q(x), r(x)$ with

$$p(x) = a(x)q(x) + r(x) \tag{5.2}$$

and $0 \leq \text{degree } r(x) < \text{degree } a(x)$.

412 Example If $x^5 + x^4 + 1$ is divided by $x^2 + 1$ we obtain

$$x^5 + x^4 + 1 = (x^3 + x^2 - x - 1)(x^2 + 1) + x + 2,$$

and so the quotient is $q(x) = x^3 + x^2 - x - 1$ and the remainder is $r(x) = x + 2$.

413 Example Find the remainder when $(x+3)^5 + (x+2)^8 + (5x+9)^{1997}$ is divided by $x+2$.

Solution: As we are dividing by a polynomial of degree 1, the remainder is a polynomial of degree 0, that is, a constant. Therefore, there is a polynomial $q(x)$ and a constant r with

$$(x+3)^5 + (x+2)^8 + (5x+9)^{1997} = q(x)(x+2) + r$$

Letting $x = -2$ we obtain

$$(-2+3)^5 + (-2+2)^8 + (5(-2)+9)^{1997} = q(-2)(-2+2) + r = r.$$

As the sinistral side is 0 we deduce that the remainder $r = 0$.

414 Example A polynomial leaves remainder -2 upon division by $x-1$ and remainder -4 upon division by $x+2$. Find the remainder when this polynomial is divided by $x^2 + x - 2$.

Solution: From the given information, there exist polynomials $q_1(x), q_2(x)$ with $p(x) = q_1(x)(x-1) - 2$ and $p(x) = q_2(x)(x+2) - 4$. Thus $p(1) = -2$ and $p(-2) = -4$. As $x^2 + x - 2 = (x-1)(x+2)$ is a polynomial of degree 2, the remainder $r(x)$ upon dividing $p(x)$ by $x^2 + x - 1$ is of degree 1 or smaller, that is $r(x) = ax + b$ for some constants a, b which we must determine. By the Division Algorithm,

$$p(x) = q(x)(x^2 + x - 1) + ax + b.$$

Hence

$$-2 = p(1) = a + b$$

and

$$-4 = p(-2) = -2a + b.$$

From these equations we deduce that $a = 2/3, b = -8/3$. The remainder sought is

$$r(x) = \frac{2}{3}x - \frac{8}{3}.$$

415 Theorem (Ruffini's Factor Theorem) The polynomial $p(x)$ is divisible by $x - a$ if and only if $p(a) = 0$. Thus if p is a polynomial of degree n, then $p(a) = 0$ if and only if $p(x) = (x-a)q(x)$ for some polynomial q of degree $n-1$.

Proof: *As $x - a$ is a polynomial of degree 1, the remainder after diving $p(x)$ by $x - a$ is a polynomial of degree 0, that is, a constant. Therefore*

$$p(x) = q(x)(x-a) + r.$$

From this we gather that $p(a) = q(a)(a-a) + r = r$, from where the theorem easily follows. ❑

416 Example Find the value of a so that the polynomial

$$t(x) = x^3 - 3ax^2 + 2$$

be divisible by $x + 1$.

Solution: By Ruffini's Theorem 415, we must have

$$0 = t(-1) = (-1)^3 - 3a(-1)^2 + 2 \implies a = \frac{1}{3}.$$

417 Definition Let a be a root of a polynomial p. We say that a is a root of *multiplicity* m if $p(x)$ is divisible by $(x-a)^m$ but not by $(x-a)^{m+1}$. This means that p can be written in the form $p(x) = (x-a)^m q(x)$ for some polynomial q with $q(a) \neq 0$.

418 Corollary If a polynomial of degree n had any roots at all, then it has at most n roots.

> **Proof:** *If it had at least $n+1$ roots then it would have at least $n+1$ factors of degree 1 and hence degree $n+1$ at least, a contradiction.* ❑

Notice that the above theorem only says that if a polynomial has any roots, then it must have at most its degree number of roots. It does not say that a polynomial must possess a root. That all polynomials have at least one root is much more difficult to prove. We will quote the theorem, without a proof.

419 Theorem (Fundamental Theorem of Algebra) A polynomial of degree at least one with complex number coefficients has at least one complex root.

> STOP *The Fundamental Theorem of Algebra implies then that a polynomial of degree n has* exactly *n roots (counting multiplicity).*

A more useful form of Ruffini's Theorem is given in the following corollary.

420 Corollary If the polynomial p with integer coefficients,

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

has a rational root $\frac{s}{t} \in \mathbb{Q}$ (here $\frac{s}{t}$ is assumed to be in lowest terms), then s divides a_0 and t divides a_n.

> **Proof:** *We are given that*
>
> $$0 = p\left(\frac{s}{t}\right) = a_n\left(\frac{s^n}{t^n}\right) + a_{n-1}\left(\frac{s^{n-1}}{t^{n-1}}\right) + \cdots + a_1\left(\frac{s}{t}\right) + a_0.$$
>
> *Clearing denominators,*
>
> $$0 = a_n s^n + a_{n-1} s^{n-1} t + \cdots + a_1 s t^{n-1} + a_0 t^n.$$
>
> *This last equality implies that*
>
> $$-a_0 t^n = s(a_n s^{n-1} + a_{n-1} s^{n-2} t + \cdots + a_1 t^{n-1}).$$
>
> *Since both sides are integers, and since s and t have no factors in common, then s must divide a_0. We also gather that*
>
> $$-a_n s^n = t(a_{n-1} s^{n-1} + \cdots + a_1 s t^{n-2} + a_0 t^{n-1}),$$
>
> *from where we deduce that t divides a_n, concluding the proof.* ❑

421 Example Factorise $a(x) = x^3 - 3x - 5x^2 + 15$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: By Corollary 420, if $a(x)$ has integer roots then they must be in the set $\{-1,1,-3,3,-5,5\}$. We test $a(\pm 1), a(\pm 3), a(\pm 5)$ to see which ones vanish. We find that $a(5) = 0$. By the Factor Theorem, $x-5$ divides $a(x)$. Using long division,

$$\begin{array}{r|l} & x^2 \qquad -3 \\ \hline x-5 & x^3-5x^2-3x+15 \\ & -x^3+5x^2 \\ \hline & \qquad -3x+15 \\ & \qquad 3x-15 \\ \hline & \qquad 0 \end{array}$$

we find

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x^2-3),$$

which is the required factorisation over $\mathbb{Z}[x]$. The factorisation over $\mathbb{R}[x]$ is then

$$a(x) = x^3 - 3x - 5x^2 + 15 = (x-5)(x-\sqrt{3})(x+\sqrt{3}).$$

422 Example Factorise $b(x) = x^5 - x^4 - 4x + 4$ over $\mathbb{Z}[x]$ and over $\mathbb{R}[x]$.

Solution: By Corollary 420, if $b(x)$ has integer roots then they must be in the set $\{-1,1,-2,2,-4,4\}$. We quickly see that $b(1) = 0$, and so, by the Factor Theorem, $x-1$ divides $b(x)$. By long division

$$\begin{array}{r|l} & x^4 \qquad -4 \\ \hline x-1 & x^5-x^4-4x+4 \\ & -x^5+x^4 \\ \hline & \qquad -4x+4 \\ & \qquad 4x-4 \\ \hline & \qquad 0 \end{array}$$

we see that

$$b(x) = (x-1)(x^4-4) = (x-1)(x^2-2)(x^2+2),$$

which is the desired factorisation over $\mathbb{Z}[x]$. The factorisation over $\mathbb{R}$ is seen to be

$$b(x) = (x-1)(x-\sqrt{2})(x+\sqrt{2})(x^2+2).$$

Since the discriminant of x^2+2 is $-8 < 0$, x^2+2 does not split over $\mathbb{R}$.

423 Lemma Complex roots of a polynomial with real coefficients occur in conjugate pairs, that is, if p is a polynomial with real coefficients and if $u+vi$ is a root of p, then its conjugate $u-vi$ is also a root for p. Here $i = \sqrt{-1}$ is the imaginary unit.

Proof: *Assume*

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

and that $p(u+vi) = 0$. Since the conjugate of a real number is itself, and conjugation is multiplicative (Theorem 795), we have

$$\begin{aligned} 0 &= \overline{0} \\ &= \overline{p(u+vi)} \\ &= \overline{a_0 + a_1(u+vi) + \cdots + a_n(u+vi)^n} \\ &= \overline{a_0} + \overline{a_1(u+vi)} + \cdots + \overline{a_n(u+vi)^n} \\ &= a_0 + a_1(u-vi) + \cdots + a_n(u-vi)^n \\ &= p(u-vi), \end{aligned}$$

whence $u-vi$ is also a root. ❑

Since the complex pair root $u \pm vi$ would give the polynomial with real coefficients

$$(x-u-vi)(x-u+vi) = x^2 - 2ux + (u^2+v^2),$$

we deduce the following theorem.

424 Theorem Any polynomial with real coefficients can be factored in the form

$$A(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k}(x^2+a_1x+b_1)^{n_1}(x^2+a_2x+b_2)^{n_2}\cdots(x^2+a_lx+b_l)^{n_l},$$

where each factor is distinct, the m_i, l_k are positive integers and A, r_i, a_i, b_i are real numbers.

Homework

425 Problem Find the cubic polynomial p having zeroes at $x = -1, 2, 3$ and satisfying $p(1) = -24$.

426 Problem How many cubic polynomials with leading coefficient -2 are there splitting in the set $\{1,2,3\}$?

427 Problem Find the cubic polynomial c having a root of $x = 1$, a root of multiplicity 2 at $x = -3$ and satisfying $c(2) = 10$.

428 Problem A cubic polynomial p with leading coefficient 1 satisfies $p(1) = 1, p(2) = 4, p(3) = 9$. Find the value of $p(4)$.

429 Problem The polynomial $p(x)$ has integral coefficients and $p(x) = 7$ for four different values of x. Shew that $p(x)$ never equals 14.

430 Problem Find the value of a so that the polynomial

$$t(x) = x^3 - 3ax^2 + 12$$

be divisible by $x+4$..

431 Problem Let $f(x) = x^4+x^3+x^2+x+1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

432 Problem If $p(x)$ is a cubic polynomial with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$, find $p(6)$.

433 Problem The polynomial $p(x)$ satisfies $p(-x) = -p(x)$. When $p(x)$ is divided by $x-3$ the remainder is 6. Find the remainder when $p(x)$ is divided by x^2-9.

434 Problem Factorise $x^3+3x^2-4x+12$ over $\mathbb{Z}[x]$.

435 Problem Factorise $3x^4+13x^3-37x^2-117x+90$ over $\mathbb{Z}[x]$.

436 Problem Find a, b such that the polynomial x^3+6x^2+ax+b be divisible by the polynomial x^2+x-12.

5.5 Graphs of Polynomials

We start with the following theorem, which we will state without proof.

437 Theorem A polynomial function $x \mapsto p(x)$ is an everywhere continuous function.

438 Theorem Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \quad a_n \neq 0$, be a polynomial with real number coefficients. Then

$$p(-\infty) = (\text{signum}\,(a_n))(-1)^n\infty, \qquad p(+\infty) = (\text{signum}\,(a_n))\infty.$$

Thus a polynomial of odd degree will have opposite signs for values of large magnitude and different sign, and a polynomial of even degree will have the same sign for values of large magnitude and different sign.

Proof: *If $x \neq 0$ then*

$$p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = a_nx^n\left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \sim a_nx^n,$$

since as $x \to \pm\infty$, the quantity in parenthesis tends to 1 and so the eventual sign of $p(x)$ is determined by a_nx^n, which gives the result. ❑

439 Corollary A polynomial of odd degree with real number coefficients always has a real root.

Proof: *Since a polynomial of odd degree eventually changes sign, since it is continuous, the corollary follows from Bolzano's Intermediate Value Theorem 266.* ❑

We now consider polynomials with real number coefficients and that split in $\mathbb{R}$. Such polynomials have the form

$$p(x) = a(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k},$$

where $a \neq 0$ and the r_i are real numbers and the $m_i \geq 1$ are integers. Graphing such polynomials will be achieved by referring to the following theorem.

440 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the graph of the polynomial

$$p(x) = a(x-r_1)^{m_1}(x-r_2)^{m_2}\cdots(x-r_k)^{m_k},$$

- crosses the x-axis at $x = r_i$ if m_i is odd.
- is tangent to the x-axis at $x = r_i$ if m_i is even.
- has a convexity change at $x = r_i$ if $m_i \geq 3$ and m_i is odd.

Proof: *Since the local behaviour of $p(x)$ is that of $c(x-r_i)^{m_i}$ (where c is a real number constant) near r_i, the theorem follows at once from our work in section 5.1.* ❑

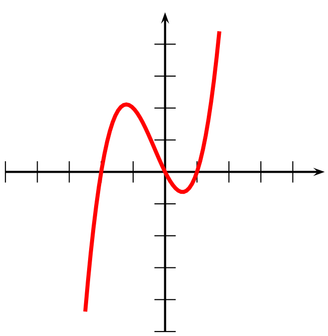


Figure 5.20: $y = (x+2)x(x-1)$.

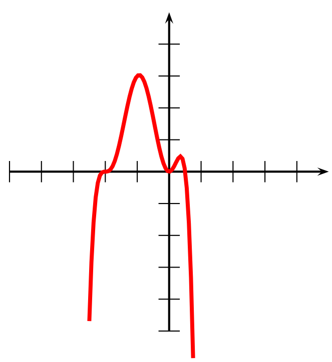


Figure 5.21: $y = (x+2)^3x^2(1-2x)$.

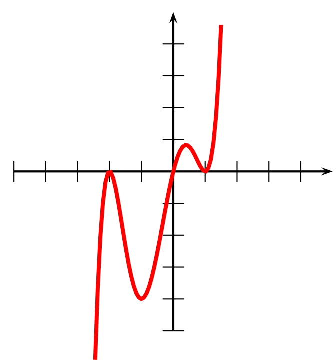


Figure 5.22: $y = (x+2)^2x(1-x)^2$.

441 Example Make a rough sketch of the graph of $y = (x+2)x(x-1)$. Determine where it achieves its local extrema and their values. Determine where it changes convexity.

Solution: We have $p(x) = (x+2)x(x-1) \sim (x)\cdot x(x) = x^3$, as $x \to +\infty$. Hence $p(-\infty) = (-\infty)^3 = -\infty$ and $p(+\infty) = (+\infty)^3 = +\infty$. This means that for large negative values of x the graph will be on the negative side of the y-axis and that for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 440, the graph crosses the x-axis at $x = -2$, $x = 0$, and $x = 1$. The graph is shewn in figure 5.20.

442 Example Make a rough sketch of the graph of $y = (x+2)^3x^2(1-2x)$.

Solution: We have $(x+2)^3x^2(1-2x) \sim x^3 \cdot x^2(-2x) = -2x^6$. Hence if $p(x) = (x+2)^3x^2(1-2x)$ then $p(-\infty) = -2(-\infty)^6 = -\infty$ and $p(+\infty) = -2(+\infty)^6 = -\infty$, which means that for both large positive and negative values of x the graph will be on the negative side of the y-axis. By Theorem 440, in a neighbourhood of $x = -2$, $p(x) \sim 20(x+2)^3$, so the graph crosses the x-axis changing convexity at $x = -2$. In a neighbourhood of 0, $p(x) \sim 8x^2$ and the graph is tangent to the x-axis at $x = 0$. In a neighbourhood of $x = \dfrac{1}{2}$, $p(x) \sim \dfrac{25}{16}(1-2x)$, and so the graph crosses the x-axis at $x = \frac{1}{2}$. The graph is shewn in figure 5.21.

443 Example Make a rough sketch of the graph of $y = (x+2)^2x(1-x)^2$.

Solution: The dominant term of $(x+2)^2x(1-x)^2$ is $x^2\cdot x(-x)^2 = x^5$. Hence if $p(x) = (x+2)^2x(1-x)^2$ then $p(-\infty) = (-\infty)^5 = -\infty$ and $p(+\infty) = (+\infty)^5 = +\infty$, which means that for large negative values of x the graph will be on the negative side of the y-axis and for large positive values of x the graph will be on the positive side of the y-axis. By Theorem 440, the graph crosses the x-axis changing convexity at $x = -2$, it is tangent to the x-axis at $x = 0$ and it crosses the x-axis at $x = \frac{1}{2}$. The graph is shewn in figure 5.22.

Homework

444 Problem Make a rough sketch of the following curves: (i) $y = (x-1)^3$, (ii) $y = (1-x)^3$, (iii) $y = (x-1)(x-2)^2$, (iv) $y = (1-x)(x-2)^2$, (v) $y = (1-x)^2(x-2)$, (vi) $y = (x-1)^2(2-x)$, (vii) $y = (x-1)(x-2)(x-3)$, (viii) $y = (x-1)(x-2)(3-x)$

445 Problem Sketch the graphs of the following curves: (i) $y = (x-1)(x-2)^2(x-3)^3$, (ii) $y = x^2(x-1)^2(x+1)^4$, (iii) $y = x(x-1)^3(x+5)^5$ (iv) $y = -x^2(x-1)(x+2)(x-3)^3$.

446 Problem The polynomial in figure 5.23 has degree 4.

❶ Determine $p(0)$.

❷ Find the equation of $p(x)$.

❸ Find $p(-3)$.

❹ Find $p(2)$.

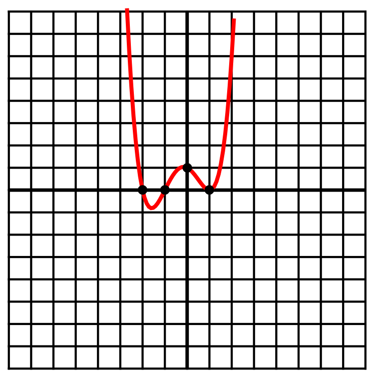


Figure 5.23: Problem 446.

Answers

387 (i) $y = (x+3)^2$ vertex at $(-3,0)$, (ii) $y = (x+6)^2-1$ vertex at $(-6,-1)$, (iii) $y = (x+1)^2-16$, vertex at $(-1,-16)$ (iv) $y = -(x-\frac{1}{2})^2+\frac{1}{4}$, vertex at $(\frac{1}{2},\frac{1}{4})$ (v) $y = 2(x-3)^2+5$, vertex at $(3,5)$, (vi) $3(x-\frac{1}{3})^2+\frac{5}{9}$, vertex at $(\frac{1}{3},\frac{5}{9})$ (vii) $y = \frac{1}{5}(x+5)^2+8$, vertex at $(-5,8)$

388 $(3,-9)$

389 $y = 2x^2-1$

390 $y = -2(x+3)(x-4)$

391 Observe that $x(1-x) = \dfrac{1}{4}-(x-\dfrac{1}{2})^2 \le \dfrac{1}{4}$ and that for $x\in[0,1], 0\le x(1-x)$. Thus if all these products are $> \frac{1}{4}$ we obtain $\dfrac{1}{4^3} < a(1-b)b(1-c)c(1-a) = a(1-a)b(1-b)c(1-c) \le \dfrac{1}{4^3}$, a contradiction. Thus one of the products must be $\le \frac{1}{4}$.

392 $P(x) = 21025-25(x-1)^2$; \$21025

393 We have

$$\begin{aligned} |x^2-2x| = |x^2+1| &\iff (x^2-2x = x^2+1) \text{ or } (x^2+2x = -x^2-1)\\ &\iff (-2x-1=0) \text{ or } (2x^2+2x+1=0)\\ &\iff \left(x=-\frac{1}{2}\right) \text{ or } \left(x = -\frac{1}{2}\pm\frac{i}{2}\right), \end{aligned}$$

whence the solution set is $\left\{-\frac{1}{2}\right\}$.

394 We have

$$\begin{aligned} (x^2+2x-3)^2 = 2 &\iff (x^2+2x-3=\sqrt{2}) \text{ or } (x^2+2x-3=-\sqrt{2})\\ &\iff (x^2+2x-3-\sqrt{2}=0) \text{ or } (x^2+2x-3+\sqrt{2}=0)\\ &\iff \left(x = \frac{-2\pm\sqrt{4-4(-3-\sqrt{2})}}{2}\right)\\ &\qquad \text{or } \left(x = \frac{-2\pm\sqrt{4-4(-3+\sqrt{2})}}{2}\right)\\ &\iff \left(x = \frac{-2\pm\sqrt{16+4\sqrt{2}}}{2}\right)\\ &\qquad \text{or } \left(x = \frac{-2\pm\sqrt{16-4\sqrt{2}}}{2}\right)\\ &\iff (x = -1\pm\sqrt{4+\sqrt{2}}) \text{ or } (x=-1\pm\sqrt{4-\sqrt{2}}). \end{aligned}$$

Since each of $4\pm\sqrt{2} > 0$, all four solutions found are real. The set of solutions is $\left\{-1\pm\sqrt{4\pm\sqrt{2}}\right\}$.

395

$$\begin{aligned} x^3-x^2-9x+9=0 &\iff x^2(x-1)-9(x-1)=0\\ &\iff (x-1)(x^2-9)=0\\ &\iff (x-1)(x-3)(x+3)=0\\ &\iff x\in\{-3,1,3\}. \end{aligned}$$

396

$$\begin{aligned} x^3-2x^2-11x+12=0 &\iff x^3-x^2-x^2+x-12x+12=0 \\ &\iff x^2(x-1)-x(x-1)-12(x-1)=0 \\ &\iff (x-1)(x^2-x-12)=0 \\ &\iff (x-1)(x+3)(x-4)=0 \\ &\iff x\in\{-3,1,4\}. \end{aligned}$$

397 $x^3-1=(x-1)(x^2+x+1)$. If $x\neq 1$, the two solutions to $x^2+x+1=0$ can be obtained using the quadratic formula, getting $x=1/2\pm i\sqrt{3}/2$. There is only one real solution, namely $x=1$.

398 Observe that

$$x^{-4}-10x^{-2}+9=(x^{-2}-9)(x^{-2}-1).$$

Thus $\frac{1}{x^2}=9$ and $\frac{1}{x^2}=1$, whence $x=\pm\frac{1}{3}$ and $x=\pm 1$.

399 Rearranging,

$$(t^2-81)x=3(t-9) \implies (t-9)(t+9)x=3(t-9). \qquad (5.3)$$

If $t=9$, (5.3) becomes $0=0$, which will be true for all values of x. If $t=-9$, (5.3) becomes $0=-54$, which is clearly nonsense. If $t\in\mathbb{R}\setminus\{-9,9\}$, then

$$x=\frac{3}{t+9}$$

is the unique solution to the equation.

400 Let x and $50-x$ be the numbers. We seek to maximise the product $P(x)=x(50-x)$. But $P(x)=50x-x^2=-(x^2-50x)=-(x^2-50x+625)+625=625-(x-25)^2$. We deduce that $P(x)\leq 625$, as the square of any real number is always positive. The maximum product is thus 625 occurring when $x=25$.

401 If b,h are the base and height, respectively, of the rectangle, then we have $20=2b+2h$ or $10=b+h$. The area of the rectangle is then $A(h)=bh=h(10-h)=10h-h^2=25-(h-5)^2$. This shows that $A(h)\leq 25$, and equality occurs when $h=5$. In this case $b=10-h=5$. The height is the same as the base, and so the rectangle yielding maximum area is a square.

402

❶ The current production is $25\times 600=15000$ fruits.

❷ If x more trees are planted, the production of each tree will be $600-15x$.

❸ Let $P(x)$ be the total production after planting x more trees. Then $P(x)=(25+x)(600-15x)=-15x^2+225x+15000$. A good function modelling this problem is

$$P: \begin{array}{ccc} \{x\in\mathbb{N}\,|\,x\geq 25\} & \to & \mathbb{N} \\ x & \mapsto & -15x^2+225x+15000 \end{array}.$$

This model assumes that the amount of trees is never fewer than 25.

❹ We maximise $P(x)=-15x^2+225x+15000=15000-15(x^2-15x)=15843.75-15(x-7.5)^2$. The production is maximised if either 7 or 8 more trees are added, in which case the production will be $15843.75-15(7-7.5)^2=15840$ fruits.

425 Such polynomial must have the form $p(x)=a(x+1)(x-2)(x-3)$, and so we must determine a. But $-24=p(1)=a(2)(-1)(-2)=4a$. Hence $a=-6$. We thus find $p(x)=-6(x+1)(x-2)(x-3)$.

426 There are ten such polynomials. They are $p_1(x)=-2(x-1)^3$, $p_2(x)=-2(x-2)^3$, $p_3(x)=-2(x-3)^3$, $p_4(x)=-2(x-1)(x-2)^2$, $p_5(x)=-2(x-1)^2(x-2)$, $p_6(x)=-2(x-1)(x-3)^2$, $p_7(x)=-2(x-1)^2(x-3)$, $p_8(x)=-2(x-2)(x-3)^2$, $p_9(x)=-2(x-2)^2(x-3)$, $p_{10}(x)=-2(x-1)(x-2)(x-3)$.

427 This polynomial must have the form $c(x)=a(x-1)(x+3)^2$. Now $10=c(2)=a(2-1)(2+3)^2=25a$, whence $a=\frac{2}{5}$. The required polynomial is thus $c(x)=\frac{2}{5}(x-1)(x+3)^2$.

428 Put $g(x)=p(x)-x^2$. Observe that g is also a cubic polynomial with leading coefficient 1 and that $g(x)=0$ for $x=1,2,3$. This means that $g(x)=(x-1)(x-2)(x-3)$ and hence $p(x)=(x-1)(x-2)(x-3)+x^2$. This yields $p(4)=(3)(2)(1)+4^2=22$.

429 The polynomial $g(x)=p(x)-7$ vanishes at the 4 different integer values a,b,c,d. In virtue of the Factor Theorem,

$$g(x)=(x-a)(x-b)(x-c)(x-d)q(x),$$

where $q(x)$ is a polynomial with integral coefficients. Suppose that $p(t)=14$ for some integer t. Then $g(t)=p(t)-7=14-7=7$. It follows that

$$7=g(t)=(t-a)(t-b)(t-c)(t-d)q(t),$$

that is, we have factorised 7 as the product of at least 4 different factors, which is impossible since 7 can be factorised as $7(-1)1$, the product of at most 3 distinct integral factors. From this contradiction we deduce that such an integer t does not exist.

430 By the Factor Theorem, we must have

$$\begin{aligned} 0=t(-4)=(-4)^3-3a(-4)^2+40 \\ \iff 0=-24-48a \\ \iff a=-\frac{1}{2}. \end{aligned}$$

431 Observe that $f(x)(x-1)=x^5-1$ and

$$f(x^5)=x^{20}+x^{15}+x^{10}+x^5+1=(x^{20}-1)+(x^{15}-1)+(x^{10}-1)+(x^5-1)+5.$$

Each of the summands in parentheses is divisible by x^5-1 and, a fortiori, by $f(x)$. The remainder sought is thus 5.

432 Put $g(x)=p(x)-x$, then $p(6)=16$.

434 $(x-2)(x+2)(x-3)$

435 $(x-3)(x+3)(x+5)(3x-2)$

436 $a=-7, b=-60$

Chapter 6

Rational Functions and Algebraic Functions

6.1 Inverse Power Functions

We now proceed to investigate the behaviour of functions of the type $x \mapsto \frac{1}{x^n}$, where $n > 0$ is an integer.

447 Theorem Let $n > 0$ be an integer. Then

- if n is even, $x \mapsto \frac{1}{x^n}$ is increasing for $x < 0$, decreasing for $x > 0$ and convex for all $x \neq 0$.
- if n is odd, $x \mapsto \frac{1}{x^n}$ is decreasing for all $x \neq 0$, concave for $x < 0$, and convex for $x > 0$.

Thus $x \mapsto \frac{1}{x^n}$ has a pole of order n at $x = 0$ and a horizontal asymptote at $y = 0$.

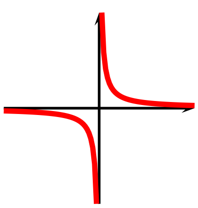


Figure 6.1: $x \mapsto \frac{1}{x}$

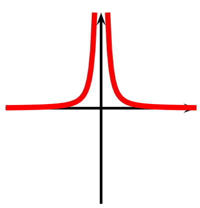


Figure 6.2: $x \mapsto \frac{1}{x^2}$

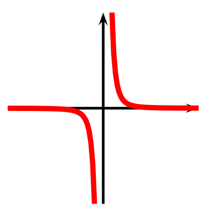


Figure 6.3: $x \mapsto \frac{1}{x^3}$

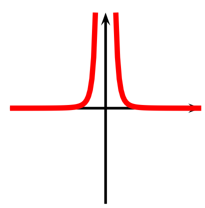


Figure 6.4: $x \mapsto \frac{1}{x^4}$

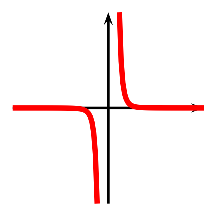


Figure 6.5: $x \mapsto \frac{1}{x^5}$

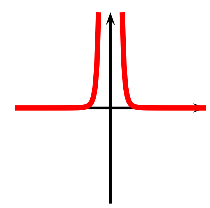


Figure 6.6: $x \mapsto \frac{1}{x^6}$

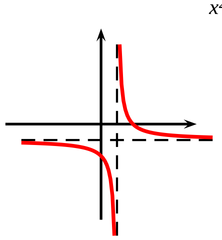


Figure 6.7: $x \mapsto \frac{1}{x-1} - 1$

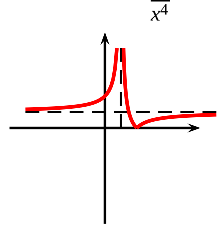


Figure 6.8: $x \mapsto \left|\frac{1}{x-1} - 1\right|$

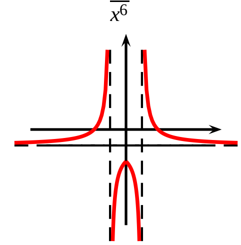


Figure 6.9: $x \mapsto \frac{1}{|x|-1} - 1$

448 Example A few functions $x \mapsto \frac{1}{x^n}$ are shewn in figures 6.1 through 6.6.

449 Example Figures 6.7 through 6.9 shew a few transformations of $x \mapsto \frac{1}{x}$.

6.2 Rational Functions

450 Definition By a *rational function* $x \mapsto r(x)$ we mean a function r whose assignment rule is of the $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x) \neq 0$ are polynomials.

We now provide a few examples of graphing rational functions. Analogous to theorem 440, we now consider rational functions $x \mapsto r(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials with no factors in common and splitting in $\mathbb{R}$.

451 Theorem Let $a \neq 0$ and the r_i are real numbers and the m_i be positive integers. Then the rational function with assignment rule

$$r(x) = K\frac{(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_k)^{m_k}}{(x-b_1)^{n_1}(x-b_2)^{n_2}\cdots(x-b_l)^{n_l}},$$

- has zeroes at $x = a_i$ and poles at $x = b_j$.
- crosses the x-axis at $x = a_i$ if m_i is odd.
- is tangent to the x-axis at $x = a_i$ if m_i is even.
- has a convexity change at $x = a_i$ if $m_i \geq 3$ and m_i is odd.
- both $r(b_j-)$ and $r(b_j+)$ blow to infinity. If n_i is even, then they have the same sign infinity: $r(b_i+) = r(b_i-) = +\infty$ or $r(b_i+) = r(b_i-) = -\infty$. If n_i is odd, then they have different sign infinity: $r(b_i+) = -r(b_i-) = +\infty$ or $r(b_i+) = -r(b_i-) = -\infty$.

Proof: *Since the local behaviour of $r(x)$ is that of $c(x-r_i)^{t_i}$ (where c is a real number constant) near r_i, the theorem follows at once from Theorem 374 and 447.* ❑

452 Example Draw a rough sketch of $x \mapsto \frac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$.

Solution: Put $r(x) = \frac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$. By Theorem 451, r has zeroes at $x = 1$, and $x = -2$, and poles at $x = -1$ and $x = 2$. As $x \to 1$, $r(x) \sim \frac{3}{2}(x-1)^2$, hence the graph of r is tangent to the axes, and positive, around $x = 2$. As $x \to -2$, $r(x) \sim -\frac{9}{16}(x+2)$, hence the graph of r crosses the x-axis at $x = -2$, coming from positive y-values on the left of $x = -2$ and going to negative y=values on the right of $x = -2$. As $x \to -1$, $r(x) \sim \frac{4}{9(x+1)}$, hence the graph of r blows to $-\infty$ to the left of $x = -1$ and to $+\infty$ to the right of $x = -1$. As $x \to 2$, $r(x) \sim \frac{4}{3(x-2)^2}$, hence the graph of r blows to $+\infty$ both from the left and the right of $x = 2$. Also we observe that

$$r(x) \sim \frac{(x)^2(x)}{(x)(x)^2} = \frac{x^3}{x^3} = 1,$$

and hence r has the horizontal asymptote $y = 1$. The graph of r can be found in figure 6.10.

453 Example Draw a rough sketch of $x \mapsto \frac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$.

Solution: Put $r(x) = \frac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$. First observe that $r(x) = r(-x)$, and so r is even. By Theorem 451, r has zeroes at $x = \pm\frac{3}{4}$, and poles at $x = \pm 1$. As $x \to \frac{3}{4}$, $r(x) \sim -\frac{36}{7}(x-3/4)^2$, hence the graph of r is tangent to the axes, and negative, around $x = 3/4$, and similar behaviour occurs around $x = -\frac{3}{4}$. As $x \to 1$, $r(x) \sim \frac{49}{512(x-1)}$, hence the graph of r blows to $-\infty$ to the left of $x = 1$ and to $+\infty$ to the right of $x = 1$. As $x \to -1$, $r(x) \sim -\frac{49}{512(x-1)}$, hence the graph of r blows to $+\infty$ to the left of $x = -1$ and to $-\infty$ to the right of $x = -1$. Also, as $x \to +\infty$,

$$r(x) \sim \frac{(x)^2(x)^2}{(x)(x)} = x^2,$$

so $r(+\infty) = +\infty$ and $r(-\infty) = +\infty$. The graph of r can be found in figure 6.11.

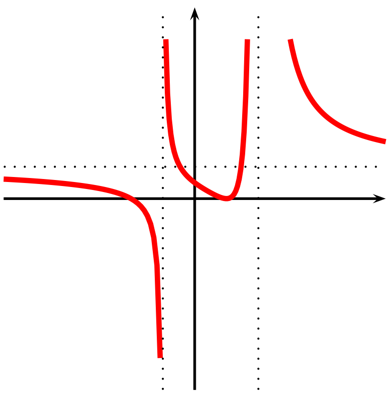


Figure 6.10: $x \mapsto \frac{(x-1)^2(x+2)}{(x+1)(x-2)^2}$

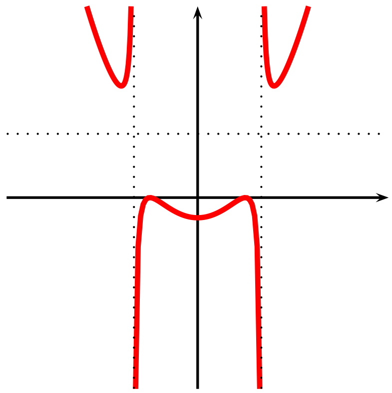


Figure 6.11: $x \mapsto \frac{(x-3/4)^2(x+3/4)^2}{(x+1)(x-1)}$

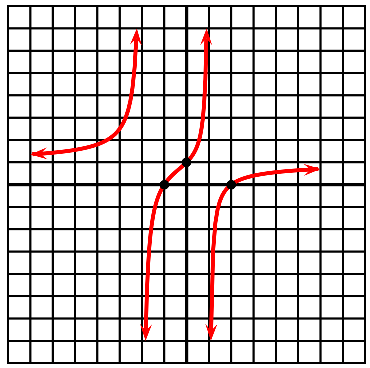


Figure 6.12: Problem 454.

Homework

454 Problem The rational function q in figure 6.12 has only two simple poles and satisfies $q(x) \to 1$ as $x \to \pm\infty$. You may assume that the poles and zeroes of q are located at integer points.

1. Find $q(0)$.
2. Find $q(x)$ for arbitrary x.
3. Find $q(-3)$.
4. Find $\lim_{x \to -2+} q(x)$.

455 Problem Find the condition on the distinct real numbers a, b, c such that the function $x \mapsto \frac{(x-a)(x-b)}{x-c}$ takes all real values for real values of x. Sketch two scenarios to illustrate a case when the condition is satisfied and a case when the condition is not satisfied.

6.3 Algebraic Functions

456 Definition We will call *algebraic function* a function whose assignment rule can be obtained from a rational function by a finite combination of additions, subtractions, multiplications, divisions, exponentiations to a rational power.

457 Theorem Let $|q| \geq 2$ be an integer. If

- if q is even then $x \mapsto x^{1/q}$ is increasing and concave for $q \geq 2$ and decreasing and convex for $q \leq -2$ for all $x > 0$ and it is undefined for $x < 0$.
- if q is odd then $x \mapsto x^{1/q}$ is everywhere increasing and convex for $x < 0$ but concave for $x > 0$ if $q \geq 3$. If $q \leq -3$ then $x \mapsto x^{1/q}$ is decreasing and concave for $x < 0$ and increasing and convex for $x > 0$.

A few of the functions $x \mapsto x^{1/q}$ are shewn in figures 6.13 through 6.24.

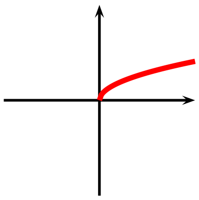


Figure 6.13: $x \mapsto x^{1/2}$

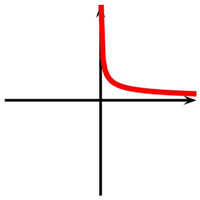


Figure 6.14: $x \mapsto x^{-1/2}$

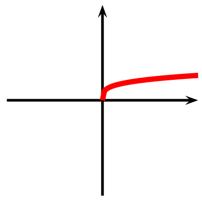


Figure 6.15: $x \mapsto x^{1/4}$

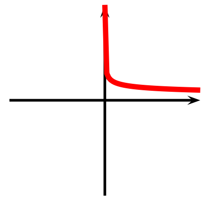


Figure 6.16: $x \mapsto x^{-1/4}$

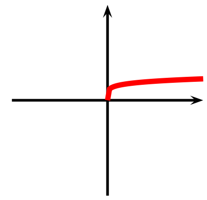


Figure 6.17: $x \mapsto x^{1/6}$

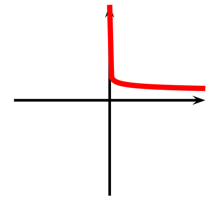


Figure 6.18: $x \mapsto x^{-1/6}$

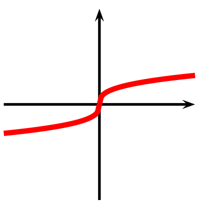


Figure 6.19: $x \mapsto x^{1/3}$

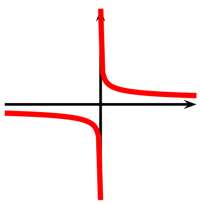


Figure 6.20: $x \mapsto x^{-1/3}$

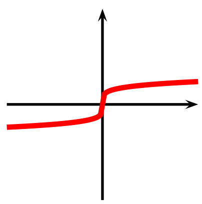


Figure 6.21: $x \mapsto x^{1/5}$

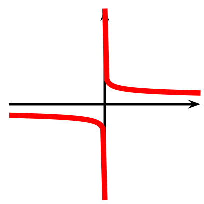


Figure 6.22: $x \mapsto x^{-1/5}$

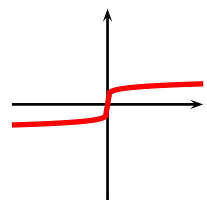


Figure 6.23: $x \mapsto x^{1/7}$

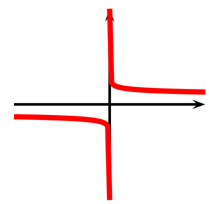


Figure 6.24: $x \mapsto x^{-1/7}$

Homework

458 Problem Draw the graph of each of the following curves.

1. $x \mapsto (1+x)^{1/2}$
2. $x \mapsto (1-x)^{1/2}$
3. $x \mapsto 1+(1+x)^{1/3}$
4. $x \mapsto 1-(1-x)^{1/3}$
5. $x \mapsto \sqrt{x}+\sqrt{-x}$

Chapter 7

Exponential Functions

7.1 Exponential Functions

459 Definition Let $a > 0, a \neq 1$ be a fixed real number. The function

$$\begin{array}{ccc} \mathbb{R} & \to &]0;+\infty[\\ x & \mapsto & a^x \end{array},$$

is called the *exponential function* of *base a*.

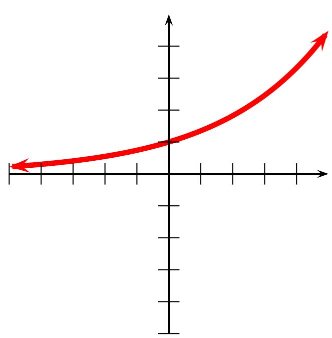


Figure 7.1: $x \mapsto a^x,\ a > 1$.

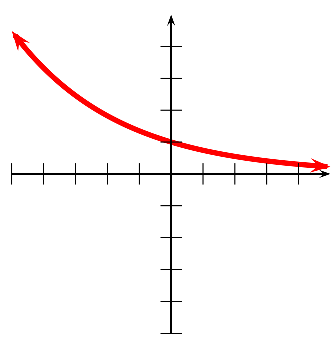


Figure 7.2: $x \mapsto a^x,\ 0 < a < 1$.

We will now prove that the generic graphs of the exponential function resemble those in figures 7.1 and 7.2.

460 Theorem If $a > 1,\ x \mapsto a^x$ is strictly increasing and convex, and if $0 < a < 1$ then $x \mapsto a^x$ is strictly decreasing and convex.

Proof: *Put $f(x) = a^x$. Recall that a function f is strictly increasing or decreasing depending on whether the ratio*

$$\frac{f(t)-f(s)}{t-s} > 0 \quad \text{or} \quad < 0$$

for $t \neq s$. Now,

$$\frac{f(t)-f(s)}{t-s} = \frac{a^t - a^s}{t-s} = (a^s) \cdot \frac{a^{t-s}-1}{t-s}.$$

If $a>1$, and $t-s>0$ then also $a^{t-s}>1$.[1] *If $a>1$, and $t-s<0$ then also $a^{t-s}<1$. Thus regardless on whether $t-s>0$ or <0 the ratio*

$$\frac{a^{t-s}-1}{t-s}>0,$$

whence f is increasing for $a>1$. A similar argument proves that for $0<a<1$, f would be decreasing.

To prove convexity will be somewhat more arduous. Recall that f is convex if for arbitrary $0\leq\lambda\leq 1$ we have

$$f(\lambda s+(1-\lambda)t)\leq \lambda f(s)+(1-\lambda)f(t),$$

that is, a straight line joining any two points of the curve lies above the curve. We will not be able to prove this quickly, we will just content with proving midpoint convexity: we will prove that

$$f\left(\frac{s+t}{2}\right)\leq \frac{1}{2}f(s)+\frac{1}{2}f(t).$$

This is equivalent to

$$a^{\frac{s+t}{2}}\leq \frac{1}{2}a^s+\frac{1}{2}a^t,$$

which in turn is equivalent to

$$2\leq a^{\frac{s-t}{2}}+a^{\frac{t-s}{2}}.$$

But the square of a real number is always non-negative, hence

$$\left(a^{\frac{s-t}{4}}-a^{\frac{t-s}{4}}\right)^2\geq 0 \implies a^{\frac{s-t}{2}}+a^{\frac{t-s}{2}}\geq 2,$$

proving midpoint convexity. ❏

STOP *The line $y=0$ is an asymptote for $x\mapsto a^x$. If $a>1$, then $a^x\to 0$ as $x\to-\infty$ and $a^x\to+\infty$ as $x\to+\infty$. If $0<a<1$, then $a^x\to+\infty$ as $x\to-\infty$ and $a^x\to 0$ as $x\to+\infty$.*

Homework

461 Problem Make rough sketches of the following curves.

❶ $x\mapsto 2^x$

❷ $x\mapsto 2^{|x|}$

❸ $x\mapsto 2^{-|x|}$

❹ $x\mapsto 2^x+3$

❺ $x\mapsto 2^{x+3}$

7.2 The number e

Consider now the following problem, first studied by the Swiss mathematician Jakob Bernoulli around the 1700s: Query: If a creditor lends money at interest under the condition that during each individual moment the proportional part of the annual interest be added to the principal, what is the balance at the end of a full year?[2]

Suppose a dollars are deposited, and the interest is added n times a year at a rate of x. After the first time period, the balance is

$$b_1=\left(1+\frac{x}{n}\right)a.$$

After the second time period, the balance is

$$b_2=\left(1+\frac{x}{n}\right)b_1=\left(1+\frac{x}{n}\right)^2a.$$

[1] The alert reader will find this argument circular! I have tried to prove this theorem from first principles without introducing too many tools. Alas, I feel tired. . .

[2] “Quæritur, si creditor aliquis pecuniam suam fœnori exponat, ea lege, ut singulis momentis pars proportionalis usuræ annuæ sorti annumeretur; quantum ipsi finito anno debeatur?”

Proceeding recursively, after the n-th time period, the balance will be

$$b_n = \left(1+\frac{x}{n}\right)^n a.$$

The study of the sequence

$$e_n = \left(1+\frac{1}{n}\right)^n$$

thus becomes important. It was Bernoulli's pupil, Leonhard Euler, who shewed that the sequence $\left(1+\frac{1}{n}\right)^n, n = 1,2,3,\ldots$ converges to a finite number, which he called e. In other words, Euler shewed that

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n. \tag{7.1}$$

It must be said, in passing, that Euler did not rigourously shewed the existence of the above limit. He, however, gave other formulations of the irrational number

$$e = 2.71828182845904523536028747135\ldots,$$

among others, the infinite series

$$e = 2+\frac{1}{2!}++\frac{1}{3!}++\frac{1}{4!}++\frac{1}{5!}+\cdots, \tag{7.2}$$

and the infinite continued fraction

$$e = 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{\ldots}}}}}}}}}. \tag{7.3}$$

We will now establish a series of results in order to prove that the limit in 7.1 exists.

462 Lemma Let n be a positive integer. Then

$$x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}).$$

Proof: *The lemma follows by direct multiplication of the dextral side.* ❑

463 Lemma If $0 \le a < b,\ n \in \mathbb{N}$

$$na^{n-1} < \frac{b^n - a^n}{b-a} < nb^{n-1}.$$

Proof: *By Lemma 462*

$$\begin{aligned}\frac{b^n-a^n}{b-a} &= b^{n-1}+b^{n-2}a+b^{n-3}a^2+\cdots+b^2a^{n-3}+ba^{n-2}+a^{n-1}\\ &< b^{n-1}+b^{n-1}+\cdots+b^{n-1}+b^{n-1}\\ &= nb^{n-1},\end{aligned}$$

from where the dextral inequality follows. The sinistral inequality can be established similarly. ❑

464 Theorem The sequence

$$e_n = \left(1+\frac{1}{n}\right)^n, n = 1, 2, \ldots$$

is a bounded increasing sequence, and hence it converges to a limit, which we call e.

Proof: *By Lemma 463*

$$\frac{b^{n+1}-a^{n+1}}{b-a} \leq (n+1)b^n \implies b^n[(n+1)a-nb] < a^{n+1}.$$

Putting $a = 1+\frac{1}{n+1}$, $b = 1+\frac{1}{n}$ *we obtain*

$$e_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1} = e_{n+1},$$

whence the sequence $e_n, n = 1, 2, \ldots$ *increases. Again, by putting* $a = 1$, $b = 1+\frac{1}{2n}$ *we obtain*

$$\left(1+\frac{1}{2n}\right)^n < 2 \implies \left(1+\frac{1}{2n}\right)^{2n} < 4 \implies e_{2n} < 4.$$

Since $e_n < e_{2n} < 4$ *for all n, the sequence is bounded. In view of Theorem 849 the sequence converges to a limit. We call this limit e.* ❑

Since the sequence increases towards e we have

$$2 = \left(1+\frac{1}{1}\right)^1 < e.$$

From the proof of Theorem 464 it stems that $2 < e < 4$*. In fact, in can be shewn that* $e \approx 2.718281828459045235360287471352\ldots$ *and so* $2 < e < 3$.

e is called the natural exponential base*. The function* $x \mapsto e^x$ *has the property that any tangent drawn to the curve at the point x has slope* e^x*. The notation* $\exp(x) = \exp x = e^x$ *is often used.*

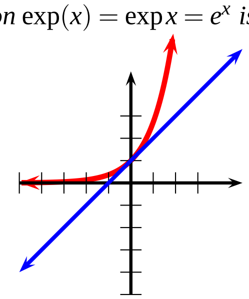


Figure 7.3: $1+x \leq e^x \ \forall x \in \mathbb{R}$.

465 Theorem If $x \in \mathbb{R}$ then

$$1+x \leq e^x,$$

with equality only for $x = 0$.

Proof: *From figure 7.3, the line* $y = 1+x$ *lies below the graph of* $y = e^x$*, proving the theorem.*❑

Replacing x by $x-1$ we obtain,

466 Corollary

$$x \leq e^{x-1}, \ \forall x \in \mathbb{R}.$$

Equality occurs if and only if $x = 1$.

Homework

467 Problem True or False.

1. $\exists t \in \mathbb{R}$ such that $e^t = -9$.
2. As $x \to -\infty$, $2^x \to -\infty$.
3. $\forall x \in \mathbb{R}$, $10 + x^2 + x^4 > 2^x$.
4. $x \mapsto e^x$ is increasing over $\mathbb{R}$.
5. $x \mapsto \frac{e^x}{\pi^x}$ is increasing over $\mathbb{R}$.
6. $ex \leq e^x, \quad \forall x \in \mathbb{R}$.

468 Problem By using Theorem 465, and the fact that $\pi > e$, prove that $e^\pi > \pi^e$.

(Hint: Put $x = \frac{\pi}{e} - 1$.)

469 Problem Make a rough sketch of each of the following.

1. $x \mapsto 2^x$
2. $x \mapsto e^x$
3. $x \mapsto \left(\frac{1}{2}\right)^x$
4. $x \mapsto -1 + 2^x$
5. $x \mapsto e^{|x|}$
6. $x \mapsto e^{-|x|}$

470 Problem Let $n \in \mathbb{N}, n > 1$. Prove that

$$n! < \left(\frac{n+1}{2}\right)^n.$$

471 Problem Put

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Prove that

$$\cosh^2 x - \sinh^2 x = 1.$$

The function $x \mapsto \cosh x$ is known as the *hyperbolic cosine*. The function $x \mapsto \sinh x$ is known as the *hyperbolic sine*.

472 Problem Prove that for $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

and

$$\left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}.$$

(Hint: Use a suitable choice of a and b in Lemma 463.)

7.3 Arithmetic Mean-Geometric Mean Inequality

Using Corollary 466, we may prove, à la Pólya, the Arithmetic-Mean-Geometric-Mean Inequality (AM-GM Inequality, for short).

473 Theorem (Arithmetic-Mean-Geometric-Mean Inequality) Let

$$a_1, a_2, \ldots, a_n$$

be non-negative real numbers. Then

$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Equality occurs if and only if $a_1 = a_2 = \ldots = a_n$.

Proof: *Put*

$$A_k = \frac{na_k}{a_1 + a_2 + \cdots + a_n},$$

and $G_n = a_1 a_2 \cdots a_n$. *Observe that*

$$A_1 A_2 \cdots A_n = \frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n},$$

and that

$$A_1 + A_2 + \cdots + A_n = n.$$

By Corollary 466, *we have*

$$A_1 \leq \exp(A_1 - 1),$$

$$A_2 \le \exp(A_2 - 1),$$
$$\vdots$$
$$A_n \le \exp(A_n - 1).$$

Since all the quantities involved are non-negative, we may multiply all these inequalities together, to obtain,

$$A_1 A_2 \cdots A_n \le \exp(A_1 + A_2 + \cdots + A_n - n).$$

In view of the observations above, the preceding inequality is equivalent to

$$\frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n} \le \exp(n - n) = e^0 = 1.$$

We deduce that

$$G_n \le \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n,$$

which is equivalent to

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now, for equality to occur, we need each of the inequalities $A_k \le \exp(A_k - 1)$ to hold. This occurs, in view of Corollary 466 if and only if $A_k = 1$, $\forall k$, which translates into $a_1 = a_2 = \ldots = a_n$.. This completes the proof. ❑

474 Corollary (Harmonic-Mean-Geometric-Mean Inequality) If

$$a_1, a_2, \ldots, a_n$$

are positive real numbers, then

$$\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \le \sqrt[n]{a_1 a_2 \cdots a_n}.$$

Proof: *By the AM-GM Inequality,*

$$\sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_n}} \le \frac{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}{n},$$

from where the result follows by rearranging. ❑

475 Example The sum of two positive real numbers is 100. Find their maximum product.

Solution: Let x and y be the numbers. We use the AM-GM Inequality for $n = 2$. Then

$$\sqrt{xy} \le \frac{x + y}{2}.$$

In our case, $x + y = 100$, and so

$$\sqrt{xy} \le 50,$$

which means that the maximum product is $xy \le 50^2 = 2500$. If we take $x = y = 50$, we see that the maximum product is achieved for this choice of x and y.

476 Example From a rectangular cardboard piece measuring 75×45 a square of side x is cut from each of its corners in order to make an open box. See figure 7.4. Find the function $x \mapsto V(x)$ that gives the volume of the box as a function of x, and obtain an upper bound for the volume of this box.

Solution: From the diagram shewn, the height of the box is x, its length $75 - 2x$ and its width $45 - 2x$. Hence

$$V(x) = x(75 - 2x)(45 - 2x).$$

Now, if we used the AM-GM Inequality for the three quantities $x,\ 80-2x$, and $50-2x$, we would obtain

$$\begin{array}{rcl} V(x) & = & x(75-2x)(45-2x) \\ & < & \left(\frac{x+75-2x+45-2x}{3}\right)^3 \\ & = & \left(\frac{120-3x}{3}\right)^3 \\ & = & (40-x)^3. \end{array}$$

(We use the strict inequality sign because we know that equality will never be achieved: $75-2x$ never equals $45-2x$.) This has the disadvantage of depending on x. In order to overcome this, we use the following trick. Consider, rather, the three quantities $4x,\ 75-2x$, and $45-2x$. Then

$$\begin{array}{rcl} 4V(x) & = & 4x(75-2x)(45-2x) \\ & < & \left(\frac{4x+75-2x+45-2x}{3}\right)^3 \\ & = & \left(\frac{120}{3}\right)^3 \\ & = & 64000. \end{array}$$

This means that

$$V(x) < \frac{64000}{4} = 16000.$$

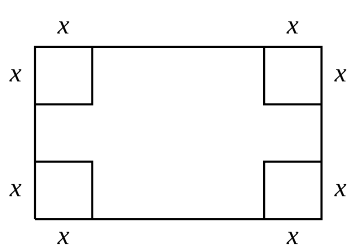


Figure 7.4: Example 476.

Later in calculus, you will see that the volume is

$$V(x) \leq (20 - \frac{5}{2}\sqrt{19})(35+5\sqrt{19})(5+5\sqrt{19}),$$

and that the maximum is achieved when

$$x = 20 - \frac{5}{2}\sqrt{19}.$$

477 Example Find the maximum value of the function $f: \begin{array}{ccc} [0;1] & \rightarrow & \mathbb{R} \\ x & \mapsto & x(1-x)^2 \end{array}$.

Solution: Observe that for $x \in [0;1]$ both x and $1-x$ are non-negative. We maximise, rather, $2f(x)$ via the AM-GM Inequality.

$$2f(x) = 2x(1-x)^2 = 2x(1-x)(1-x) \leq \left(\frac{2x+1-x+1-x}{3}\right)^3 = \frac{8}{27}.$$

Thus

$$f(x) \leq \frac{1}{2} \cdot \frac{8}{27} = \frac{4}{27}.$$

The maximum value is attained when $2x = 1-x$, that is, when $x = \frac{1}{3}$.

Homework

478 Problem Let x, y, z be any real numbers. Prove that

$$3x^2y^2z^2 \le x^6 + y^6 + z^6.$$

479 Problem The sum of 5 positive real numbers is S. What is their maximum product?

480 Problem Use AM-GM to prove that $\cosh x \ge 1, \quad \forall x \in \mathbb{R}$.

481 Problem Maximise the following functions over $[0;1]$.

1. $a : x \mapsto x(1-x)^3$.
2. $b : x \mapsto x^2(1-x)^2$.
3. $c : x \mapsto x^2(1-x)^3$.

482 Problem Prove that of all rectangular boxes with a given surface area, the cube has the largest volume.

Answers

467 F; F; F; T; F; T

479 $\frac{S^5}{3125}$.

481 (1) $a(x) \le \frac{27}{256}$ achieved at $x = \frac{1}{4}$,
(2) $b(x) \le \frac{1}{16}$ achieved at $x = \frac{1}{2}$,
(3) $c(x) \le \frac{108}{3125}$ achieved at $x = \frac{2}{5}$, (Hint: Consider $\frac{9}{4}c(x) = (\frac{3}{2}x)(\frac{3}{2}x)(1-x)^3$.)

Chapter 8

Logarithmic Functions

8.1 Logarithms

Recall that if $a > 0, a \neq 1$ is a fixed real number, $\begin{array}{ccc} \mathbb{R} & \to &]0;+\infty[\\ x & \mapsto & a^x \end{array}$ maps a real number x to a positive number y, i.e., $a^x = y$.

We call x the *logarithm* of y to the base a, and we write $x = \log_a y$. In other words, the function $\begin{array}{ccc} \mathbb{R} & \to &]0;+\infty[\\ x & \mapsto & a^x \end{array}$ has inverse $\begin{array}{ccc}]0;+\infty[& \to & \mathbb{R} \\ x & \mapsto & \log_a x \end{array}$.[1]

483 Example $\log_5 25 = 2$ since $5^2 = 25$.

484 Example $\log_2 1024 = 10$ since $2^{10} = 1024$.

485 Example $\log_3 27 = 3$ since $3^3 = 27$.

486 Example $\log_{190123456} 1 = 0$ as $190123456^0 = 1$.

If $a > 0, a \neq 1$, it should be clear that $\log_a 1 = 0$, $\log_a a = 1$, and in general $\log_a a^t = t$, where t is any real number.

487 Example $\log_{\sqrt{2}} 8 = \log_{2^{1/2}} (2^{1/2})^6 = 6$.

488 Example $\log_{\sqrt{2}} 32 = \log_{2^{1/2}} (2^{1/2})^{10} = 10$.

489 Example $\log_{3\sqrt{3}} 81\sqrt[8]{27} = \log_{3^{3/2}} (3^{3/2})^{(2/3)(35/8)} = \dfrac{2}{3} \cdot \dfrac{35}{8} = \dfrac{35}{12}$.

Aliter: We seek a solution x to

$$(3\sqrt{3})^x = 81\sqrt[8]{27}$$

[1] In higher mathematics, and in many computer algebra programmes like Maple ®, the notation "log" without indicating the base, is used for the natural logarithm of base e. Misguided authors, enemies of the State, communists,Al-Qaeda members, vegetarians and other vile criminals use "log" in calculators and in lower mathematics to denote the logarithm of base 10, and use "ln" to denote the natural logarithm. This makes things somewhat confusing. In these notes we will denote the logarithm base 10 by "$\log_{10}$" and the natural logarithm by "$\log_e$", which is hardly original but avoids confusion.

Expressing the sinistral side as powers of 3, we have

$$\begin{aligned}(3\sqrt{3})^x &= (3\cdot 3^{1/2})^x \\ &= (3^{1+1/2})^x \\ &= (3^{3/2})^x \\ &= 3^{3x/2}\end{aligned}$$

Also, the dextral side equals

$$\begin{aligned}81\sqrt[8]{27} &= 3^4\cdot(3^3)^{1/8} \\ &= 3^{4+3/8} \\ &= 3^{35/8}\end{aligned}$$

Thus $(3\sqrt{3})^x = 81\sqrt[8]{27}$ implies that $3^{3x/2} = 3^{35/8}$ or $\dfrac{3x}{2} = \dfrac{35}{8}$ from where $x = \frac{35}{12}$.

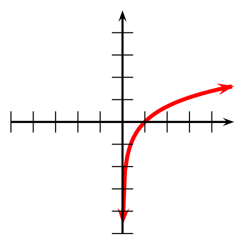


Figure 8.1: $x \mapsto \log_a x, a > 1$

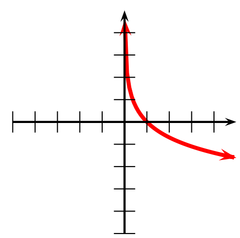


Figure 8.2: $x \mapsto \log_a x, 0 < a < 1$.

Since $x\mapsto a^x$ and $x\mapsto \log_a x$ are inverses, the graph of $x\mapsto \log_a x$ is symmetric with respect to the line $y = x$ to the graph of $x\mapsto a^x$. For $a > 1, x\mapsto a^x$ is increasing and convex, $x\mapsto \log_a\ x,\ a > 1$ will be increasing and concave, as in figure 8.1. Also, for $0 < a < 1, x\mapsto a^x$ is decreasing and convex, $x\mapsto \log_a\ x,\ 0 < a < 1$ will be decreasing and concave, as in figure 8.2.

490 Example Between which two integers does $\log_2\ 1000$ lie?

Solution: Observe that $2^9 = 512 < 1000 < 1024 = 2^{10}$. Since $x\mapsto \log_2 x$ is increasing, we deduce that $\log_2 1000$ lies between 9 and 10.

491 Example Find $\lfloor \log_3\ 201\rfloor$.

Solution: $3^4 = 81 < 201 < 243 = 3^5$. Hence $\lfloor \log_3 201\rfloor = 4$.

492 Example Which is greater $\log_5 7$ or $\log_8 3$?

Solution: Clearly $\log_5 7 > 1 > \log_8 3$.

493 Example Find the integer that equals

$$\lfloor \log_2 1\rfloor + \lfloor \log_2 2\rfloor + \lfloor \log_2 3\rfloor + \lfloor \log_2 4\rfloor + \cdots + \lfloor \log_2 66\rfloor.$$

Solution: Firstly, $\log_2\ 1 = 0$. We may decompose the interval $[2;66]$ into dyadic blocks, as

$$[2;66] = [2;4[\cup[4;8[\cup[8;16[\cup[16,;32[\cup[32,;64[\cup[64;66].$$

On the first interval there are $4-2=2$ integers with $\lfloor\log_2\ x\rfloor = 1, x\in[2;4[$. On the second interval there are $8-4=4$ integers with $\lfloor\log_2\ x\rfloor = 2, x\in[4;8[$. On the third interval there are $16-8=8$ integers with $\lfloor\log_2\ x\rfloor = 3, x\in[8;16[$. On the fourth interval there are $32-16=16$ integers with $\lfloor\log_2\ x\rfloor = 4, x\in[16;32[$. On the fifth interval there are $64-32=32$ integers with $\lfloor\log_2\ x\rfloor = 5, x\in[32;64[$. On the sixth interval there are $66-64+1=3$ integers with $\lfloor\log_2\ x\rfloor = 6, x\in[64;66]$. Thus

$$\begin{aligned}\lfloor\log_2\ 1\rfloor + \lfloor\log_2\ 2\rfloor + \lfloor\log_2\ 3\rfloor + \\ +\lfloor\log_2\ 4\rfloor + \cdots + \lfloor\log_2\ 66\rfloor \\ &= 2(1)+4(2)+8(3)+ \\ &\quad +16(4)+32(5)+3(6) \\ &= 276.\end{aligned}$$

494 Example What is the natural domain of definition of $x\mapsto\log_2(x^2-3x-4)$?

Solution: We need $x^2-3x-4=(x-4)(x+1)>0$. By making a sign diagram, or looking at the graph of the parabola $y=(x-4)(x+1)$ we see that this occurs when $x\in]-\infty;-1[\cup]4;+\infty[$.

495 Example What is the natural domain of definition of $x\mapsto\log_{|x|-4}(2-x)$?

Solution: We need $2-x>0$ and $|x|-4\neq 1$. Thus $x<2$ and $x\neq 5, x\neq -5$. We must have $x\in]-\infty;-5[\cup]-5;2[$.

Homework

496 Problem True or False.

1. $\exists x\in\mathbb{R}$ such that $\log_4 x = 2$.
2. $\exists x\in\mathbb{R}$ such that $\log_4 x = -2$.
3. $\log_2 1 = 0$.
4. $\log_2 0 = 1$.
5. $\log_2 2 = 1$.
6. $x\mapsto\log_{1/5} x$ is increasing over $\mathbb{R}_+^*$.
7. $\forall x>0, (\log_5 x)^2 = \log_5 x^2$.
8. $\log_3\ 201 = 4$.

497 Problem Compute the following.

1. $\log_{1/3}\ 243$
2. $\log_{10}\ .00001$
3. $\log_{.001}\ 100000$
4. $\log_9\ \dfrac{1}{3}$
5. $\log_{1024}\ 64$
6. $\log_{5^{2/3}}\ 625$
7. $\log_{2\sqrt{2}}\ 32\sqrt[5]{2}$
8. $\log_2\ .0625$
9. $\log_{.0625}\ 2$
10. $\log_3\ \sqrt[4]{729\sqrt[3]{9^{-1}27^{-4/3}}}$

498 Problem Let $a>0, a\neq 1$. Compute the following.

1. $\log_a\ \sqrt[4]{a^{8/5}}$
2. $\log_a\ \sqrt[3]{a^{-15/2}}$
3. $\log_a\ \dfrac{1}{a^{1/2}}$
4. $\log_{a^3}\ a^6$
5. $\log_{a^2}\ a^3$
6. $\log_{a^{5/6}}\ a^{7/25}$

499 Problem Make a rough sketch of the following.

1. $x\mapsto\log_2\ x$
2. $x\mapsto\log_2\ |x|$
3. $x\mapsto 4+\log_{1/2}\ x$
4. $x\mapsto 5-\log_3\ x$
5. $x\mapsto 2-\log_{1/4}\ x$
6. $x\mapsto\log_5\ x$
7. $x\mapsto\log_5\ |x|$
8. $x\mapsto|\log_5\ x|$
9. $x\mapsto|\log_5\ |x||$
10. $x\mapsto 2+\log_e|x|$
11. $x\mapsto -3+\log_{1/2}|x|$
12. $x\mapsto 5-|\log_4 x|$

500 Problem Prove that for $x>0$,

$$1-x\leq -\log_e x.$$

501 Problem Prove that for $x > 0$ we have
$$x^e \leq e^x.$$
Use this to prove that for $x > 0$,
$$\log_e x \leq \frac{x}{e}.$$

502 Problem Find the natural domain of definition of the following.

❶ $x \mapsto \log_2(x^2 - 4)$
❷ $x \mapsto \log_2(x^2 + 4)$
❸ $x \mapsto \log_2(4 - x^2)$
❹ $x \mapsto \log_2(\frac{x+1}{x-2})$
❺ $x \mapsto \log_{x^2+1}(x^2 + 1)$
❻ $x \mapsto \log_{1-x^2} x$

8.2 Simple Exponential and Logarithmic Equations

Recall that for $a > 0,\ a \neq 1,\ b > 0$ the relation $a^x = b$ entails $x = \log_a\ b$. This proves useful in solving the following equations.

503 Example Solve the equation
$$\log_4 x = -3.$$

Solution: $x = 4^{-3} = \dfrac{1}{64}$.

504 Example Solve the equation
$$\log_2 x = 5.$$

Solution: $x = 2^5 = 32$.

505 Example Solve the equation
$$\log_x\ 16 = 2.$$

Solution: $16 = x^2$. Since the base must be positive, we have $x = 4$.

506 Example Solve the equation $3^x = 2$.

Solution: By definition, $x = \log_3\ 2$.

507 Example Solve the equation $9^x - 5 \cdot 3^x + 6 = 0$.

Solution: We have
$$9^x - 5 \cdot 3^x + 6 = (3^x)^2 - 5 \cdot 3^x + 6 = (3^x - 2)(3^x - 3).$$
Thus either $3^x - 2 = 0$ or $3^x - 3 = 0$. This implies that $x = \log_3\ 2$ or $x = 1$.

508 Example Solve the equation $25^x - 5^x - 6 = 0$.

Solution: We have
$$25^x - 5^x - 6 = (5^x)^2 - 5^x - 6 = (5^x + 2)(5^x - 3),$$
whence $5^x - 3 = 0$ or $x = \log_5\ 3$ as $5^x + 2 = 0$ does not have a real solution. (Why?)

Since $x \mapsto a^x$ and $x \mapsto \log_a x$ are inverses, we have
$$x = a^{\log_a x}\ \forall a > 0,\ a \neq 1,\ \forall x > 0 \tag{8.1}$$
Thus for example, $5^{\log_5 4} = 4,\ 26^{\log_{26} 8} = 8$. This relation will prove useful in solving some simple equations.

509 Example Solve the equation
$$\log_2 \log_4 x = -1.$$

Solution: As $\log_2 \log_4 x = -1$, we have

$$\log_4 x = 2^{\log_2 \log_4 x} = 2^{-1} = \frac{1}{2}.$$

Hence $x = 4^{\log_4 x} = 4^{1/2} = \sqrt{4} = 2$.

510 Example Solve the equation

$$\log_2 \log_3 \log_5 x = 0$$

Solution: Since $\log_2 \log_3 \log_5 x = 0$ we have

$$\log_3 \log_5 x = 2^{\log_2 \log_3 \log_5 x} = 2^0 = 1.$$

Hence

$$\log_5 x = 3^{\log_3 \log_5 x} = 3^1 = 3.$$

Finally $x = 5^{\log_5 x} = 5^3 = 125$.

511 Example Solve the equation

$$\log_2 x(x-1) = 1.$$

Solution: We have $x(x-1) = 2^1 = 2$. Hence $x^2 - x - 2 = 0$. This gives $x = 2$ or $x = -1$. Check that both are indeed solutions!

512 Example Solve the equation $\log_{e+x} e^8 = 2$.

Solution: We have $(e+x)^2 = e^8$ or $e+x = \pm e^4$. Now the base $e+x$ cannot be negative, so we discard the minus sign alternative. The only solution is when $e+x = e^4$, that is, $x = e^4 - e$.

Homework

513 Problem Find real solutions to the following equations for x.

1. $\log_x 3 = 4$
2. $\log_3 x = 4$
3. $\log_4 x = 3$
4. $\log_{x-2} 9 = 2$
5. $\log_{|x|} 16 = 4$
6. $23^x - 2 = 0$
7. $(2^x - 3)(3^x - 2)(6^x - 1) = 0$
8. $4^x - 9 \cdot 2^x + 14 = 0$
9. $49^x - 2 \cdot 7^x + 1 = 0$
10. $36^x - 2 \cdot 6^x = 0$
11. $36^x + 6^x - 6 = 0$
12. $5^x + 12 \cdot 5^{-x} = 7$
13. $\log_2 \log_3 x = 2$
14. $\log_3 \log_5 x = -1$

8.3 Properties of Logarithms

A few properties of logarithms that will simplify operations with them will now be deduced.

514 Theorem If If $a > 0, a \neq 1$, $M > 0$, and α is any real number, then

$$\log_a M^\alpha = \alpha \log_a M \tag{8.2}$$

Proof: *Let $x = \log_a M$. Then $a^x = M$. Raising both sides of this equality to the exponent α, one gathers $a^{\alpha x} = M^\alpha$. But this entails that $\log_a M^\alpha = \alpha x = \alpha(\log_a M)$, which proves the theorem.* ❑

515 Example How many digits does 8^{330} have?

Solution: Let n be the integer such that $10^n < 8^{330} < 10^{n+1}$. Clearly then 8^{330} has $n+1$ digits. Since $x \mapsto \log_{10} x$ is increasing, taking logarithms base 10 one has $n < 330\log_{10} 8 < n+1$. Using a calculator, we see that $298.001 < 330\log_{10} 8 < 298.02$, whence $n = 298$ and so 8^{330} has 299 digits.

516 Example If $\log_a t = 2$, then $\log_a t^3 = 3\log_a t = 3(2) = 6$.

517 Example $\log_5 125 = \log_5 5^3 = 3\log_5 5 = 3(1) = 3$.

518 Theorem Let $a > 0, a \neq 1, M > 0$, and let $\beta \neq 0$ be a real number. Then

$$\log_{a^\beta} M = \frac{1}{\beta}\log_a M. \tag{8.3}$$

> **Proof:** *Let $x = \log_a M$. Then $a^x = M$. Raising both sides of this equality to the power $\frac{1}{\beta}$ we gather $a^{x/\beta} = M^{1/\beta}$. But this entails that*
>
> $$\log_a M^{1/\beta} = \frac{x}{\beta} = \frac{1}{\beta}(\log_a M),$$
>
> *which proves the theorem.* ❑

519 Example Given that $\log_{8\sqrt{2}} 1024$ is a rational number, find it.

Solution: We have

$$\log_{8\sqrt{2}} 1024 = \log_{2^{7/2}} 1024 = \frac{2}{7}\log_2 2^{10} = \frac{2}{7}\cdot 10\log_2 2 = \frac{20}{7}.$$

520 Theorem If $a > 0, a \neq 1, M > 0, N > 0$ then

$$\log_a MN = \log_a M + \log_a N \tag{8.4}$$

In words, the logarithm of a product is the sum of the logarithms.

> **Proof:** *Let $x = \log_a M$ and let $y = \log_a N$. Then $a^x = M$ and $a^y = N$. This entails that $a^{x+y} = a^x a^y = MN$. But $a^{x+y} = MN$ entails $x + y = \log_a MN$, that is*
>
> $$\log_a M + \log_a N = x + y = \log_a MN,$$
>
> *as required.* ❑

521 Example If $\log_a t = 2$, $\log_a p = 3$ and $\log_a u^3 = 21$, find $\log_a t^3 pu$.

Solution: First observe that $\log_a t^3 pu = \log_a t^3 + \log_a p + \log_a u$. Now, $\log_a t^3 = 3\log_a t = 6$. Also, $21 = \log_a u^3 = 3\log_a u$, from where $\log_a u = 7$. Hence

$$\log_a t^3 pu = \log_a t^3 + \log_a p + \log_a u = 6 + 3 + 7 = 16.$$

522 Example Solve the equation

$$\log_2 x + \log_2 (x-1) = 1.$$

Solution: If $x > 1$ then

$$\log_2 x + \log_2 (x-1) = \log_2 x(x-1).$$

This entails $x(x-1) = 2$, from where $x = -1$ or $x = 2$. The solution $x = -1$ must be discarded, as we need $x > 1$.

523 Theorem If $a > 0, a \neq 1,\ M > 0,\ N > 0$ then

$$\log_a \frac{M}{N} = \log_a M - \log_a N \tag{8.5}$$

Proof: *Let $x = \log_a M$ and let $y = \log_a N$. Then $a^x = M$ and $a^y = N$. This entails that $a^{x-y} = \frac{a^x}{a^y} = \frac{M}{N}$. But $a^{x-y} = \frac{M}{N}$ entails $x - y = \log_a \frac{M}{N}$, that is*

$$\log_a M - \log_a N = x - y = \log_a \frac{M}{N},$$

as required. ❑

524 Example Let $\log_a t = 2,\ \log_a p = 3$ and $\log_a u^3 = 21$, find $\log_a \frac{p^2}{tu}$.

Solution: First observe that

$$\log_a \frac{p^2}{tu} = \log_a p^2 - \log_a tu = 2\log_a p - (\log_a t + \log_a u).$$

This entails that

$$\log_a \frac{p^2}{tu} = 2(3) - (2+21) = -17.$$

525 Theorem If $a > 0, a \neq 1,\ b > 0, b \neq 1$ and $M > 0$ then

$$\log_a M = \frac{\log_b M}{\log_b a}. \tag{8.6}$$

Proof: *From the identity $b^{\log_b M} = M$, we obtain, upon taking logarithms base a on both sides*

$$\log_a \left(b^{\log_b M}\right) = \log_a M.$$

By Theorem 3.4.1

$$\log_a \left(b^{\log_b M}\right) = (\log_b M)(\log_a b),$$

whence the theorem follows. ❑

526 Example Given that

$$(\log_2 3) \cdot (\log_3 4) \cdot (\log_4 5) \cdots (\log_{511} 512)$$

is an integer, find it.

Solution: Choose $a > 0, a \neq 1$. Then

$$\begin{aligned}(\log_2 3) \cdot (\log_3 4) \cdot (\log_4 5) \cdots (\log_{511} 512) &= \frac{\log_a 3}{\log_a 2} \cdot \frac{\log_a 4}{\log_a 3} \cdot \frac{\log_a 5}{\log_a 4} \cdots \frac{\log_a 512}{\log_a 511} \\ &= \frac{\log_a 512}{\log_a 2}.\end{aligned}$$

But

$$\frac{\log_a 512}{\log_a 2} = \log_2 512 = \log_2 2^9 = 9,$$

so the integer sought is 9.

527 Corollary If $a > 0, a \neq 1,\ b > 0, b \neq 1$ then

$$\log_a b = \frac{1}{\log_b a}. \tag{8.7}$$

Proof: *Let $M = b$ in the preceding theorem.* ❑

528 Example Given that $\log_n t = 2a,\ \log_s n = 3a^2$, find $\log_t s$ in terms of a.

Solution: We have

$$\log_t s = \frac{\log_n s}{\log_n t}.$$

Now, $\log_n s = \dfrac{1}{\log_s n} = \dfrac{1}{3a^2}$. Hence

$$\log_t s = \frac{\log_n s}{\log_n t} = \frac{\frac{1}{3a^2}}{2a} = \frac{1}{6a^3}.$$

529 Example Given that $\log_a 3 = s^{-3},\ \log_{\sqrt{3}} b = s^2 + 2,\ \log_9 c = s^3$, write $\log_3 \dfrac{a^2 b^5}{c^4}$ as a polynomial in s.

Solution: Observe that

$$\log_3 \frac{a^2 b^5}{c^4} = 2\log_3 a + 5\log_3 b - 4\log_3 c,$$

so we seek information about $\log_3 a, \log_3 b$ and $\log_3 c$. Now,

$$\log_3 a = \frac{1}{\log_a 3} = s^3, \quad \log_3 b = \frac{1}{2}\log_{\sqrt{3}} b = \frac{1}{2}s^2 + 1$$

and $\log_3 c = 2\log_9 c = 2s^3$. Hence

$$\log_3 \frac{a^2 b^5}{c^4} = 2s^3 + \frac{5}{2}s^2 + 5 - 8s^3 = -6s^3 + \frac{5}{2}s^2 + 5.$$

530 Example Given that $.63 < \log_3 2 < .631$, find the smallest positive integer a such that $3^a > 2^{102}$.

Solution: Since $x \mapsto \log_3 x$ is an increasing function, we have $a\log_3 3 > 102\log_3 2$, that is, $a > 102\log_3 2$. Using the given information, $64.26 < 102\log_3 2 < 64.362$, which means that $a = 65$ is the smallest such integer.

531 Example Assume that there is a positive real number x such that

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2,$$

where there is an infinite number of x's. What is the value of x?

Solution: Since $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$, one has

$$2 = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x^2,$$

whence, as x is positive, $x = \sqrt{2}$.

STOP *Euler shewed that the equation*

$$a^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = x$$

has real roots only for $a \in [e^{-e}; e^{1/e}]$.

532 Example How many real positive solutions does the equation

$$x^{(x^x)} = (x^x)^x$$

have?

Solution: Assuming $x > 0$ we have $x^x \log_e x = x \log_e x^x$ or $x^x \log_e x = x^2 \log_e x$. Thus $(\log_e x)(x^x - x^2) = 0$. Thus either $\log_e x = 0$, in which case $x = 1$, or $x^x = x^2$, in which case $x = 2$. The equation has therefore only two positive solutions.

533 Example The non-negative integers smaller than 10^n are split into two subsets A and B. The subset A contains all those integers whose decimal expansion does not contain a 5, and the set B contains all those integers whose decimal expansion contains at least one 5. Given n, which subset, A or B is the larger set? One may use the fact that $\log_{10} 2 := .3010$ and that $\log_{10} 3 := .4771$.

Solution: The set B contains $10^n - 9^n$ elements and the set A contains 9^n elements. Now if $10^n - 9^n > 9^n$ then $10^n > 2 \cdot 9^n$ and taking logarithms base 10 we deduce

$$n > \log_{10} 2 + 2n \log_{10} 3.$$

Thus

$$n > \frac{\log_{10} 2}{1 - 2\log_{10} 3} := 6.57\ldots$$

Therefore, if $n \le 6$, A has more elements than B and if $n > 6$, B has more elements than A.

534 Example Shew that if a, b, c, are real numbers with $a^2 = b^2 + c^2, a+b > 0, a+b \neq 1, a-b > 0, a-b \neq 1$, then

$$\log_{a-b} c + \log_{a+b} c = 2(\log_{a-b} c)(\log_{a+b} c).$$

Solution: As $c^2 = a^2 - b^2 = (a-b)(a+b)$, upon taking logarithms base $a+b$ we have

$$2\log_{a+b} c = \log_{a+b} (a-b)(a+b) = 1 + \log_{a+b} (a-b) \tag{8.8}$$

Similarly, taking logarithms base $a-b$ on the identity $c^2 = (a-b)(a+b)$ we obtain

$$2\log_{a-b} c = \log_{a-b} (a-b)(a+b) = 1 + \log_{a-b} (a+b) \tag{8.9}$$

Multiplying these last two identities,

$$\begin{aligned} 4(\log_{a-b} c)(\log_{a+b} c) &= (1 + \log_{a+b} (a-b))(1 + \log_{a-b} (a+b)) \\ &= 1 + \log_{a-b} (a+b) + \log_{a+b} (a-b) \\ &\quad + (\log_{a-b} (a+b))(\log_{a+b} (a-b)) \\ &= 2 + \log_{a-b} (a+b) + \log_{a+b} (a-b) \\ &= 2 + \log_{a-b} \tfrac{c}{a-b} + \log_{a+b} \tfrac{c}{a+b} \\ &= \log_{a-b} c + \log_{a+b} c, \end{aligned}$$

as we wanted to shew.

535 Example If $\log_{12} 27 = a$ prove that $\log_6 16 = \dfrac{4(3-a)}{3+a}$.

Solution: First notice that $a = \log_{12} 27 = 3\log_{12} 3 = \dfrac{3}{\log_3 12} = \dfrac{3}{1 + 2\log_3 2}$, whence $\log_3 2 = \dfrac{3-a}{2a}$ or $\log_2 3 = \dfrac{2a}{3-a}$. Also

$$\begin{aligned} \log_6 16 &= 4\log_6 2 \\ &= \tfrac{4}{\log_2 6} \\ &= \tfrac{4}{1+\log_2 3} \\ &= \tfrac{4}{1+\frac{2a}{3-a}} \\ &= \tfrac{4(3-a)}{3+a}, \end{aligned}$$

as required.

536 Example Solve the system

$$5\left(\log_x y + \log_y x\right) = 26$$
$$xy = 64$$

Solution: Clearly we need $x > 0, y > 0, x \neq 1, y \neq 1$. The first equation may be written as $5\left(\log_x y + \frac{1}{\log_x y}\right) = 26$ which is the same as $(\log_x y - 5)(\log_y x - \frac{1}{5}) = 0$. Thus the system splits into the two equivalent systems (I) $\log_x y = 5, xy = 64$ and (II) $\log_x y = 1/5, xy = 64$. Using the conditions $x > 0, y > 0, x \neq 1, y \neq 1$ we obtain the two sets of solutions $x = 2, y = 32$ or $x = 32, y = 2$.

Homework

537 Problem Find the exact value of

$$\frac{1}{\log_2 1996!} + \frac{1}{\log_3 1996!} + \frac{1}{\log_4 1996!} + \cdots + \frac{1}{\log_{1996} 1996!}.$$

538 Problem

1. $\log_4 MN = \log_4 M + \log N \ \forall M, N \in \mathbb{R}$.
2. $\log_5 M^2 = 2\log_5 M \forall M \in \mathbb{R}$.
3. $\exists\, M \in \mathbb{R}$ such that $\log_5 M^2 = 2\log_5 M$.

539 Problem Given that $\log_a p = 2,\ \log_a m = 9,\ \log_a n = -1$ find

1. $\log_a p^7$
2. $\log_{a^7} p$
3. $\log_{a^4} p^2 n^3$
4. $\log_{a^6} \dfrac{m^3 n}{p^6}$

540 Problem Which number is larger, 3^{1000} or 5^{600}?

541 Problem Find $(\log_3\ 169)(\log_{13}\ 243)$ without recourse of a calculator or tables.

542 Problem Find $\dfrac{1}{\log_2\ 36} + \dfrac{1}{\log_3\ 36}$ without recourse of a calculator or tables.

543 Problem Given that $\log_a\ p = b,\ \log_q a = 3b^{-2}$, find $\log_p q$ in terms of b.

544 Problem Given that $\log_2 a = s,\ \log_4 b = s^2,\ \log_{c^2} 8 = \frac{2}{s^3+1}$, write $\log_2 \dfrac{a^2 b^5}{c^4}$ as a function of s.

545 Problem Given that $\log_{a^2}(a^2+1) = 16$, find the value of

$$\log_{a^{32}}\left(a + \tfrac{1}{a}\right).$$

546 Problem Write without logarithms. Assume the proper restrictions on the variables wherever necessary.

1. $(a^{\alpha})^{-\beta \log_{\alpha^\beta}\ N^{\gamma}}$
2. $-\log_8\ \log_4\ \log_2\ 16$
3. $\log_{0.75}\ \log_2\ \sqrt{\sqrt[-2]{0.125}}$
4. $\left(5^{(\log_7\ 5)^{-1}} + (-\log_{10}\ 0.1)^{-1/2}\right)^{1/3}$
5. $b^{a^{(\log_b \log_b N)/(\log_b a)}}$
6. $2^{(\log_3\ 5)} - 5^{(\log_3\ 2)}$
7. $\left(\dfrac{1}{49}\right)^{1+(\log_7\ 2)} + 5^{-(\log_{1/5}\ 7)}$

547 Problem A sheet of paper has approximately 0.1 mm of thickness. Suppose you fold the sheet by halves, thirty times consecutively. (1) What is the thickness of the folded paper?, (2) How many times should you fold the sheet in order to obtain the distance from Earth to the Moon? (the distance from Earth to the Moon is about 384 000 km.)

548 Problem How many digits does 11^{2000} have?

549 Problem Let $A = \log_6 16, B = \log_{12} 27$. Find integers a, b, c such that $(A+a)(B+b) = c$.

550 Problem Given that $\log_{ab} a = 4$, find

$$\log_{ab} \frac{\sqrt[3]{a}}{\sqrt{b}}.$$

551 Problem The number 5^{100} is written in binary (base-2) notation. How many binary digits does it have?

552 Problem Prove that if $x > 0, a > 0, a \neq 1$ then $x^{1/\log_a\ x} = a$.

553 Problem Let a, b, x be positive real numbers distinct from 1. When is it true that

$4(\log_a\ x)^2 + 3(\log_b\ x)^2 = 8(\log_a x)(\log_b\ x)$?

554 Problem Prove that $\log_3 \pi + \log_\pi 3 > 2$.

555 Problem Solve the equation

$$4 \cdot 9^{x-1} = 3\sqrt{2^{2x+1}}$$

556 Problem Solve the equation

$$5^{x-1} + 5(0.2)^{x-2} = 26$$

557 Problem Solve the equation

$$25^x - 12 \cdot 2^x - (6.25)(0.16)^x = 0$$

558 Problem Solve the equation

$$\log_3(3^x - 8) = 2 - x$$

559 Problem Solve the equation

$$\log_4(x^2 - 6x + 7) = \log_4(x - 3)$$

560 Problem Solve the equation

$$\log_3(2-x) - \log_3(2+x) - \log_3 x + 1 = 0$$

561 Problem Solve the equation

$$2\log_4(2x) = \log_4(x^2 + 75)$$

562 Problem Solve the equation

$$\log_2(2x) = \frac{1}{4}\log_2(x-15)^4$$

563 Problem Solve the equation

$$\frac{\log_2 x}{\log_4 2x} = \frac{\log_8 4x}{\log_{16} 8x}$$

564 Problem Solve the equation

$$\log_3 x = 1 + \log_x 9$$

565 Problem Solve the equation

$$25^{\log_2 x} = 5 + 4x^{\log_2 5}$$

566 Problem Solve the equation

$$x^{\log_{10} 2x} = 5$$

567 Problem Solve the equation

$$|x-3|^{(x^2-8x+15)/(x-2)} = 1$$

568 Problem Solve the equation

$$\log_{2x-1} \frac{x^4+2}{2x+1} = 1$$

569 Problem Solve the equation

$$\log_{3x} x = \log_{9x} x$$

570 Problem Solve

$$\log_2 x + \log_4 y + \log_4 z = 2,$$

$$\log_3 x + \log_9 y + \log_9 z = 2,$$

$$\log_4 x + \log_{16} y + \log_{16} z = 2.$$

571 Problem Solve the equation

$$x^{0.5 \log_{\sqrt{x}}(x^2-x)} = 3^{\log_9 4}.$$

Answers

496 T; T; T; F; T; T; F; F

497 (1) -5, (2) -5, (3) $-\frac{5}{3}$, (4) $-\frac{1}{2}$, (5) $\frac{3}{5}$, (6) 6, (7) $\frac{52}{15}$, (8) -4 , (9) $-\frac{1}{4}$, (10) 1

498 (1) $\frac{2}{5}$, (2) $-\frac{5}{2}$, (3) $-\frac{1}{2}$, (4) 2, (5) $\frac{42}{125}$

513 (1) $\sqrt[4]{3}$, (2) 81, (3) 64 (4) 5, (5) ± 2, (6) $\log_{23} 2$, (7) $\log_2 3,\ \log_3 2,\ 0$, (8) $\log_2 7, 1$, (9) 0, (10) $\log_6 2$, (11) $\log_6 2$, (12) $\log_5 4, \log_5 3$, (13) 81, (14) $\sqrt[3]{5}$

537 1

538 F; F; T

539 (1) 14, (2) $\frac{2}{7}$, (3) $\frac{1}{4}$, (4) $\frac{7}{3}$

540 3^{1000}

541 10

542 $\frac{1}{2}$

543 $\frac{b}{3}$

544 $-3s^3 + 10s^2 + 2s - 3$

545 $\frac{31}{32}$

546 (1) $N^{-\alpha\beta\gamma/s}$, (2) 0, (3) 1 (4) 2, (5) N, (6) 0, (7) $\frac{1373}{196}$

547 (1) About 107.37 km (2) 42 times.

548 2083

549 $a = 4, b = 3, c = 24$

550 $\frac{17}{6}$

Chapter 9

Circular Functions

9.1 The Winding Function

Recall that a circle of radius r has a circumference of $2\pi r$ units of length. Hence a unit circle, i.e., one with $r = 1$, has circumference 2π.

572 Definition A *radian* is a $\dfrac{1}{2\pi}$th part of the circumference of a unit circle.

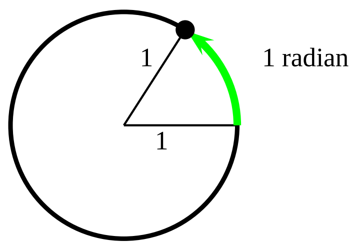


Figure 9.1: A radian.

Since $\frac{1}{2\pi} \approx 0.16$, a radian is about $\dfrac{4}{25}$ of the circumference of the unit circle. A quadrant or quarter part of a circle has arc length of $\frac{\pi}{4}$ radians. A semicircle has arc length $\frac{2\pi}{2} = \pi$ radians.

1. *A radian is simply a real number!*
2. *If a central angle of a unit circle cuts an arc of x radians, then the central angle measures x radians.*
3. *The sum of the internal angles of a triangle is π radians.*

Suppose now that we cut a unit circle into a "string" and use this string to mark intervals of length 2π on the real line. We put an endpoint 0, mark off intervals to the right of 0 with endpoints at $2\pi, 4\pi, 6\pi, \ldots$, etc. We start again, this time going to the left and marking off intervals with endpoints at $-2\pi, -4\pi, -6\pi, \ldots$, etc., as shewn in figure 9.2.

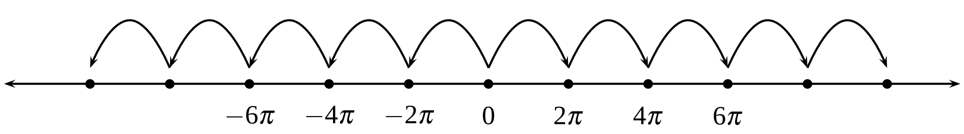


Figure 9.2: The Real Line modulo 2π.

We have decomposed the real line into the union of disjoint intervals

$$\ldots \cup [-6\pi; -4\pi[\cup [-4\pi; -2\pi[\cup [-2\pi; 0[\cup [0; 2\pi[\cup [2\pi; 4\pi[\cup [4\pi; 6\pi[\cup \ldots$$

Observe that each real number belongs to one, and only one of these intervals, that is, there is a unique integer k such that if $x \in \mathbb{R}$ then $x \in [2\pi k; (2k+2)\pi[$. For example $100 \in [30\pi; 32\pi[$ and $-9 \in [-4\pi; -2\pi[$.

There was no need for us to take 0 *as point of departure. We could have started with any real number* a*, as shewn in figure 9.3.*

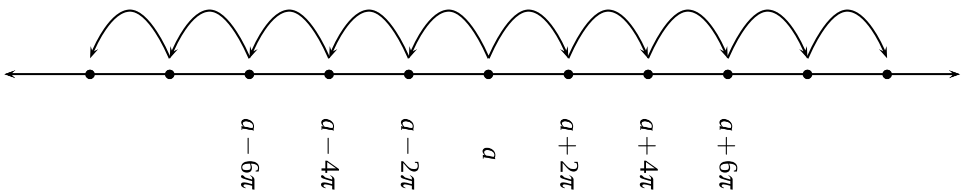


Figure 9.3: The Real Line modulo 2π, general version.

573 Definition Given two real numbers a and b, we say that *a is congruent to b* modulo 2π, written $a \equiv b \mod 2\pi$, if $\frac{a-b}{2\pi}$ is an integer. If $\frac{a-b}{2\pi}$ is not an integer, we say that a and b are *incongruent* modulo 2π and we write $a \not\equiv b \mod 2\pi$.

For example, $5\pi \equiv -7\pi \mod 2\pi$, since $\frac{5\pi-(-7\pi)}{2\pi} = \frac{12\pi}{2\pi} = 6$, an integer. However, $5\pi \not\equiv 2\pi \mod 2\pi$ as $\frac{5\pi-2\pi}{2\pi} = \frac{3\pi}{2\pi} = \frac{3}{2}$, which is not an integer.

574 Definition If $a \equiv b \mod 2\pi$, we say that a and b belong to the same *residue class* $\mod 2\pi$. We also say that a and b are *representatives* of the same residue class modulo 2π.

575 Theorem Given a real number a, all the numbers of the form $a + 2\pi k$, $k \in \mathbb{Z}$ belong to the same residue class modulo 2π.

Proof: *Take two numbers of this form,* $a + 2\pi k_1$ *and* $a + 2\pi k_2$*, say, with integers* k_1, k_2*. Then*

$$\frac{(a+2\pi k_1)-(a+2\pi k_2)}{2\pi} = k_1 - k_2,$$

which being the difference of two integers is an integer. This shews that $a + 2\pi k_1 \equiv a + 2\pi k_2 \mod 2\pi$. ❑

576 Example Take $x = \frac{\pi}{3}$. Then

$$\begin{aligned} \frac{\pi}{3} &\equiv \frac{\pi}{3} + 2\pi \equiv \frac{7\pi}{3} \mod 2\pi \\ &\equiv \frac{\pi}{3} - 2\pi \equiv -\frac{5\pi}{3} \mod 2\pi \\ &\equiv \frac{\pi}{3} + 4\pi \equiv \frac{13\pi}{3} \mod 2\pi \\ &\equiv \frac{\pi}{3} - 4\pi \equiv -\frac{11\pi}{3} \mod 2\pi \end{aligned}$$

Thus all of

$$\frac{\pi}{3}, \frac{7\pi}{3}, -\frac{5\pi}{3}, \frac{13\pi}{3}, -\frac{11\pi}{3}$$

belong to the same residue class $\mod 2\pi$.

If $a \equiv b \mod 2\pi$ *then there exists an integer* k *such that* $a = b + 2\pi k$.

Given a real number x, it is clear that there are infinitely many representatives of the class to which x belongs, as we can add any integral multiple of 2π to x and still lie in the same class. However, exactly one representative x_0 lies in the interval $[0,2\pi[$, as we saw above. We call x_0 the *canonical* representative of the class (to which x belongs modulo 2π).

To find the canonical representative of the class of x, we simply look for the integer k such that $2k\pi \leq x < (2k+2)\pi$. Then then $0 \leq x - 2k\pi < 2\pi$ and so $x - 2\pi k$ is the canonical representative of the class of x.

577 Definition We will call the procedure of finding a canonical representative for the class of x, *reduction* modulo 2π.

578 Example Reduce $5\pi \mod 2\pi$.

Solution: Since $4\pi < 5\pi < 6\pi$, we have $5\pi \equiv 5\pi - 4\pi \equiv \pi \mod 2\pi$. Thus π is the canonical representative of the class to which 5π belongs, modulo 2π.

To speed up the computations, we may avail of the fact that $2\pi k \equiv 0 \mod 2\pi$, that is, any integral multiple of 2π is congruent to $0 \mod 2\pi$.

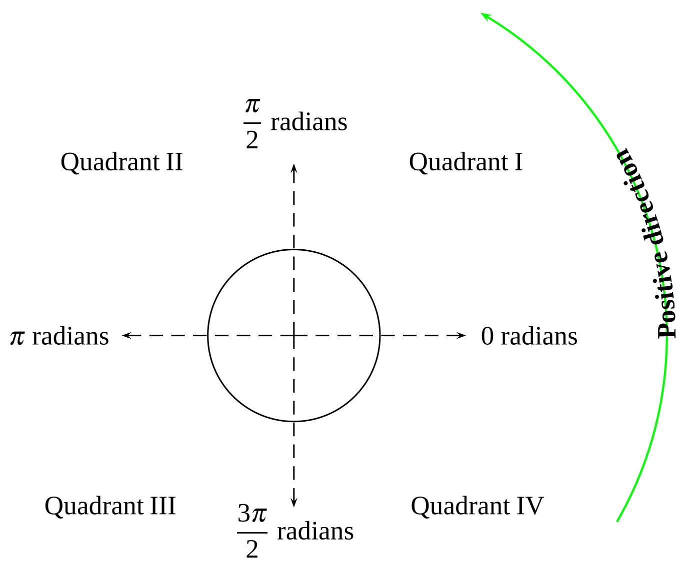


Figure 9.4: The unit circle on the Cartesian Plane.

579 Example Reduce $\dfrac{200\pi}{7}$ modulo 2π.

Solution: $\dfrac{200\pi}{7} \equiv \dfrac{196\pi + 4\pi}{7} \equiv 28\pi + \dfrac{4\pi}{7} \equiv \dfrac{4\pi}{7} \mod 2\pi$.

580 Example Reduce $-\dfrac{5\pi}{7}$ modulo 2π.

Solution: $-\dfrac{5\pi}{7} \equiv 2\pi - \dfrac{5\pi}{7} \equiv \dfrac{9\pi}{7} \mod 2\pi$.

581 Example Reduce $7 \mod 2\pi$.

Solution: Since $2\pi < 6.29 < 7 < 4\pi$, the largest even multiple of π smaller than 7 is 2π, whence $7 \equiv 7 - 2\pi \mod 2\pi$..

Place now the centre of a unit circle at the origin of the Cartesian Plane. Choosing the point $(1,0)$ as our departing point (a completely arbitrary choice), we traverse the circumference of the unit circle counterclockwise (again, the choice is completely arbitrary). If we traverse 0 units, we are still at $(1,0)$, on the positive portion of the x-axis. If we traverse a number of units in the interval $\left]0\,;\frac{\pi}{2}\right[$, we are in the first quadrant.

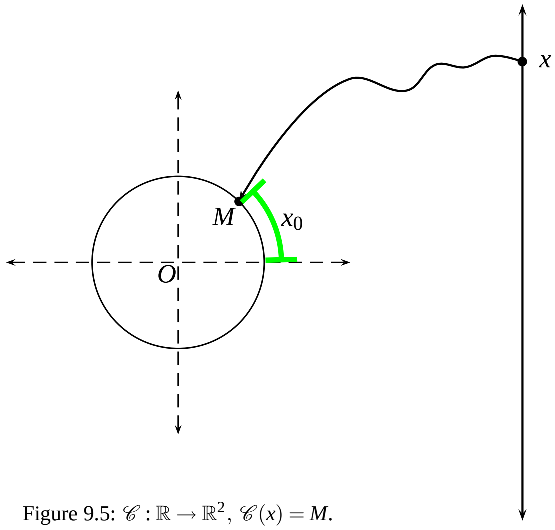


Figure 9.5: $\mathscr{C}:\mathbb{R}\to\mathbb{R}^2$, $\mathscr{C}(x) = M$.

If we have traversed exactly $\dfrac{\pi}{2}$ units, we are at $(0,1)$, on the positive portion of the y-axis. Traversing a number of units in the interval $\left]\frac{\pi}{2}\,;\pi\right[$, puts us in the second quadrant. If we travel exactly π units, we are at $(-1,0)$, the negative portion of the x-axis. Traversing a number of units in the interval $\left]\pi\,;\frac{3\pi}{2}\right[$, puts us in the third quadrant. Traversing exactly $\frac{3\pi}{2}$ units puts us at the point $(0,-1)$, the negative portion of the y-axis. Travelling a number of units in the interval $\left]\frac{3\pi}{2}\,;2\pi\right[$, puts us in the fourth quadrant. Finally, travelling exactly 2π units brings us back to $(1,0)$. So, after one revolution around the unit circle, we are back in already travelled territory. See figure 9.4.

If we traverse the unit circle clockwise, *then the arc length is measured negatively.*

We now define a function $\mathscr{C}:\mathbb{R}\to\mathbb{R}^2$ in the following fashion. Given a real number x, let x_0 be its canonical representative modulo 2π. Starting at $(1,0)$, traverse the circumference of the unit circle x_0 units counterclockwise. Your final destination is a point on the Cartesian Plane, call it M. We let $\mathscr{C}(x) = M$. See figure 9.5. The function $\mathscr{C}$ is called the *winding function*.

582 Example In what quadrant does $\mathscr{C}(-\dfrac{283\pi}{5})$ lie?

Solution: Observe that

$$\begin{aligned} -\tfrac{283\pi}{5} &\equiv \tfrac{-280\pi-3\pi}{5} \\ &\equiv -56\pi - \tfrac{3\pi}{5} \\ &\equiv -\tfrac{3\pi}{5} \\ &\equiv 2\pi - \tfrac{3\pi}{5} \\ &\equiv \tfrac{7\pi}{5} \mod 2\pi. \end{aligned}$$

Since $\frac{7\pi}{5} \in]\pi;\frac{3\pi}{2}[$, $\mathscr{C}(-\dfrac{283\pi}{5})$ lies in quadrant III.

583 Example In what quadrant does $\mathscr{C}(451)$ lie?

Solution: Since $71 < \frac{451}{2\pi} < 71.8$, $142\pi < 451 < 144\pi$, and hence $451 \equiv 451 - 142\pi \mod 2\pi$. Now, $451 - 142\pi \approx 4.89 \in \left]\frac{3\pi}{2}; 2\pi\right[$, and so $\mathscr{C}(451)$ lies in the fourth quadrant.

584 Example In which quadrant does $\mathscr{C}(\pi^2)$ lie?

Solution: We multiply the inequality $2 < \pi < 4$ through by π, obtaining $2\pi < \pi^2 < 4\pi$, whence the largest even multiple of π less than π^2 is 2π. Therefore $\pi^2 \equiv \pi^2 - 2\pi \mod 2\pi$. Now we claim that

$$\pi < \pi^2 - 2\pi < \frac{3\pi}{2}.$$

The sinistral inequality is easily deduced from the obvious inequality $3\pi < \pi^2$. The dextral inequality is deduced from the fact that $\pi^2 < 3.5\pi$. The inequality $\pi < \pi^2 - 2\pi < \frac{3\pi}{2}$ is thus proven, which means that $\mathscr{C}(\pi^2)$ lies in the third quadrant.

585 Example Find the members of the set $\left\{\frac{\pi}{2} + \frac{k\pi}{3} : k \in \mathbb{Z}\right\}$ that belong to the interval $[8\pi; 10\pi[$.

Solution: The problem is asking for all integers k such that

$$8\pi \le \frac{\pi}{2} + \frac{k\pi}{3} < 10\pi.$$

Now,

$$\begin{aligned} 8\pi \le \tfrac{\pi}{2} + \tfrac{k\pi}{3} < 10\pi &\iff 8\pi - \tfrac{\pi}{2} \le \tfrac{k\pi}{3} < 10\pi - \tfrac{\pi}{2} \\ &\iff \tfrac{15\pi}{2} \le \tfrac{k\pi}{3} < \tfrac{19\pi}{2} \\ &\iff 22.5 \le k < 28.5. \end{aligned}$$

Since k is an integer, $k \in \{23, 24, 25, 26, 27, 28\}$. The required elements are thus

$$\frac{\pi}{2} + \frac{23\pi}{3} = \frac{49\pi}{6}, \qquad \frac{\pi}{2} + \frac{26\pi}{3} = \frac{55\pi}{6},$$

$$\frac{\pi}{2} + \frac{24\pi}{3} = \frac{17\pi}{2}, \qquad \frac{\pi}{2} + \frac{27\pi}{3} = \frac{19\pi}{2},$$

$$\frac{\pi}{2} + \frac{25\pi}{3} = \frac{53\pi}{6}, \qquad \frac{\pi}{2} + \frac{28\pi}{3} = \frac{59\pi}{6}.$$

586 Example Is $\dfrac{275\pi}{6} \in \left\{\frac{\pi}{2} + \frac{k\pi}{3} : k \in \mathbb{Z}\right\}$?

Solution: The problem is asking whether there is an integer k such that

$$\frac{275\pi}{6} = \frac{\pi}{2} + \frac{k\pi}{3}.$$

Solving for k we find $k = 136$, which is an integer. The answer is affirmative and indeed,

$$\frac{275\pi}{6} = \frac{\pi}{2} + \frac{136\pi}{3}.$$

Homework

587 Problem True or False.

1. $10 \equiv 8 \mod 2\pi$.
2. $-\frac{9\pi}{7} \equiv \frac{5\pi}{7} \mod 2\pi$.
3. $\frac{1}{\pi} \equiv \frac{2}{\pi} \mod 2\pi$.
4. $\frac{7\pi}{6} \equiv \frac{\pi}{6} \mod 2\pi$.
5. $\frac{-8\pi}{41} \equiv -\frac{500\pi}{41} \mod 2\pi$.
6. $x \in [-1;0[$ then $\mathscr{C}(x)$ is in quadrant IV.

588 Problem Reduce the following real numbers $\mod 2\pi$. Determine the quadrant in which their image under $\mathscr{C}$ would lie.

1. $\dfrac{3\pi}{5}$;
2. $-\dfrac{3\pi}{5}$;
3. $\dfrac{7\pi}{5}$,;
4. $\dfrac{8\pi}{57}$;
5. $\dfrac{57\pi}{8}$;
6. $\dfrac{6\pi}{79}$;
7. $\dfrac{790\pi}{7}$;
8. 1;
9. 2;
10. 3;
11. 4;
12. 5;
13. 6,;
14. 100,;
15. -3.14;
16. -3.15

589 Problem Find all the members of the set $\{\frac{3\pi}{4} + \frac{k\pi}{5} : k \in \mathbb{Z}\}$ that lie in the interval (i) $[0;\pi[$; (ii) $[-\pi;0[$.

590 Problem Is $\dfrac{279\pi}{20} \in \{\dfrac{3\pi}{4} + \dfrac{3k\pi}{5} | k \in \mathbb{Z}\}$? Is $-\dfrac{251\pi}{20} \in \{\dfrac{3\pi}{4} + \dfrac{3k\pi}{5} | k \in \mathbb{Z}\}$?

591 Problem Prove that congruence modulo 2π is reflexive, that is, if $a \in \mathbb{R}$, then $a \equiv a \mod 2\pi$.

592 Problem Prove that congruence modulo 2π is symmetric, that is, if $a, b \in \mathbb{R}$, and if $a \equiv b \mod 2\pi$ then $b \equiv a \mod 2\pi$.

593 Problem Prove that congruence modulo 2π is transitive, that is if if $a, b, c \in \mathbb{R}$, then $a \equiv b \mod 2\pi$ and $b \equiv c \mod 2\pi$ imply $a \equiv c \mod 2\pi$.

9.2 Cosines and Sines: Definitions

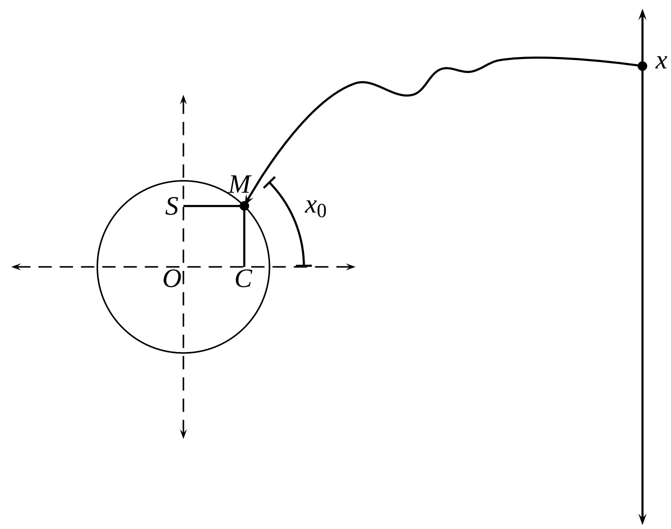


Figure 9.6: Geometric construction of the cosine and sine functions.

Consider any real number x. We find its canonical representative $x_0 \mod 2\pi$ and use this to find $\mathscr{C}(x) = M$, as in figure 9.6. We now project the point M so obtained onto C and S on the axes. The *cosine* function $\begin{array}{rcl} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \cos x \end{array}$ is given by $\cos(x) = \cos x = OC$ (the algebraic length of the segment OC, that is, the signed distance from O to C) and the *sine* function $\begin{array}{rcl} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin x \end{array}$ is given by $\sin(x) = \sin x = OS$ (the algebraic length of the segment OS).

STOP

1. *The farthest right M can go is to $(1,0)$ and the farthest left is to $(-1,0)$. Thus $-1 \le \cos x \le 1$. Similarly, the farthest up M can go is to $(0,1)$ and the farthest down it can go is to $(0,-1)$. Hence $-1 \le \sin x \le 1$.*

2. *The sine and cosine functions are defined for all real numbers.*
3. *If* $a \equiv b \mod 2\pi$ *then* $\cos a = \cos b$ *and* $\sin a = \sin b$*. In other words, the cosine and sine functions are periodic with period* 2π*, that is*
$$\sin(2\pi + x) = \sin x \ \forall x \in \mathbb{R}, \tag{9.1}$$
$$\cos(2\pi + x) = \cos x \ \forall x \in \mathbb{R}. \tag{9.2}$$
4. *The point* M *has abscissa* $\cos x$ *and ordinate* $\sin x$*, that is,* $M = (\cos x, \sin x)$.
5. *The functions* $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \cos x \end{array}$ *and* $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin x \end{array}$ *are surjective (onto) but not injective (one-to-one).*

We may now compute some simple sines and cosines.

594 Example If $x = 0$ then the point M is $(1,0)$. Thus $\cos 0 = 1$, $\sin 0 = 0$. See figure 9.7.

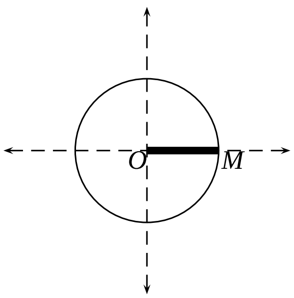


Figure 9.7: $\cos 0 = 1, \sin 0 = 0$

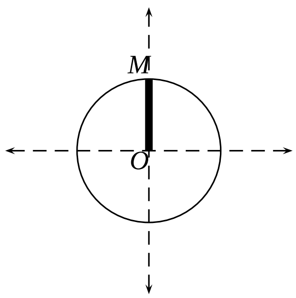


Figure 9.8: $\cos \frac{\pi}{2} = 0, \sin \frac{\pi}{2} = 1$.

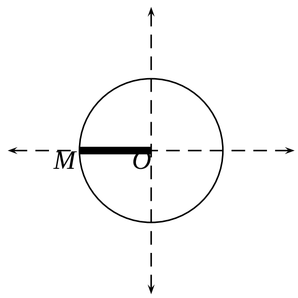


Figure 9.9: $\cos \pi = -1, \sin \pi = 0$.

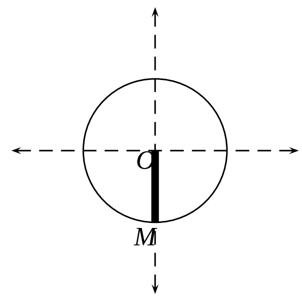


Figure 9.10: $\cos \frac{3\pi}{2} = 0, \sin \frac{3\pi}{2} = -1$.

595 Example If $x = \dfrac{\pi}{2}$ the point M is $(0,1)$. From this we gather that $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$. See figure 9.8.

596 Example If $x = \pi$ then the point M is $(-1,0)$. Thus $\cos \pi = -1$, $\sin \pi = 0$. See figure 9.9.

597 Example If $x = \dfrac{3\pi}{2}$ the point M is $(0,1)$. From this we gather that $\cos \frac{3\pi}{2} = 0$ and $\sin \frac{3\pi}{2} = -1$. See figure 9.10.

598 Definition If $K \neq -1$, we write $\sin^K x, \cos^K x$ to denote $(\sin x)^K, (\cos x)^K$, respectively. $\sin^{-1} x, \cos^{-1} x$, a are reserved for when we study inversion later in these notes.

The following relation, known as the *Pythagorean Relation* is fundamental in the study of circular functions.

599 Theorem (Pythagorean Relation) Let x be any real number. Then
$$\cos^2 x + \sin^2 x = 1. \tag{9.3}$$

Proof: *Let* $\mathscr{C}(x) = M = (\cos x, \sin x)$*, as in figure 9.11., where* $O = (0,0)$*, and* S, C *are the projections of* M *onto the axes. In* $\triangle OCM$*,* $\cos x = OC$*, and* $\sin x = OS = CM$*. As* $\triangle OCM$ *is a right triangle and* $OM = 1$*, by the Pythagorean Theorem, we have*
$$\cos^2 x + \sin^2 x = OC^2 + CM^2 = OM^2 = 1^2 = 1,$$
which completes the proof. ❑

Pay attention to the notation $\cos^2 x$ for $(\cos x)^2$ and respectively to $\sin^2 x$ for $(\sin x)^2$. Do not confuse these with $\cos x^2$ and $\sin x^2$. For example, if $x = \pi$ then $\cos^2 \pi = (-1)^2 = 1$ and $\sin^2 \pi = 0^2 = 0$. Since $\mathscr{C}(\pi^2)$ lies in the third quadrant, $\cos \pi^2 < 0$ and $\sin \pi^2 < 0$. Hence $\cos^2 \pi \neq \cos \pi^2$ and $\sin^2 \pi \neq \sin \pi^2$.

From the Pythagorean Relation,

$$\cos x = \pm\sqrt{1 - \sin^2 x}$$

and

$$\sin x = \pm\sqrt{1 - \cos^2 x}.$$

The ambiguity in sign is resolved by specifying in which quadrant $\mathscr{C}(x)$ lies, see figure 9.12.

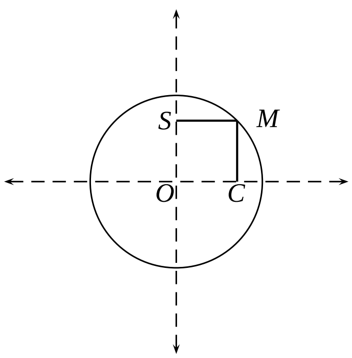


Figure 9.11: Pythagorean Relation.

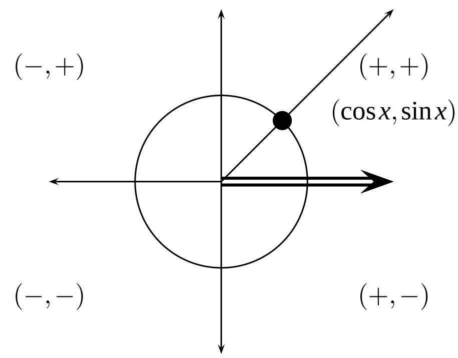


Figure 9.12: Signs of cos and sin.

600 Example Let $\frac{3\pi}{2} < x < 2\pi$ and $\cos x = \frac{1}{3}$. Find $\sin x$.

Solution: $\mathscr{C}(x)$ lies in the fourth quadrant, where $\sin x < 0$. We have

$$\sin x = -\sqrt{1 - \cos^2 x} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}.$$

601 Example Given that $\frac{\pi}{2} < x < \pi$, and that $\sin x = \frac{3}{5}$, find $\cos x$.

Solution: Since $\mathscr{C}(x)$ lies in the second quadrant, the cosine is negative. Hence

$$\cos x = -\sqrt{1 - \sin^2 x} = -\sqrt{1 - \left(\frac{3}{5}\right)^2} = -\frac{4}{5}.$$

602 Theorem (Symmetry Identities) Let $x \in \mathbb{R}$. Then the following are identities.

$$\cos(-x) = \cos x, \tag{9.4}$$

$$\sin(-x) = -\sin x, \tag{9.5}$$

$$\cos(\pi - x) = -\cos x, \tag{9.6}$$

$$\sin(\pi - x) = \sin x, \tag{9.7}$$

$$\cos(\pi + x) = -\cos x, \tag{9.8}$$

$$\sin(\pi + x) = -\sin x, \tag{9.9}$$

Proof: *The first identity says that the cosine is an even function; the second that the sine is an odd function. The third and fourth identities are "supplementary angle" identities. The fifth and the sixth identities are a "reflexion about the origin." All of these identities can be derived at once from figure 9.13.* ❑

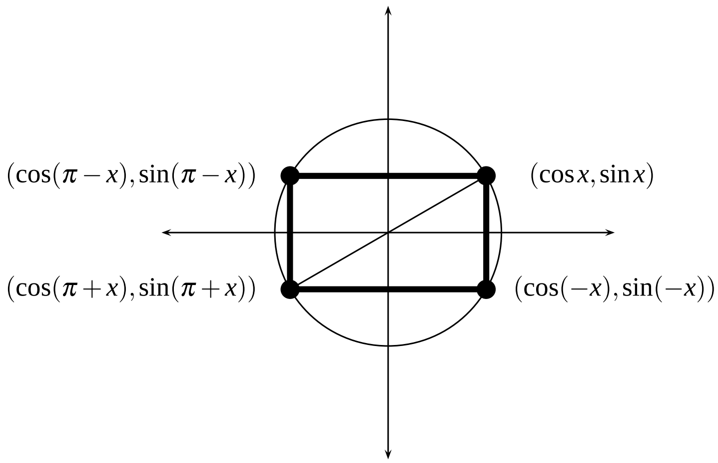


Figure 9.13: Identities deduced from symmetry.

Only the first two and the last two are worth committing to memory.

By the 2π-periodicity of the cosine and sine we have

$$\cos(2\pi k + x) = \cos x, \ \forall x \in \mathbb{R} \ \forall k \in \mathbb{Z} \tag{9.10}$$

$$\sin(2\pi k + x) = \sin x, \ \forall x \in \mathbb{R} \ \forall k \in \mathbb{Z}. \tag{9.11}$$

Now,

$$\cos((2k+1)\pi + x) = \cos(2\pi k + \pi + x) = \cos(\pi + x) = -\cos x$$

and

$$\sin((2k+1)\pi + x) = \sin(2\pi k + \pi + x) = \sin(\pi + x) = -\sin x,$$

whence the following corollary is proved.

603 Corollary Let $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Then

$$\cos((2k+1)\pi + x) = -\cos x \tag{9.12}$$

and

$$\sin((2k+1)\pi + x) = -\sin x \tag{9.13}$$

In other words, if we add even multiples of π to a real number, we get back the same cosine and the sine of the real number. If we add odd multiples of π to a real number, we get minus the cosine or sine of the real number.

604 Example Write

$$\sin(32\pi + x) - 18\cos(19\pi - x) + \cos(56\pi + x) - 9\sin(x + 17\pi)$$

in the form $a\sin x + b\cos x$.

Solution: The even multiples of π addends give

$$\sin(32\pi + x) = \sin x$$

and

$$\cos(56\pi + x) = \cos x.$$

Examining the odd multiples of π addends we see that $\cos(19\pi - x) = -\cos(-x)$. But $\cos(-x) = \cos x$, as the cosine is an even function and so

$$\cos(19\pi - x) = -\cos x.$$

Similarly,

$$\sin(17\pi + x) = -\sin x.$$

Upon gathering all of these equalities, we deduce that

$$\begin{aligned} \sin(32\pi + x) - 18\cos(19\pi - x) \\ +\cos(56\pi + x) - 9\sin(x + 17\pi) \\ &= \sin x - 18(-\cos x) \\ &\quad +\cos x - 9\sin x \\ &= -8\sin x + 19\cos x. \end{aligned}$$

605 Example Prove that $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

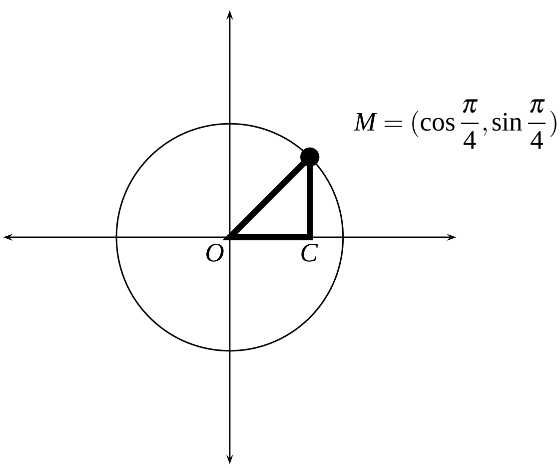


Figure 9.14: $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = 1$.

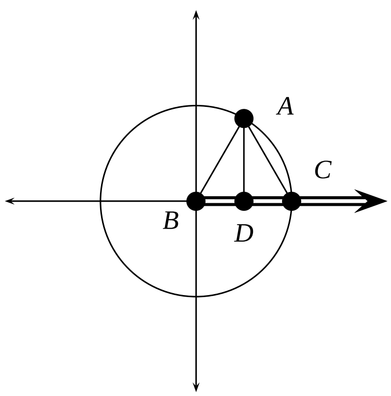


Figure 9.15: $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

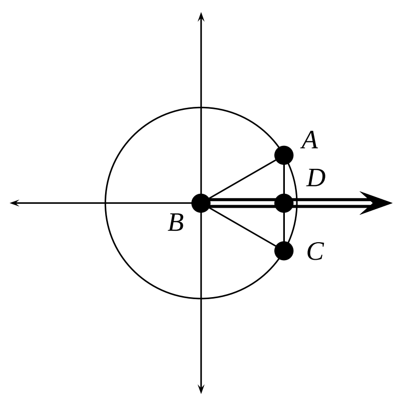


Figure 9.16: $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$

Solution: $\mathscr{C}\left(\dfrac{\pi}{4}\right)$ is half-way between $\mathscr{C}(0)$ and $\mathscr{C}\left(\dfrac{\pi}{2}\right)$. Thus $\triangle OCM$ in figure 9.14 is an isosceles right triangle. As $OC = CM$, we have

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4}.$$

By the Pythagorean Relation,

$$\cos^2\frac{\pi}{4} + \sin^2\frac{\pi}{4} = 1,$$

and so $2\cos^2\frac{\pi}{4} = 1$. Since $\mathscr{C}\left(\frac{\pi}{4}\right)$ lies in the first quadrant, we take the positive square root. We deduce $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. This implies that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

606 Example Prove that $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and that $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

Solution: In figure 9.15 , $A = (\cos\frac{\pi}{3}, \sin\frac{\pi}{3})$, $B = (0,0)$ and $C = (1,0)$. Since $BA = BC = 1$, $\triangle BAC$ is isosceles. Thus $\angle A = \angle C$. Moreover, since the sum of the angles of a triangle is π radians and central $\angle B$ measures $\dfrac{\pi}{3}$ radians, the triangle is equilateral. Let D denote the foot of the perpendicular from A to the side BC. Since $\triangle BAC$ is equilateral, D is halfway of the distance between B and C, which means that $= \dfrac{1}{2}$. Thus

$$\cos\frac{\pi}{3} = \frac{1}{2}.$$

Also, taking the positive square root (why?)

$$\sin\frac{\pi}{3} = \sqrt{1 - \cos^2\frac{\pi}{3}} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2},$$

as we wanted to shew.

607 Example Prove that $\cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$ and that $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$.

Solution: Reflect the point $A = (\cos\frac{\pi}{6}, \sin\frac{\pi}{6})$ about the x-axis to the point $C = (\cos\frac{\pi}{6}, -\sin\frac{\pi}{6})$, as in figure 9.16. Observe that since $\angle DBA = \angle CBD = \dfrac{\pi}{6}$ then $\angle CBA = \dfrac{\pi}{3}$. Thus $\triangle ABC$ is equilateral, and so $AD = \frac{1}{2}$, which implies that

$$\sin\frac{\pi}{6} = \frac{1}{2}.$$

We deduce that

$$\cos\frac{\pi}{6} = \sqrt{1 - \sin^2\frac{\pi}{6}} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}.$$

The student will do well in memorising the special values deduced above, which are conveniently gathered in the table below.

x	$\sin x$	$\cos x$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

608 Example Find $\cos(-\dfrac{\pi}{6})$ and $\sin(-\dfrac{\pi}{6})$.

Solution: Since $x \mapsto \cos x$ is an even function, we have

$$\cos(-\frac{\pi}{6}) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}.$$

Since $x \mapsto \sin x$ is an odd function, we have

$$\sin(-\frac{\pi}{6}) = -\sin(\frac{\pi}{6}) = -\frac{1}{2}.$$

609 Example Find $\cos\dfrac{7\pi}{6}$ and $\sin\dfrac{7\pi}{6}$.

Solution: By the reflexion about the origin identities

$$\cos\frac{7\pi}{6} = \cos(\pi + \frac{\pi}{6}) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

and

$$\sin\frac{7\pi}{6} = \sin(\pi + \frac{\pi}{6}) = -\sin\frac{\pi}{6} = -\frac{1}{2}.$$

610 Example

$$\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(-\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

and

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = -\sin\left(-\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

611 Example Find the exact value of

$$\cos\left(-\frac{32}{3}\pi\right)$$

Solution:

$$\begin{aligned}\cos\left(-\tfrac{32}{3}\pi\right) &= \cos\left(\tfrac{32\pi}{3}\right)\\ &= \cos(10\pi + \tfrac{2\pi}{3})\\ &= \cos(\tfrac{2\pi}{3})\\ &= -\tfrac{1}{2}\end{aligned}$$

Aliter:

$$\begin{aligned}\cos\left(-\tfrac{32}{3}\pi\right) &= \cos\left(\tfrac{32\pi}{3}\right)\\ &= \cos(11\pi - \tfrac{\pi}{3})\\ &= -\cos(-\tfrac{\pi}{3})\\ &= -\cos\left(\tfrac{\pi}{3}\right)\\ &= -\tfrac{1}{2}\end{aligned}$$

612 Example Find the exact value of

$$\sin\left(-\frac{31}{3}\pi\right)$$

Solution:

$$\begin{aligned}\sin\left(-\tfrac{31}{3}\pi\right) &= -\sin\left(\tfrac{31\pi}{3}\right)\\ &= -\sin(10\pi + \tfrac{\pi}{3})\\ &= -\sin(\tfrac{\pi}{3})\\ &= -\tfrac{\sqrt{3}}{2}\end{aligned}$$

613 Theorem (Complementary Angle Identities) The following identities hold:

$$\cos(\frac{\pi}{2} - x) = \sin x \ \forall x \in \mathbb{R} \tag{9.14}$$

$$\sin(\frac{\pi}{2} - x) = \cos x \ \forall x \in \mathbb{R} \tag{9.15}$$

Proof: *We will prove the result for* $x \in [0; \frac{\pi}{2}[$. *The extension of these identities to all real numbers depends on Theorem 602 and we leave it as an exercise. In figure 9.17 assume that arc MA (read counterclockwise) measures* x *and that* $x \in [0; \frac{\pi}{4}[$. *Reflect point* $A = (\cos x, \sin x)$ *about the line* $y = x$, *to point* $B = (\sin x, \cos x)$ *as in figure 9.17. Arc BT (read counterclockwise) measures* x, *and so arc MAB measures* $\frac{\pi}{2} - x$. *This means that* $B = (\cos(\frac{\pi}{2} - x), \sin(\frac{\pi}{2} - x))$, *from where the Theorem follows for* $x \in]0; \frac{\pi}{4}[$. *Assume now that* $x \in [\frac{\pi}{4}; \frac{\pi}{2}[$. *Then* $\frac{\pi}{2} - x \in [0; \frac{\pi}{4}[$, *and so we apply the result just obtained to* $\frac{\pi}{2} - x$:

$$\cos\left(\frac{\pi}{2} - x\right) = \sin\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \sin x,$$

and

$$\sin\left(\frac{\pi}{2} - x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) = \cos x.$$

So, we have established the result for $x \in [0; \frac{\pi}{2}[$. ❑

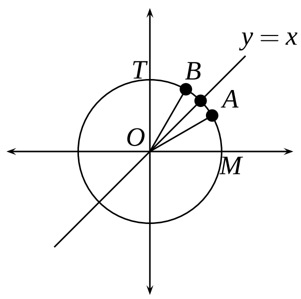


Figure 9.17: Complementary Angle Identities.

Using the complementary angle identities,

$$\sin\frac{\pi}{6} = \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2},$$

and

$$\cos\frac{\pi}{6} = \sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2},$$

for instance.

614 Example Prove that

$$\sin x = \cos\left(x - \frac{\pi}{2}\right), \ \forall x \in \mathbb{R}.$$

Solution: Since the cosine is an even function,

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) = \cos\left(-\left(\frac{\pi}{2} - x\right)\right) = \cos\left(x - \frac{\pi}{2}\right).$$

615 Example Prove that the following hold identically.

$$\cos x = \sin\left(x + \frac{\pi}{2}\right), \ \forall x \in \mathbb{R}.$$

$$-\sin x = \cos\left(x + \frac{\pi}{2}\right), \ \forall x \in \mathbb{R}.$$

Solution: Using the fact the fact that the cosine is an even function, and using the complementary angle identity for the cosine,

$$\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} - (-x)\right) = \sin\left(\frac{\pi}{2} + x\right).$$

Since the sine is an odd function,

$$\sin x = -\sin(-x) = -\cos\left(\frac{\pi}{2} - (-x)\right) = -\cos\left(\frac{\pi}{2} + x\right).$$

616 Example Let $0 < \theta < \frac{\pi}{2}$. Given that $\sin 2\theta = \cos 3\theta$ find $\sin 5\theta$.

Solution: Since $\sin 2\theta = \cos 3\theta$, these two quantities have the same sign. Since $0 < 2\theta < \pi$, then both $\mathscr{C}(2\theta)$ and $\mathscr{C}(3\theta)$ must be in quadrant I. By the complementary angle identities, we have $\sin 2\theta = \cos(\frac{\pi}{2} - 2\theta)$. Thus $\cos(\frac{\pi}{2} - 2\theta) = \cos 3\theta$, and so, $\frac{\pi}{2} - 2\theta = 3\theta$ or $5\theta = \frac{\pi}{2}$. Hence $\sin 5\theta = 1$.

617 Example Write in the form $a\sin\alpha + b\sin\alpha$:

$$\sin(\pi - \alpha) + \cos\left(\frac{\pi}{2} + \alpha\right) - \cos(\pi + \alpha)$$

By reflexion about the origin, $\sin(\pi - \alpha) = -\sin(-\alpha)$. Since the sine is an odd function, $-\sin(-\alpha) = -(-\sin\alpha) = \sin\alpha$. By the complementary angle identities, and since the sine is an odd function

$$\cos\left(\frac{\pi}{2} + \alpha\right) = \cos\left(\frac{\pi}{2} - (-\alpha)\right) = \sin(-\alpha) = -\sin\alpha.$$

Finally, by reflexion about the origin, $\cos(\pi + \alpha) = -\cos\alpha$. Upon collecting all of these equalities,

$$\sin(\pi - \alpha) + \cos\left(\frac{\pi}{2} + \alpha\right) - \cos(\pi + \alpha) = \cos\alpha.$$

618 Example Given that

$$3\sin x + 4\cos x = 5,$$

find $\sin x$ and $\cos x$.

Solution: We have

$$3\sin x + 4\cos x = 5 \iff \sin x = \frac{5 - 4\cos x}{3}.$$

Putting this in the identity $\cos^2 x + \sin^2 x = 1$ we obtain

$$\cos^2 x + \left(\frac{5 - 4\cos x}{3}\right)^2 = 1$$

$$\cos^2 x + \frac{25 - 40\cos x + 16\cos^2 x}{9} = 1$$

$$9\cos^2 x + 25 - 40\cos x + 16\cos^2 x = 9$$

$$25\cos^2 x - 40\cos x + 16 = 0$$

$$(5\cos x - 4)^2 = 0$$

$$\cos x = \frac{4}{5}$$

Substituting this value we obtain

$$\sin x = \frac{5 - 4\cos x}{3} = \frac{5 - \frac{16}{5}}{3} = \frac{3}{5}.$$

619 Example Find k such that the expression

$$(\sin x+\cos x)^2+k\sin x\cos x=1$$

becomes an identity.

Solution: We have

$$\begin{aligned}1 &= (\sin x+\cos x)^2+k\sin x\cos x\\ &= \sin^2 x+2\sin x\cos x+\cos^2 x+k\sin x\cos x\\ &= 1+(k+2)\sin x\cos x\end{aligned}$$

We thus have $(k+2)\sin x\cos x=0$. This will hold for all real numbers x if $k=-2$.

Homework

620 Problem Write in the form $a\sin x+b\cos x$, with real constants a,b.

$$A(x)=\sin\left(\frac{\pi}{2}-x\right)+\cos(5\pi-x)+\cos\left(\frac{3\pi}{2}-x\right)+\sin\left(\frac{3\pi}{2}+x\right)$$

621 Problem True or False.

1. $\sin\frac{7\pi}{6}=1/2$.
2. $\cos(\frac{\pi}{2}+99)=\sin 99$.
3. $\cos(-1993)=\cos 1993$.
4. $\sin(-1993)=-\sin 1993$.
5. If $\sin x=1$, then $x=\pi/2$.
6. $\cos(\cos\pi)=\cos(\cos 0)$.
7. $\forall x\in\mathbb{R},\ \sin 2x=2\sin x$.
8. $\exists x\in\mathbb{R}$ such that $\cos x=2$.
9. $\exists x\in\mathbb{R}$ such that $\cos^2 x=\cos x^2$
10. $(\sin x+\cos x)^2=1,\ \forall x\in\mathbb{R}$.
11. $\cos x=\sin(x+\frac{\pi}{2}),\ \forall x\in\mathbb{R}$.
12. $\sin x=\cos(x-\frac{\pi}{2}),\ \forall x\in\mathbb{R}$.
13. $\sin x=\cos(x+\frac{\pi}{2}),\ \forall x\in\mathbb{R}$.
14. $-\frac{1}{2}\le\cos\frac{x}{2}\le\frac{1}{2},\ \forall x\in\mathbb{R}$.
15. $1\le-2\cos\frac{x}{2}+3\le 5,\ \forall x\in\mathbb{R}$.
16. $\exists A\in\mathbb{R}$ such that the equation $\cos x=A$ has exactly 7 real solutions.
17. $\cos^2 x-\sin^2 x=-1,\forall x\in\mathbb{R}$.

622 Problem Given that $\sin t=-0.8$ and $\mathscr{C}(t)$ lies in the fourth quadrant, find $\cos t$.

623 Problem Given that $\cos u=-0.9$ and $\mathscr{C}(u)$ lies in the second quadrant, find $\sin u$.

624 Problem Given that $\sin t=\frac{\sqrt{7}}{5}$ and $\mathscr{C}(t)$ lies in the first quadrant, find $\cos t$.

625 Problem Given that $\cos u=\frac{\sqrt{13}}{4}$ and $\mathscr{C}(u)$ lies in the third quadrant, find $\sin u$.

626 Problem Using the fact that $\frac{5\pi}{6}=\pi-\frac{\pi}{6}$, find $\cos\frac{5\pi}{6}$ and $\sin\frac{5\pi}{6}$.

627 Problem Using the fact that $\frac{3\pi}{4}=\pi-\frac{\pi}{4}$, find $\cos\frac{3\pi}{4}$ and $\sin\frac{3\pi}{4}$.

628 Problem Find $\sin(\dfrac{31\pi}{6})$ and $\cos(\dfrac{31\pi}{6})$.

629 Problem Find $\sin(\dfrac{20\pi}{3})$ and $\cos(\dfrac{20\pi}{3})$.

630 Problem Find $\sin(\dfrac{17\pi}{4})$ and $\cos(\dfrac{17\pi}{4})$.

631 Problem Find $\sin(\dfrac{-15\pi}{4})$ and $\cos(-\dfrac{15\pi}{4})$.

632 Problem Find $\sin(\dfrac{202\pi}{3})$ and $\cos(\dfrac{202\pi}{3})$.

633 Problem Find $\sin(\dfrac{171\pi}{4})$ and $\cos(\dfrac{171\pi}{4})$.

634 Problem If $|\sin\theta|<1$ and $|\cos\theta|>0$, prove that

$$\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}=\frac{2}{\cos\theta}$$

holds identically.

635 Problem Given that

$$\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4},$$

find $\sin\frac{2\pi}{5}$, $\cos\frac{3\pi}{5}$ and $\sin\frac{3\pi}{5}$

636 Problem Given that $\cos\alpha+\sin\alpha=A$ and $\sin\alpha\cos\alpha=B$, prove that $A^2-2B=1$

637 Problem Given that $\cos\alpha+\sin\alpha=A$ and $\sin\alpha\cos\alpha=B$, prove that $\sin^3\alpha+\cos^3\alpha=A-AB$.

638 Problem Demonstrate that for all real numbers x, the following is an identity

$$(\sin x+4\cos x)^2+(4\sin x-\cos x)^2=17$$

639 Problem Prove that $\cos^4 x-\sin^4 x=\cos^2 x-\sin^2 x$ is an identity.

640 Problem Prove that

$$\sqrt{1+2\sin x\cos x}=|\sin x+\cos x|, \quad \forall x\in\mathbb{R}.$$

641 Problem Prove that $\forall x\in\mathbb{R}$,

$$\sin^4 x+\cos^4 x+2(\sin x\cos x)^2=1.$$

642 Problem Prove, by recurrence, that

$$\sin(x+n\pi)=(-1)^n\sin x,$$

and

$$\cos(x+n\pi)=(-1)^n\cos x.$$

643 Problem Prove that $\forall x\in\mathbb{R}$,

$$\sin^6 x+\cos^6 x+3(\sin x\cos x)^2=1.$$

644 Problem Prove that

$$\frac{\sin x-\cos x+1}{\sin x+\cos x-1}=\frac{\sin x+1}{\cos x}$$

$\forall x\in\mathbb{R}$ such that $\sin x+\cos x\neq 1$ and $\cos x\neq 0$.

645 Problem (AHSME 1976) If $\sin x+\cos x=\frac{1}{5}$ and $x\in]0;\pi[$, find $\cos x$ and $\sin x$.

646 Problem (AIME 1983) Find the minimum value of the function

$$x\mapsto\frac{9x^2\sin^2 x+4}{x\sin x}$$

over the interval $]0;\pi[$.

9.3 The Graphs of Sine and Cosine

To obtain the graph of $x \mapsto \sin x$, we traverse the circumference of the unit circle, starting from $(1,0)$, in a levogyrate (counterclockwise) sense, recording each time the abscissa of the point visited. See figure 9.18.

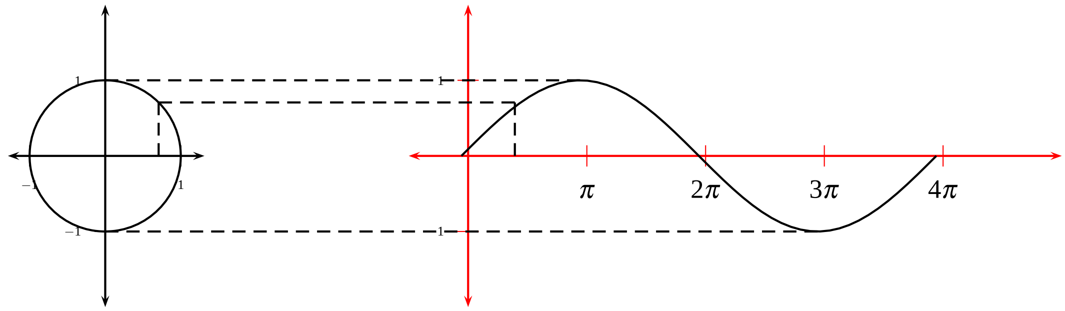


Figure 9.18: The graph $y = \sin x$ for $x \in [0;2\pi[$.

Since $x \mapsto \sin x$ is periodic with period 2π and an odd function, we may now graph $x \mapsto \sin x$ for all values of x. See figure 9.19.

647 Example Give a graphical argument justifying the inequality $\frac{2}{\pi}x \leq \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

Solution: The equation of the straight line joining $(0,0)$ and $(\frac{\pi}{2},1)$ is $y = \frac{2}{\pi}x$. From the graphs below, the graph of $y = \frac{2}{\pi}x$ lies below that of $y = \sin x$ in the interval $[0;\frac{\pi}{2}]$. See figure 9.20.

648 Example Graph $x \mapsto 2\sin x$.

Solution: Recall that if $y = f(x)$, then $y = 2f(x)$ is a distortion of the graph of $y = f(x)$, in which the y-coordinate is doubled. The graph of $x \mapsto 2\sin x$ is shewn in figure 9.21. Observe that $-2 \leq 2\sin x \leq 2$, so the least value that $x \mapsto 2\sin x$ could attain is -2 and the largest value is 2.

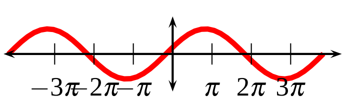


Figure 9.19: The graph of $x \mapsto \sin x$.

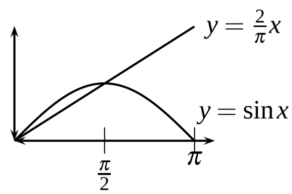


Figure 9.20: $\frac{2}{\pi}x \leq \sin x$ for $0 \leq x \leq \frac{\pi}{2}$.

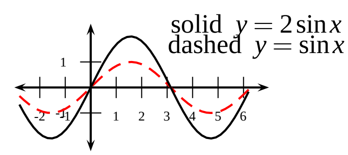


Figure 9.21: $x \mapsto 2\sin x$

649 Definition The average between the least value and largest value of of a periodic function its *amplitude*.

650 Theorem Let $A \in \mathbb{R} \setminus \{0\}$. Then $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \sin Ax \end{array}$ and $\begin{array}{ccc} \mathbb{R} & \to & [-1;1] \\ x & \mapsto & \cos Ax \end{array}$ have period $\dfrac{2\pi}{|A|}$.

Proof: *Since $x \mapsto \sin x$ and $x \mapsto \cos x$ have period 2π, then, if $A \in \mathbb{R} \setminus \{0\}$ is constant, we have*

$$\sin A\left(x + \frac{2\pi}{|A|}\right) = \sin(Ax \pm 2\pi) = \sin Ax,$$

and

$$\cos A\left(x + \frac{2\pi}{|A|}\right) = \cos(Ax \pm 2\pi) = \cos Ax,$$

whence $x \mapsto \sin Ax$ and $x \mapsto \cos Ax$ have period at most $\frac{2\pi}{|A|}$.

Could the period of $x \mapsto \sin Ax, A \neq 0$ and $x \mapsto \cos Ax, A \neq 0$ be smaller than $\dfrac{2\pi}{|A|}$? Assume $0 < P < \dfrac{2\pi}{|A|}$ is a period for these functions. Then $0 < P|A| < 2\pi$ and $\sin Ax = \sin A(x \pm P)$ and $\cos Ax = \cos A(x \pm P)$. In particular,

$$0 = \sin 0 = \sin \pm AP.$$

This means that $|A|P$ is a zero of $x \mapsto \sin x$. Since $0 < |A|P < 2\pi$, we must have $|A|P = \pi$. Now

$$1 = \cos 0 = \cos \pm AP = \cos |A|P = \cos \pi = -1,$$

a contradiction. Thus the period of $x \mapsto \sin Ax, A \neq 0$ and $x \mapsto \cos Ax, A \neq 0$ is precisely $\dfrac{2\pi}{|A|}$, as we wanted to shew. ❑

651 Example Graph $x \mapsto \sin 2x$.

Solution: Since $-1 \leq \sin 2x \leq 1$, the amplitude of $x \mapsto \sin 2x$ is $\frac{1-(-1)}{2} = 1$. The period of $x \mapsto \sin 2x$ is $2\pi \div 2 = \pi$. Recall that if $y = f(2x)$, then $y = f(2x)$ is a distortion of the graph of $y = f(x)$, in which the x-coordinate is halved. The graph of $x \mapsto \sin 2x$ is shewn in figure 9.22.

652 Example Graph $x \mapsto \sin\left(x + \dfrac{\pi}{2}\right)$

Solution: Recall that if $a > 0$ the graph of $x \mapsto f(x+a)$ is a translation a units to the left of the graph $x \mapsto f(x)$. Now, the cosine is an even function, and by the complementary angle identities, we have

$$\cos x = \cos(-x) = \sin\left(\frac{\pi}{2} - (-x)\right) = \sin\left(\frac{\pi}{2} + x\right),$$

and so this graph is the same as that of the cosine function. The graph of $y = \sin(x + \frac{\pi}{2}) = \cos x$ is shewn in in figure 9.23.

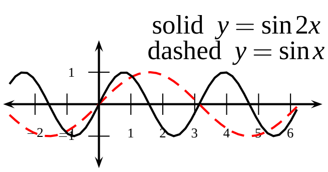


Figure 9.22: $x \mapsto \sin 2x$

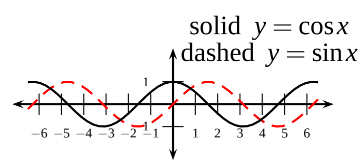


Figure 9.23: $x \mapsto \cos x$

653 Example Give a purely graphical argument (no calculators allowed!) justifying $\cos 1 < \sin 1$.

Solution: At $x = \frac{\pi}{4}$, the graphs of the sine and the cosine coincide. For $x \in [\frac{\pi}{4}; \frac{\pi}{2}]$, the values of the sine increase from $\frac{\sqrt{2}}{2}$ to 1, whereas the values of the cosine decrease from $\frac{\sqrt{2}}{2}$ to 0. Since $\frac{\pi}{4} < 1 < \frac{\pi}{2}$, we have $\cos 1 < \sin 1$.

654 Example Graph $x \mapsto -2\cos\dfrac{x}{2} + 3$

Solution: Since $-1 \leq \cos\frac{x}{2} \leq 1$, we have $1 \leq -2\cos\frac{x}{2} + 3 \leq 5$. The amplitude of $x \mapsto -2\cos\frac{x}{2} + 3$ is therefore $\frac{5-1}{2} = 2$. The period of $x \mapsto -2\cos\dfrac{x}{2} + 3$ is $\dfrac{2\pi}{\frac{1}{2}} = 4\pi$. The graph is shewn in figure 9.24.

655 Example Draw the graph of $x \mapsto -3\sin\frac{x}{4}$. What is the amplitude, period, and where is the first positive real zero of this function?

Solution: Since $-3 \le -3\sin x \le 3$, the amplitude of $x \mapsto -3\sin\frac{x}{4}$ is $\frac{3-(-3)}{2} = 3$. The period is $2\pi \div \frac{1}{4} = 8\pi$, and the first positive zero occurs when $\frac{x}{4} = \pi$, i.e., at $x = 4\pi$. A portion of the graph is shewn in figure 9.25.

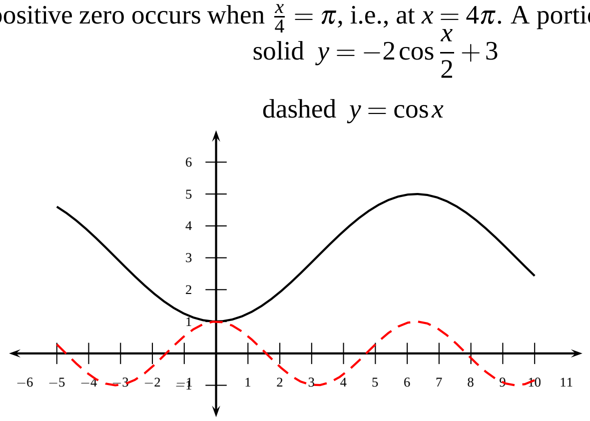


Figure 9.24: $y = -2\cos\frac{x}{2} + 3$

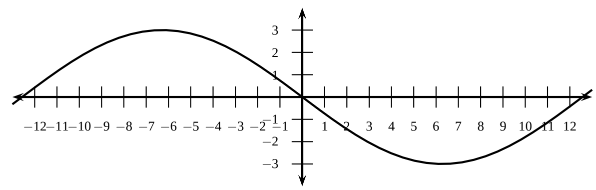


Figure 9.25: $y = -3\sin\frac{x}{4}$

656 Example For which real numbers x is $\log_{\cos x} x$ a real number?

Solution: If $\log_a t$ is defined and real, then $a > 0, a \neq 1$ and $t > 0$. Hence one must have $\cos x > 0, \cos x \neq 1$ and $x > 0$. All this happens when

$$x \in \left]0; \frac{\pi}{2}\right[\cup \left]\frac{3\pi}{2} + 2\pi n; 2\pi(n+1)\right[\cup \left]2\pi(n+1); \frac{5\pi}{2} + 2\pi n\right[,$$

for $n \ge 0, n \in \mathbb{Z}$.

657 Example For which real numbers x is $\log_x \cos x$ a real number?

Solution: In this case one must have $x > 0, x \neq 1$ and $\cos x > 0$. Hence

$$x \in]0;1[\cup \left]1; \frac{\pi}{2}\right[\cup \left]\frac{3\pi}{2} + 2\pi n; \frac{5\pi}{2} + 2\pi n\right[,$$

for $n \ge 0, n \in \mathbb{Z}$.

658 Example Find the period of $x \mapsto \sin 2x + \cos 3x$.

Solution: Let P be the period of $x \mapsto \sin 2x + \cos 3x$. The period of $x \mapsto \sin 2x$ is π and the period of $x \mapsto \cos 3x$ is $\frac{2\pi}{3}$. In one full period of length P, both $x \mapsto \sin 2x$ and $x \mapsto \cos 3x$ must go through an integral number of periods. Thus $P = s\pi = \frac{2\pi t}{3}$, for some positive integers s and t. But then $3s = 2t$. The smallest positive solutions of this is $s = 2, t = 3$. The period sought is then $P = s\pi = 2\pi$.

659 Example How many real numbers x satisfy

$$\sin x = \frac{x}{100}?$$

Solution: Plainly $x = 0$ is a solution. Also, if $x > 0$ is a solution, so is $-x < 0$. So, we can restrict ourselves to positive solutions.
If x is a solution then $|x| = 100|\sin x| \le 100$. So one can further restrict x to the interval $]0;100]$. Decompose $]0;100]$ into 2π-long intervals (the last interval is shorter):

$$]0;100] =]0;2\pi] \cup]2\pi;4\pi] \cup]4\pi;6\pi] \cup \cdots \cup]28\pi;30\pi] \cup]30\pi;100].$$

From the graphs of $y=\sin x, y=x/100$ we see that that the interval $]0;2\pi]$ contains only one solution. Each interval of the form $]2\pi k;\ 2(k+1)\pi], k=1,2,\ldots,14$ contains two solutions. As $31\pi<100$, the interval $]30\pi;100]$ contains a full wave, hence it contains two solutions. Consequently, there are $1+2\cdot 14+2=31$ positive solutions, and hence, 31 negative solutions. Therefore, there is a total of $31+31+1=63$ solutions.

Homework

660 Problem True or False. Use graphical arguments for the numerical premises. No calculators!

1. $x\mapsto\cos 3x$ has period 3.
2. $\cos 3>\sin 1$.
3. The first real zero of $x\mapsto 2\sin x+8$ occurs at $x=\pi$
4. There is a real number x for which the graph of $x\mapsto 8+\cos\frac{x}{10}$ touches the x-axis.

661 Problem Graph portions of the following. Find the amplitude, period, and the location of the first positive real zero, if there is one, of each function.

1. $x\mapsto 3\sin x$
2. $x\mapsto\sin 3x$
3. $x\mapsto\sin(-3x)$
4. $x\mapsto 3\sin 3x$
5. $x\mapsto 3\cos x$
6. $x\mapsto\cos 3x$
7. $x\mapsto\frac{1}{3}\cos x$
8. $x\mapsto\cos\frac{1}{3}x$
9. $x\mapsto -2\cos\frac{1}{3}x+13$
10. $x\mapsto\frac{1}{4}\cos\frac{1}{3}x-10$
11. $x\mapsto|\sin x|$
12. $x\mapsto\sin|x|$

662 Problem Find the period of $x\mapsto\sin 3x+\cos 5x$

663 Problem Find the period of $x\mapsto\sin x+\cos 5x$

664 Problem How many real solutions are there to

$$\sin x=\log_e x\,?$$

665 Problem Let $x\geq 0$. Justify graphically that

$$\sin x\leq x.$$

Your argument must make no appeal to graphing software.

666 Problem Let $x\in\mathbb{R}$. Justify graphically that

$$1-\frac{x^2}{2}\leq\cos x.$$

Your argument must make no appeal to graphing software.

9.4 Inversion

Since $\begin{array}{ccc}\mathbb{R}&\to&[-1;1]\\ x&\mapsto&\sin x\end{array}$ is periodic, it is not injective, and hence it does not have an inverse. We can, however, restrict the domain and in this way obtain an inverse of sorts. The choice of the restriction of the domain is arbitrary, but the interval $[-\frac{\pi}{2};\frac{\pi}{2}]$ is customarily used.

667 Definition The *Principal Sine Function*, $\begin{array}{ccc}[-\frac{\pi}{2};\frac{\pi}{2}]&\to&[-1;+1]\\ x&\mapsto&\text{Sin}\ x\end{array}$ is the restriction of the function $x\mapsto\sin x$ to the interval $[-\frac{\pi}{2};\frac{\pi}{2}]$. With such restriction

$$\begin{array}{ccc}[-\frac{\pi}{2};\frac{\pi}{2}]&\to&[-1;+1]\\ x&\mapsto&\text{Sin}\ x\end{array}$$

is bijective with inverse

$$\begin{array}{ccc}[-1;+1]&\to&[-\frac{\pi}{2};\frac{\pi}{2}]\\ x&\mapsto&\arcsin x\end{array}$$

The graph of $\begin{array}{ccc}[-1;+1]&\to&[-\frac{\pi}{2};\frac{\pi}{2}]\\ x&\mapsto&\arcsin x\end{array}$ is thus symmetric with the graph of $\begin{array}{ccc}[-\frac{\pi}{2};\frac{\pi}{2}]&\to&[-1;+1]\\ x&\mapsto&\text{Sin}\ x\end{array}$ with respect to the line $y=x$. See figure 9.26 for the graph of $y=\arcsin x$. The notation $\sin^{-1}$ is often used to represent arcsin. The function $x\mapsto\arcsin x$ is an odd function, that is,

$$\arcsin(-x)=-\arcsin x,\ \forall x\in[-1;1].$$

Also, $[-\frac{\pi}{2};\frac{\pi}{2}]$ is the smallest interval containing 0 where all the values of $x\mapsto\text{Sin}\ x$ in the interval $[-1;1]$ are attained. Moreover, $\forall(x,y)\in[-1;1]\times[-\frac{\pi}{2};\frac{\pi}{2}], y=\arcsin x\iff x=\sin y$.

1. *Whilst it is true that* $\sin\arcsin x = x, \forall x \in [-1;1]$, *the relation* $\arcsin\sin x = x$ *is not always true. For example,* $\arcsin\sin\frac{7\pi}{6} = \arcsin(-\frac{1}{2}) = -\frac{\pi}{6} \neq \frac{7\pi}{6}$.

2. $\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (\arcsin\circ\sin)(x) \end{array}$ *is a* 2π*-periodic odd function with*

$$(\arcsin\circ\sin)(x) = \begin{cases} x & \text{if } x \in \left[0;\frac{\pi}{2}\right] \\ \pi - x & \text{if } x \in \left[\frac{\pi}{2};\pi\right]. \end{cases}$$

The graph of $x \mapsto (\arcsin\circ\sin)(x)$ *is shewn in figure 9.27.*

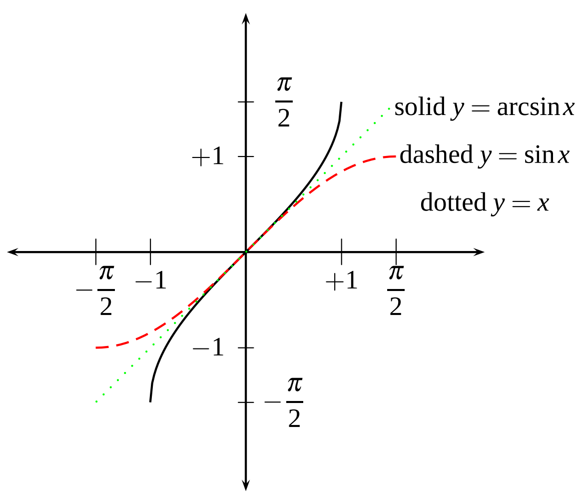


Figure 9.26: $y = \arcsin x$

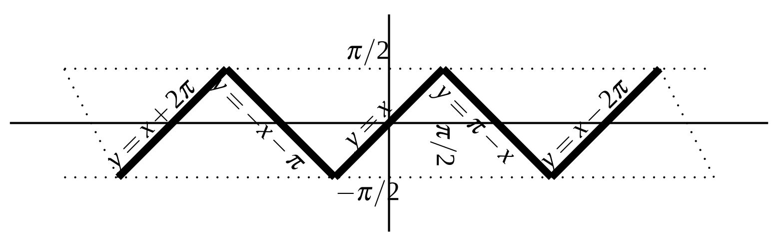


Figure 9.27: $y = (\arcsin\circ\sin)(x)$

668 Theorem The equation

$$\sin x = A$$

has (i) no real solutions if $|A| > 1$, (ii) the infinity of solutions

$$x = (-1)^n \arcsin A + n\pi,\ n \in \mathbb{Z},$$

if $|A| \leq 1$.

Proof: *Since* $-1 \leq \sin x \leq 1$ *for* $x \in \mathbb{R}$*, the first assertion is clear.*

Now, let $|A| \leq 1$*. In figure 9.28 (where we have chosen* $0 \leq A \leq 1$*, the argument for* $-1 \leq A < 0$ *being similar), the first two positive intersections of* $y = A$ *with* $y = \sin x$ *occur at* $x = \arcsin A$ *and* $x = \pi - \arcsin A$*. Since the sine function is periodic with period* 2π*, this means that*

$$x = \arcsin A + 2\pi n,\ n \in \mathbb{Z}$$

and

$$x = \pi - \arcsin A + 2\pi n = -\arcsin A + (2n+1)\pi,\ n \in \mathbb{Z}$$

are the real solutions of this equation. Both relations can be summarised by writing

$$x = (-1)^n \arcsin A + n\pi,\ n \in \mathbb{Z}.$$

This proves the theorem. ❑

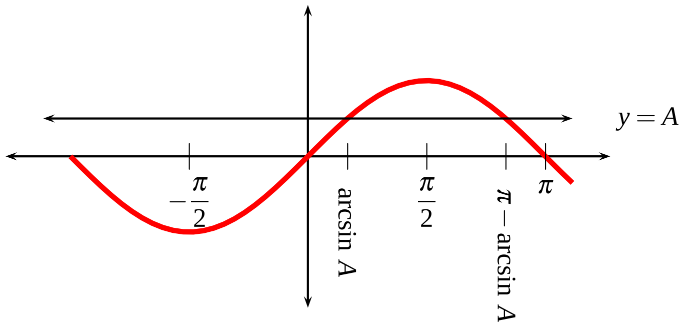


Figure 9.28: The equation $\sin x = A$

669 Example Find all real solutions to $\sin x = -\frac{1}{2}$, and then find all solutions in the interval $[12\pi; \frac{27\pi}{2}]$.

Solution: The general solution to $\sin x = -\frac{1}{2}$ is given by

$$\begin{aligned} x &= (-1)^n \arcsin\left(-\tfrac{1}{2}\right) + n\pi \\ &= (-1)^n \left(-\tfrac{\pi}{6}\right) + n\pi \\ &= (-1)^{n+1}\tfrac{\pi}{6} + n\pi \end{aligned}$$

Now, if

$$12\pi \le (-1)^{n+1}\frac{\pi}{6} + n\pi \le \frac{27\pi}{2}$$

then

$$12 - (-1)^{n+1}\frac{1}{6} \le n \le \frac{27}{2} - (-1)^{n+1}\frac{1}{6}.$$

The smallest $12 - (-1)^{n+1}\frac{1}{6}$ can be is $12 - \frac{1}{6} = \frac{71}{6} > 11$. The largest $\frac{27}{2} - (-1)^{n+1}\frac{1}{6}$ can be is $\frac{27}{2} + \frac{1}{6} = \frac{41}{3} < 14$. So possibly,

$$11 < n < 14,$$

which means that $n = 12$ or $n = 13$.
Testing $n = 12$, $x = -\frac{\pi}{6} + 12\pi = \frac{71\pi}{6}$, which falls outside the interval and $x = \frac{\pi}{6} + 13\pi = \frac{79\pi}{6}$, which falls in the interval. So the only solution in the interval $[12\pi; \frac{27\pi}{2}]$ is $\frac{79\pi}{6}$.

670 Example Find the set of all solutions of

$$\sin\frac{\pi}{x^2} = \frac{1}{2}.$$

Are there any solutions in the interval $]1;3[$?

Solution: We have

$$\begin{aligned} \frac{\pi}{x^2} &= (-1)^n \arcsin\frac{1}{2} + n\pi = (-1)^n\frac{\pi}{6} + n\pi \\ \frac{1}{x^2} &= (-1)^n\frac{1}{6} + n \\ x^2 &= \frac{1}{(-1)^n\frac{1}{6} + n} \\ x^2 &= \frac{6}{(-1)^n + 6n}. \end{aligned}$$

The expression on the right is negative for integers $n \leq -1$. Therefore

$$x = \pm\sqrt{\frac{6}{(-1)^n + 6n}}, n = 0, 1, 2, 3, \ldots.$$

The set of all solutions is thus

$$\left\{-\sqrt{\frac{6}{(-1)^n + 6n}}, \sqrt{\frac{6}{(-1)^n + 6n}}\, n = 0, 1, 2, 3, \ldots\right\}.$$

If

$$1 < \sqrt{\frac{6}{(-1)^n + 6n}} < 3,$$

then

$$1 < \frac{6}{(-1)^n + 6n} < 9,$$

$$\frac{1}{6} < \frac{1}{(-1)^n + 6n} < \frac{3}{2},$$

$$\frac{2}{3} < 6n + (-1)^n < 6,$$

$$\frac{2}{3} - (-1)^n < 6n < 6 - (-1)^n.$$

The smallest $\frac{2}{3} - (-1)^n$ can be is $-\frac{1}{3}$ and the largest $6 - (-1)^n$ can be is 7. Hence we must test n such that $-\frac{1}{3} < 6n < 7$, that is, $n = 0$ and $n = 1$. If $n = 0$, then $x = \sqrt{6} \in]1; 3[$. If $n = 1$, then $x = \sqrt{\frac{6}{5}} \in]1; 3[$. So the solutions belonging to $]1; 3[$ are $x = \sqrt{6}$ and $x = \sqrt{\frac{6}{5}}$.

671 Example Find the set of all real solutions to

$$\sin\frac{2}{2x+1} = \frac{\sqrt{2}}{2}$$

Solution: We have

$$\frac{2}{2x+1} = (-1)^n \arcsin\left(\frac{\sqrt{2}}{2}\right) + \pi n, n \in \mathbb{Z},$$

which is equivalent to each of the following equations

$$\frac{2}{2x+1} = (-1)^n \frac{\pi}{4} + \pi n,$$

$$\frac{2x+1}{2} = \frac{1}{(-1)^n \frac{\pi}{4} + \pi n},$$

$$x + \frac{1}{2} = \frac{1}{(-1)^n \frac{\pi}{4} + \pi n},$$

whence the solution set is

$$\left\{-\frac{1}{2} + \frac{4}{(-1)^n \pi + 4n\pi},\ n \in \mathbb{Z}\right\}.$$

672 Example Find the set of all real solutions to

$$2\sin^2 x - \sin x - 1 = 0.$$

Solution: Factorising

$$0 = 2\sin^2 x - \sin x - 1 = (2\sin x + 1)(\sin x - 1)$$

Hence either $\sin x = -\frac{1}{2}$ and so

$$x = (-1)^n \arcsin\frac{1}{2} + \pi n = (-1)^n(\frac{-\pi}{6}) + \pi n = (-1)^{n+1}\frac{\pi}{6} + \pi n,$$

or $\sin x = 1$ and so

$$x = (-1)^n \arcsin 1 + \pi n = (-1)^n\frac{\pi}{2} + \pi n.$$

The solution set is therefore

$$\left\{(-1)^{n+1}\frac{\pi}{6} + \pi n,\ (-1)^n\frac{\pi}{2} + \pi n, n \in \mathbb{Z}\right\}.$$

673 Definition The *Principal Cosine Function*, $\begin{array}{rcl} [0;\pi] & \to & [-1;1] \\ x & \mapsto & \text{Cos } x \end{array}$ is the restriction of the function $x \mapsto \cos x$ to the interval $[0;\pi]$. With such restriction

$$\begin{array}{rcl} [0;\pi] & \to & [-1;1] \\ x & \mapsto & \text{Cos } x \end{array}$$

is bijective with inverse

$$\begin{array}{rcl} [-1;1] & \to & [0;\pi] \\ x & \mapsto & \arccos x \end{array}.$$

1. *The notation* $\cos^{-1}$ *is often used to represent* arccos.
2. *Whilst it is true that* $\cos\arccos x = x, \forall x \in [-1;1]$, *the relation* $\arccos\cos x = x$ *is not always true. For example,* $\arccos\cos\frac{7\pi}{6} = \arccos(-\frac{\sqrt{3}}{2}) = \frac{5\pi}{6} \neq \frac{7\pi}{6}$.
3. $x \mapsto \arccos x$ *is neither an even nor an odd even function.*
4. $\begin{array}{rcl} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (\arccos\circ\cos)(x) \end{array}$ *is a* 2π*-periodic even function with*

$$(\arcsin\circ\cos)(x) = \begin{cases} x & \text{if } x \in [0;\pi] \\ -x & \text{if } x \in [-\pi;0] \end{cases}.$$

5. $\forall (x,y) \in [-1;1] \times [0;\pi], y = \arccos x \iff x = \cos y.$
6. *The graphs of* $x \mapsto \text{Cos } x$ *and* $x \mapsto \arccos x$ *are symmetric with respect to the line* $y = x$.

The graph of $x \mapsto \arccos x$ is shewn in figure 9.29.

For convenience, we provide the following table.

x	$\arcsin x$	$\arccos x$	x	$\arcsin x$	$\arccos x$
0	0	$\frac{\pi}{2}$			
1	$\frac{\pi}{2}$	0	-1	$-\frac{\pi}{2}$	π
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{2\pi}{3}$
$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$	$\frac{3\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$	$\frac{5\pi}{6}$

674 Theorem The equation
$$\cos x = A$$
has (i) no real solutions if $|A| > 1$, (ii) the infinity of solutions
$$x = \pm \arccos A + 2n\pi,\ n \in \mathbb{Z},$$
if $|A| \leq 1$.

Proof: *Since $-1 \leq \cos x \leq 1$ for $x \in \mathbb{R}$, the first assertion is clear. Now, let $|A| \leq 1$. In figure 9.30 (where we have chosen $0 \leq A \leq 1$, the argument for $-1 \leq A < 0$ being similar), the two intersections of $y = A$ with $y = \cos x$ closest to $x = 0$ occur at $x = \arccos A$ and $x = -\arccos A$. Since the cosine function is periodic with period 2π, this means that*
$$x = \arccos A + 2\pi n,\ n \in \mathbb{Z}$$
and
$$x = -\arccos A + 2\pi n,\ n \in \mathbb{Z}$$
are the real solutions of this equation. Both relations can be summarised by writing
$$x = \pm \arccos A + 2n\pi,\ n \in \mathbb{Z}.$$
This proves the theorem. ❑

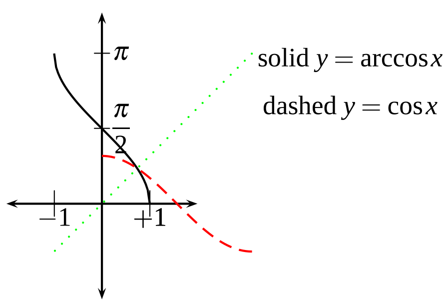


Figure 9.29: $y = \arccos x$

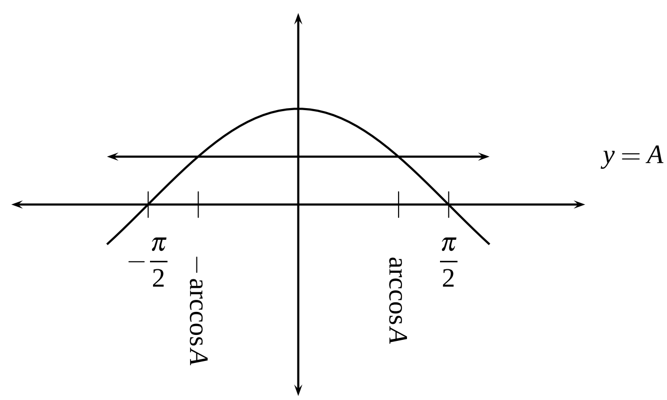


Figure 9.30: The equation $\cos x = A$

675 Example Find the set of all real solutions to
$$2\sin^2 x + 3\cos x - 3 = 0.$$

Solution: Since the equation has a cosine to the first power, we write the equation in terms of cosine only, obtaining
$$\begin{aligned} 0 &= 2\sin^2 x + 3\cos x - 3 \\ &= 2(1-\cos^2 x) + 3\cos x - 3 \\ &= 2\cos^2 x - 3\cos x + 1 \\ &= (2\cos x - 1)(\cos x - 1) \end{aligned}$$
Thus either $\cos x = \frac{1}{2}$, in which case
$$x = \pm \arccos \frac{1}{2} + 2\pi n = \pm \frac{\pi}{3} + 2\pi n$$

or $\cos x = 1$ in which case

$$x = \pm\arccos 1 + 2\pi n = 2\pi n.$$

The solution set is

$$\left\{\pm\frac{\pi}{3} + 2\pi n, 2\pi n,\ n \in \mathbb{Z}\right\}.$$

676 Example Find the solutions of the equation

$$\log_{\sqrt{2}\sin x}(1 + \cos x) = 2$$

in the interval $[0; 2\pi]$.

Solution: If the logarithmic expression is to make sense, then $\sqrt{2}\sin x > 0$, $\sqrt{2}\sin x \neq 1$ and $1 + \cos x > 0$. For this we must have

$$x \in \left]0; \frac{\pi}{4}\right[\cup \left]\frac{\pi}{4}; \frac{3\pi}{4}\right[\cup \left]\frac{3\pi}{4}; \pi\right[.$$

Now, if x belongs to this set

$$\log_{\sqrt{2}\sin x}(1 + \cos x) = 2 \iff 2\sin^2 x = 1 + \cos x.$$

Using $\sin^2 x = 1 - \cos^2 x$, the last equality occurs if and only if

$$(2\cos x - 1)(\cos x + 1) = 0.$$

If $\cos x + 1 = 0$, then $x = \pi$, a value that must be discarded (why?). If $\cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$, which is the only solution in $[0; 2\pi]$.

677 Example Find the set of all the real solutions to

$$2^{\sin^2 x} + 5(2^{\cos^2 x}) = 7$$

Solution: Observe that

$$\begin{aligned} 2^{\sin^2 x} + 5(2^{\cos^2 x}) - 7 &= 2^{\sin^2 x} + 5(2^{1-\sin^2 x}) - 7 \\ &= 2^{\sin^2 x} + 5(2^1 \cdot 2^{-\sin^2 x}) - 7 \\ &= 2^{\sin^2 x} + \left(\frac{10}{2^{\sin^2 x}}\right) - 7 \\ &= u + \frac{10}{u} - 7. \end{aligned}$$

with $u = 2^{\sin^2 x}$. From this, $0 = u^2 - 7u + 10 = (u-5)(u-2)$. Thus either $u = 2$,, meaning $2^{\sin^2 x} = 2$ which is to say $\sin x = \pm 1$ or $x = (-1)^n(\frac{\pm\pi}{2}) + n\pi$. When $2^{\sin^2 x} = 5$ one sees that $\sin^2 x = \log_2 5$. Since the sinistral side of the last equality is at most 1 and its dextral side is greater than 1, there are no real roots in this instance. The solution set is thus

$$\left\{(-1)^n(\frac{\pm\pi}{2}) + n\pi,\ n \in \mathbb{Z}\right\}.$$

678 Example Find all the real solutions of the equation

$$\cos^{2000} x - \sin^{2000} x = 1.$$

Solution: Transposing

$$\cos^{2000} x = \sin^{2000} x + 1.$$

The dextral side is ≥ 1 and the sinistral side is ≤ 1. Thus equality is only possible if both sides are equal to 1, which entails that $\cos x = 1$ or $\cos x = -1$, whence $x = \pi n, n \in \mathbb{Z}$.

679 Example Find all the real solutions of the equation

$$\cos^{2001} x - \sin^{2001} x = 1.$$

Solution: Since $|\cos x| \leq 1$ and $|\sin x| \leq 1$, we have

$$\begin{aligned} 1 &= \cos^{2001} x - \sin^{2001} x \\ &= \cos^{2001}(-x) + \sin^{2001}(-x) \\ &\leq |\cos^{2001}(-x)| + |\sin^{2001}(-x)| \\ &= |\cos^{1999}(-x)|\cos^2(-x) + |\sin^{1999}(-x)|\sin^2(-x) \\ &\leq \cos^2(-x) + \sin^2(-x) \\ &= 1. \end{aligned}$$

The inequalities are tight, and so equality holds throughout. The first inequality above is true if and only if $\cos(-x) \geq 0$ and $\sin(-x) \geq 0$. The second inequality is true if and only if $|\cos(-x)| = 1$ or $|\sin(-x)| = 1$. Hence we must have either $\cos(-x) = 1$ or $\sin(-x) = 1$.This means $x = 2n\pi$ or $x = -\frac{\pi}{2} + 2n\pi$ where $n \in \mathbb{Z}$.

680 Example What is $\sin \arccos \frac{3}{4}$?

Solution: Put $t = \arccos \frac{3}{4}$. Then $\frac{3}{4} = \cos t$ with $t \in [0; \frac{\pi}{2}]$. In the interval $[0; \frac{\pi}{2}]$, $\sin t$ is positive. Hence

$$\sin t = \sqrt{1 - \cos^2 y} = \sqrt{1 - \left(\frac{3}{4}\right)^2} = \frac{\sqrt{7}}{4}.$$

681 Example What is $\sin \arccos(-\frac{3}{7})$?

Put $t = \arccos(-\frac{3}{7})$. Then $-\frac{3}{7} = \cos t$ with $y \in [\frac{\pi}{2}; \pi]$. In the interval $[\frac{\pi}{2}; \pi]$, $\sin t$ is positive. Hence

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - \left(-\frac{3}{7}\right)^2} = \frac{2\sqrt{10}}{7}.$$

682 Example Let $x \in]-\frac{1}{5}; 0[$. Express $\sin \arccos 5x$ as a function of x.

Solution: First notice that $5x \in]-1; 0[$, which means that $\arccos 5x \in]\frac{\pi}{2}; \pi[$, an interval where the sine is positive. Put $t = \arccos 5x$. Then $5x = \cos t$. Finally,

$$\sin t = \sqrt{1 - \cos^2 t} = \sqrt{1 - 25x^2}.$$

683 Example Prove that

$$\arcsin x + \arccos x = \frac{\pi}{2}, \forall x \in [-1; 1].$$

Solution: By the complementary angle identity for the cosine,

$$\cos\left(\frac{\pi}{2} - \arcsin x\right) = \sin(\arcsin x) = x.$$

Since $-\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2}$, we have $\frac{\pi}{2} - \arcsin x \in [0; \pi]$. This means that

$$\cos\left(\frac{\pi}{2} - \arcsin x\right) = x \Longleftrightarrow \frac{\pi}{2} - \arcsin x = \arccos x,$$

whence the desired result follows.

Homework

684 Problem True or False.

1. $\arcsin\frac{\pi}{2} = 1$.
2. If $\arccos x = -\frac{1}{2}$, then $x = -\frac{\pi}{3}$.
3. If $\arcsin x \geq 0$ then $x \in [0; \frac{\pi}{2}]$.
4. $\arccos\,\cos(-\frac{\pi}{3}) = \frac{\pi}{3}$.
5. $\arccos\,\cos(-\frac{\pi}{6}) = -\frac{\pi}{6}$.
6. $\arcsin\frac{1}{2000} + \arccos\frac{1}{2000} = \frac{\pi}{2}$.
7. $\exists x \in \mathbb{R}$ such that $\arcsin x > 1$.
8. $-1 \leq \arccos x \leq 1$, $\forall x \in \mathbb{R}$.
9. $\sin\arcsin x = x$, $\forall x \in \mathbb{R}$.
10. $\arccos(\cos x) = x$, $\forall x \in [0; \pi]$.

685 Problem Find all the real solutions to $2\sin x + 1 = 0$ in the interval $[-\pi; \pi]$.

686 Problem Find the set of all real solutions to

$$\sin\left(3x - \frac{\pi}{4}\right) = 0.$$

687 Problem Find the set of all real solutions of the equation

$$-2\sin^2 x - \cos x + 1 = 0.$$

688 Problem Find all the real solutions to $\sin 3x = -1$. Find all the solutions belonging to the interval $[98\pi; 100\pi]$.

689 Problem Find the set of all real solutions to

$$5\cos^2 x - 2\cos x - 7 = 0.$$

690 Problem Find the set of all real solutions to

$$\sin x \cos x = 0.$$

691 Problem Find the set of all real solutions to

$$\cos 3x = \frac{4}{3}.$$

692 Problem Find the set of all real solutions to

$$4\sin^2 2x - 3 = 0.$$

693 Problem Find all real solutions belonging to the interval $[-2; 2]$, if any, to the following equations.

1. $4\sin^2 x - 3 = 0$
2. $2\sin^2 x = 1$
3. $\cos\frac{2x}{3} = -\frac{\sqrt{3}}{2}$
4. $\sin\frac{3}{x} = 1$
5. $\dfrac{1+\sin x}{1-\cos x} = 0$

694 Problem Find $\sin\arccos\frac{1}{3}$.

695 Problem Find $\cos\arcsin(-\frac{2}{3})$.

696 Problem Find $\sin\arccos(-\frac{2}{3})$.

697 Problem Find $\arcsin(\sin 5)$; $\arccos(\cos 10)$

698 Problem Find all the real solutions of the following equations.

1. $\cos x + \dfrac{1}{\cos x} = \frac{3}{2}$.
2. $2\cos^3 x + \cos^2 x - 2\cos x - 1 = 0$.
3. $6\cos^2\left(5x - \dfrac{\pi}{3}\right) - \cos\left(5x - \dfrac{\pi}{3}\right) = 2$.
4. $4\cos^2 x - 2(\sqrt{2}+1)\cos x + \sqrt{2} = 0$.
5. $4\cos^4 x - 17\cos^2 x + 4 = 0$.
6. $(2\cos x + 1)^2 - 4\cos^2 x + (\sin x)(2\cos x + 1) + 1 = 0$.
7. $4\sin^2 x - 2(\sqrt{3} - \sqrt{2})\sin x = \sqrt{6}$.
8. $-2\sin^2 x + 19|\sin x| + 10 = 0$.

699 Problem Demonstrate that

$$\arccos\,x + \arccos(-x) = \pi, \quad \forall x \in [-1; 1],$$

$$\arcsin x = -\arcsin(-x), \quad \forall x \in [-1; 1].$$

700 Problem Shew that

$$\arcsin x = \arccos\sqrt{1-x^2}, \ \forall x \in [0; 1],$$

$$\arccos x = \arcsin\sqrt{1-x^2}, \ \forall x \in [0; 1].$$

701 Problem Let $0 < x < \frac{1}{3}$. Find $\cos\,\arcsin 3x$ and $\cos\,\arccos 3x$ as functions of x.

702 Problem Let $-\frac{1}{2} < x < 0$. Find $\sin\,\arcsin 2x$ and $\sin\,\arccos 2x$ as functions of x.

703 Problem Find real constants a, b such that

$$(\arcsin\circ\sin)(x) = ax + b, \ \forall x \in [\frac{99\pi}{2}; \frac{101\pi}{2}].$$

704 Problem Prove that $\begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & (\arccos\circ\cos)(x) \end{array}$ is a 2π-periodic even function and graph a portion of this function for $x \in [-2\pi; 2\pi]$.

9.5 The Circular Functions

We define the *tangent*, *secant*, *cosecant* and *cotangent* of $x \in \mathbb{R}$ as follows.

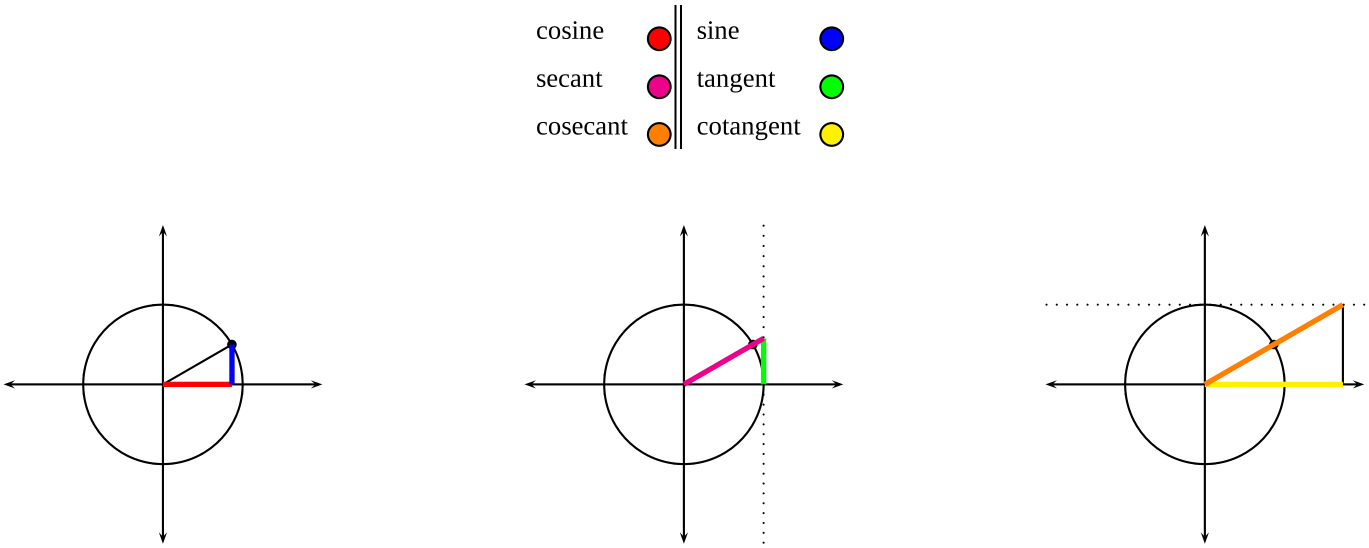
$$\tan x = \frac{\sin x}{\cos x}, \ x \neq \frac{\pi}{2} + \pi n, \ n \in \mathbb{Z}, \tag{9.16}$$

$$\sec x = \frac{1}{\cos x}, \ x \neq \frac{\pi}{2} + \pi n, \ n \in \mathbb{Z}, \tag{9.17}$$

$$\csc x = \frac{1}{\sin x}, \ x \neq \pi n, \ n \in \mathbb{Z}, \tag{9.18}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}, \ x \neq \pi n, \ n \in \mathbb{Z}. \tag{9.19}$$

The circles below have all radius 1.



1. *The image of* $x \mapsto \tan x$ *over its domain* $\mathbb{R} - \{\frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\}$ *is* $\mathbb{R}$.
2. *The image of* $x \mapsto \cot x$ *over its domain* $\mathbb{R} - \{\pi n,\ n \in \mathbb{Z}\}$ *is* $\mathbb{R}$.
3. *The image of* $x \mapsto \sec x$ *over its domain* $\mathbb{R} - \{\frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\}$ *is* $]-\infty; -1] \cup [1; +\infty[$.
4. *The image of* $x \mapsto \csc x$ *over its domain* $\mathbb{R} - \{\pi n,\ n \in \mathbb{Z}\}$ *is* $]-\infty; -1] \cup [1; +\infty[$.

705 Example Given that $\tan x = -3$ and $\mathscr{C}(x)$ lies in the fourth quadrant, find $\sin x$ and $\cos x$.

Solution: In the fourth quadrant $\sin x < 0$ and $\cos x > 0$. Now, $-3 = \tan x = \frac{\sin x}{\cos x}$ entails $\sin x = -3\cos x$. As $\sin^2 x + \cos^2 x = 1$, One gathers $9\cos^2 x + \cos^2 x = 1$ or $\cos^2 x = \frac{1}{10}$. Choosing the positive root, $\cos x = \frac{\sqrt{10}}{10}$. Finally,

$$\sin x = -3\cos x = -\frac{3\sqrt{10}}{10}.$$

706 Example Given that $\cot x = 4$ and $\mathscr{C}(x)$ lies in the third quadrant, find the values of $\tan x, \sin x, \cos x, \csc x, \sec x$.

Solution: From $\cot x = 4$, we have $\cos x = 4\sin x$. Using this and $\sin^2 x + \cos^2 x = 1$, we gather $\sin^2 x + 16\sin^2 x = 1$, and since $\mathscr{C}(x)$ lies in the third quadrant, $\sin x = -\frac{\sqrt{17}}{17}$. Moreover, $\cos x = 4\sin x = -\frac{4\sqrt{17}}{17}$. Finally, $\tan x = \frac{1}{\cot x} = \frac{1}{4}$, $\csc x = \frac{1}{\sin x} = -\sqrt{17}$ and $\sec x = \frac{1}{\cos x} = -\frac{\sqrt{17}}{4}$.

707 Theorem The function $\begin{array}{ccc} \mathbb{R} - \{\dfrac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$ is an odd function.

Proof: *If* $x \neq \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}$

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = -\frac{\sin x}{\cos x} = -\tan x,$$

which proves the assertion. ❏

708 Theorem The function $\begin{array}{ccc} \mathbb{R} - \{\dfrac{\pi}{2} + \pi n,\ n \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$ is periodic with period π.

Proof: *Since*

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \tan x,$$

the period is at most π.

Assume now that $0 < P < \pi$ *is a period for* $x \mapsto \tan x$. *Then* $\tan x = \tan(x+P)$ $\forall x \in \mathbb{R}$ *and in particular,*

$$0 = \tan 0 = \tan P = \frac{\sin P}{\cos P},$$

which entails that $\sin P = 0$. *But then* P *would be a positive zero of* $x \mapsto \sin x$ *smaller than* π, *a contradiction. Hence the period of* $x \mapsto \tan x$ *is exactly* π, *which completes the proof.* ❑

How to graph $x \mapsto \tan x$? We start with $x \in [0; \frac{\pi}{2}[$ and then appeal to theorem 707 and theorem 708 to extend this construction for all $x \in \mathbb{R}$.
In figure 9.31, choose B such that the measure of arc AB (measured counterclockwise) be x. Point $A = (1,0)$, and point $B = (\sin x, \cos x)$. Since points B and $(1,t)$ are collinear, the gradient (slope) of the line joining $(0,0)$ and B is the same as that joining $(0,0)$ and $(1,t)$. Computing gradients, we have

$$\frac{\sin x - 0}{\cos x - 0} = \frac{t-0}{1-0},$$

whence $t = \tan x$. We have thus produced a line segment measuring $\tan x$. If we let x vary from 0 to $\pi/2$ we obtain the graph of $x \mapsto \tan x$ for $x \in [0; \frac{\pi}{2}[$.

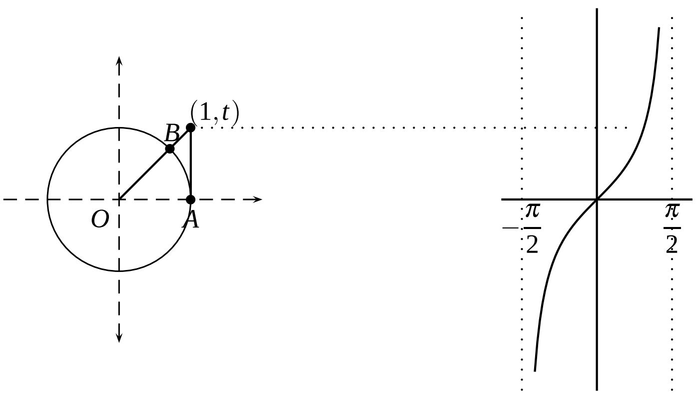


Figure 9.31: Construction of the graph of $x \mapsto \tan x$ for $x \in [0, \frac{\pi}{2}[$.

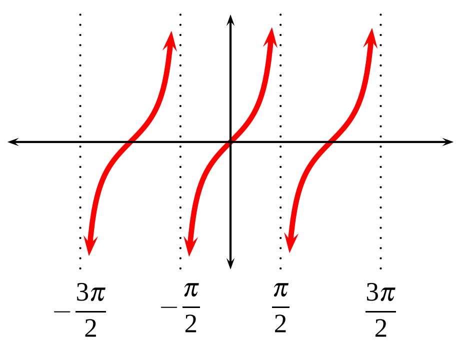


Figure 9.32: $y = \tan x$

Since $\cos x = 0$ at $x = \frac{\pi}{2}(2n+1)$, $n \in \mathbb{Z}$, $x \mapsto \tan x$ has poles at the points $x = \frac{\pi}{2}(2n+1)$, $n \in \mathbb{Z}$. The graph of $x \mapsto \tan x$ is shewn in figure 9.32.
We now define the Principal Tangent function and the arctan function.

709 Definition The *Principal Tangent Function*, $x \mapsto \text{Tan } x$ is the restriction of the function $x \mapsto \tan x$ to the interval $]-\frac{\pi}{2}; \frac{\pi}{2}[$. With such restriction

$$\begin{array}{ccc}]-\dfrac{\pi}{2}; \dfrac{\pi}{2}[& \to & \mathbb{R} \\ x & \mapsto & \text{Tan } x \end{array}$$

is bijective with inverse

$$\begin{array}{ccc} \mathbb{R} & \to &]-\dfrac{\pi}{2}; \dfrac{\pi}{2}[\\ x & \mapsto & \arctan x \end{array}$$

The graph of $x \mapsto \arctan x$ is shewn in figure 9.33. Observe that the lines $y = \pm\frac{\pi}{2}$ are asymptotes to $x \mapsto \arctan x$.

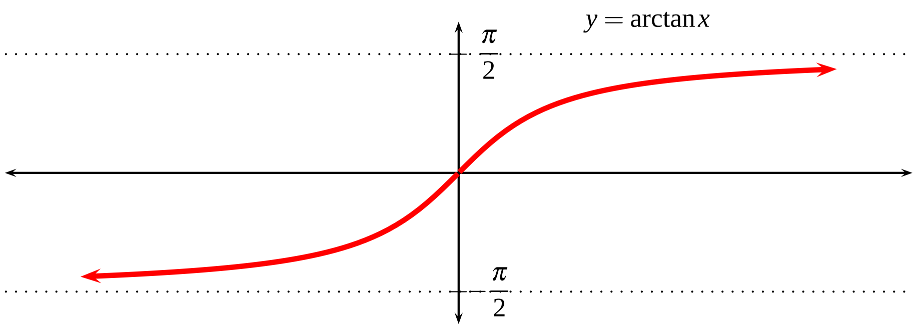


Figure 9.33: $y = \arctan x$

1. $\forall x \in \mathbb{R},\ \tan(\arctan(x)) = x$.
2. $\begin{array}{rcl} \mathbb{R} - \{\frac{\pi}{2} + n\pi, n \in \mathbb{Z}\} & \to & \mathbb{R} \\ x & \mapsto & (\arctan \circ \tan)(x) \end{array}$ *is an odd π-periodic function.*

710 Theorem The equation

$$\tan x = A,\ A \in \mathbb{R}$$

has the infinitely many solutions

$$x = \arctan A + n\pi, n \in \mathbb{Z}.$$

Proof: *Since the graph of $x \mapsto \tan x$ is increasing in $]-\frac{\pi}{2};\frac{\pi}{2}[$, it intersects the graph of $y = A$ at exactly one point,*

$$\tan x = A \Longrightarrow x = \arctan A$$

if $x \in]-\frac{\pi}{2};\frac{\pi}{2}[$. Since $x \mapsto \tan x$ is periodic with period π, each of the points

$$x = \arctan A + n\pi, n \in \mathbb{Z}$$

is also a solution. ❑

711 Example Solve the equation

$$\tan^2 x = 3$$

Solution: Either $\tan x = \sqrt{3}$ or $\tan x = -\sqrt{3}$. This means that $x = \arctan\sqrt{3} + \pi n = \frac{\pi}{3} + \pi n$ or $x = \arctan(-\sqrt{3}) + \pi n = -\frac{\pi}{3} + \pi n$. We may condense this by writing $x = \pm\frac{\pi}{3} + \pi n, n \in \mathbb{Z}$.

712 Example Solve the equation $(\tan x)^{\sin x} = (\cot x)^{\cos x}$.

Solution: For the tangent and cotangent to be defined, we must have $x \neq \frac{n\pi}{2}, n \in \mathbb{Z}$. Then

$$(\tan x)^{\sin x} = (\cot x)^{\cos x} = \frac{1}{(\tan x)^{\cos x}}$$

implies

$$(\tan x)^{\sin x + \cos x} = 1.$$

Thus either $\tan x = 1$, in which case $x = \frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}$ or $\sin x + \cos x = 0$, which implies $\tan x = -1$, but this does not give real values for the expressions in the original equation. The solution is thus

$$x = \frac{\pi}{4} + n\pi,\ n \in \mathbb{Z}.$$

713 Example Find $\sin\arctan\frac{2}{3}$.

Solution: Put $t = \arctan\frac{2}{3}$. Then $\frac{2}{3} = \tan t, t \in]0;\frac{\pi}{2}[$ and thus $\sin t > 0$. We have $\frac{3}{2}\sin t = \cos t$. As

$$1 = \cos^2 t + \sin^2 t = \frac{9}{4}\sin^2 t + \sin^2 t,$$

we gather that $\sin^2 t = \frac{4}{13}$. Taking the positive square root $\sin t = \frac{2}{13}$.

714 Example Find the exact value of $\tan\arccos(-\frac{1}{5})$.

Solution: Put $t = \arccos(-\frac{1}{5})$. As the arccosine of a negative number, $t \in [\frac{\pi}{2}, \pi]$. Now, $\cos t = -\frac{1}{5}$, and so

$$\sin t = \sqrt{1 - \left(-\frac{1}{5}\right)^2} = \sqrt{\frac{24}{25}} = \frac{2\sqrt{6}}{5}.$$

We deduce that $\tan t = \frac{\sin t}{\cos t} = -2\sqrt{6}$.

715 Example Let $x \in [0;1[$. Prove that

$$\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}}.$$

Solution: Since $x \in [0;1[$, $\arcsin x \in [0;\frac{\pi}{2}[$. Put $t = \arcsin x$, then $\sin t = x$, and $\cos t > 0$ since $t \in [0;\frac{\pi}{2}[$. Now, $\cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}$, and

$$\tan t = \frac{\sin t}{\cos t} = \frac{x}{\sqrt{1-x^2}}.$$

Since $t \in [0;\frac{\pi}{2}[$ this implies that

$$t = \arctan\frac{x}{\sqrt{1-x^2}},$$

from where the desired equality follows.

716 Theorem The following Pythagorean-like Relation holds.

$$\tan^2 x + 1 = \sec^2 x,\ \forall x \in \mathbb{R} \setminus \{(2n+1)\frac{\pi}{2},\ n \in \mathbb{Z}\}. \tag{9.20}$$

Proof: *This immediately follows from* $\sin^2 x + \cos^2 x = 1$ *upon dividing through by* $\cos^2 x$. ❑

717 Example Given that $\tan x + \cot x = a$, write $\tan^3 x + \cot^3 x$ as a polynomial in a.

Solution: Using the fact that $\tan x \cot x = 1$, and the Binomial Theorem:

$$\begin{aligned}(\tan x + \cot x)^3 &= \tan^3 x + 3\tan^2 x \cot x + 3\tan x \cot^2 x + \cot^3 x \\ &= \tan^3 x + \sin^3 x + 3\tan x \cot x(\tan x + \cot x) \\ &= \tan^3 x + \sin^3 x + 3(\tan x + \cot x)\end{aligned}$$

It follows that

$$\tan^3 x + \cot^3 x = (\tan x + \cot x)^3 - 3(\tan x + \cot x) = a^3 - 3a.$$

Aliter: Observe that $a^2 = (\tan x + \cot x)^2 = \tan^2 x + \cot^2 x + 2$, hence $\tan^2 x + \cot^2 x = a^2 - 2$. Factorising the sum of cubes

$$\tan^3 x + \cot^3 x = (\tan x + \cot x)(\tan^2 x - 1 + \cot^2 x) = a(a^2 - 1 - 2)$$

which equals $a^3 - 3a$, as before.

718 Example Prove that

$$\frac{2\sin y + 3}{2\tan y + 3\sec y} = \cos y,$$

whenever the expression on the sinistral side be defined.

Solution: Decomposing the tangent and the secant as cosines we obtain,

$$\begin{aligned} \frac{2\sin y + 3}{2\tan y + 3\sec y} &= \frac{2\sin y + 3}{2\frac{\sin y}{\cos y} + \frac{3}{\cos y}} \\ &= \frac{2\sin y\cos y + 3\cos y}{2\sin y + 3} \\ &= \frac{(\cos y)(2\sin y + 3)}{2\sin y + 3} \\ &= \cos y, \end{aligned}$$

as we wished to shew.

719 Example Prove the identity

$$\frac{\tan A + \tan B}{\sec A - \sec B} = \frac{\sec A + \sec B}{\tan A - \tan B},$$

whenever the expressions involved be defined.

Solution: We have

$$\begin{aligned} \frac{\tan A + \tan B}{\sec A - \sec B} &= \left(\frac{\tan A + \tan B}{\sec A - \sec B}\right)\left(\frac{\tan A - \tan B}{\sec A + \sec B}\right)\left(\frac{\sec A + \sec B}{\tan A - \tan B}\right) \\ &= \left(\frac{\tan^2 A - \tan^2 B}{\sec^2 A - \sec^2 B}\right)\left(\frac{\sec A + \sec B}{\tan A - \tan B}\right) \\ &= \left(\frac{(\sec^2 A - 1) - (\sec^2 B - 1)}{\sec^2 A - \sec^2 B}\right)\left(\frac{\sec A + \sec B}{\tan A - \tan B}\right) \\ &= \frac{\sec A + \sec B}{\tan A - \tan B}, \end{aligned}$$

as we wished to shew.

720 Example Given that $\sin A + \csc A = T$, express $\sin^4 A + \csc^4 A$ as a polynomial in T.

Solution: First observe that

$$T^2 = (\sin A + \csc A)^2 = \sin^2 A + \csc^2 A + 2\sin A\csc A,$$

hence

$$\sin^2 A + \csc^2 A = T^2 - 2.$$

By the Binomial Theorem

$$\begin{aligned} T^4 &= (\sin A + \csc A)^4 \\ &= \sin^4 A + 4\sin^3 A\csc A + 6\sin^2 A\csc^2 A + 4\sin A\csc^3 A + \csc^4 A \\ &= \sin^4 A + \csc^4 A + 6 + 4(\sin^2 A + \csc^2 A) \\ &= \sin^4 A + \csc^4 A + 6 + 4(T^2 - 2), \end{aligned}$$

whence $\sin^4 A + \csc^4 A = T^4 - 4T + 2$.

Homework

721 Problem Complete the following table.

x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$	x	$\sin x$	$\cos x$	$\tan x$	$\cot x$	$\sec x$	$\csc x$
0	0	1	0	∞	1	∞	$\frac{\pi}{6}$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
$\frac{\pi}{4}$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$	$\frac{\pi}{3}$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞	0	∞	1	$\frac{2\pi}{3}$						
$\frac{3\pi}{4}$							$\frac{5\pi}{6}$						
π							$\frac{7\pi}{6}$						
$\frac{5\pi}{4}$							$\frac{4\pi}{3}$						
$\frac{3\pi}{2}$							$\frac{5\pi}{3}$						
$\frac{7\pi}{4}$							$\frac{11\pi}{6}$						
2π													

722 Problem True or False.

1. $\tan x = \cot \frac{1}{x}$, $\forall x \in \mathbb{R} \setminus \{0\}$.
2. $\exists x \in \mathbb{R}$ such that $\sec x = \frac{1}{2}$.
3. $\arctan 1 = \frac{\arcsin 1}{\arccos 1}$.
4. $x \mapsto \tan 2x$ has period π.

723 Problem Given that $\csc x = -1.5$ and $\mathscr{C}(x)$ lies on the fourth quadrant, find $\sin x, \cos x$ and $\tan x$.

724 Problem Given that $\tan x = 2$ and $\mathscr{C}(x)$ lies on the third quadrant, find $\sin x$ and $\cos x$.

725 Problem Given that $\sin x = t^2$ and $\mathscr{C}(x)$ lies in the second quadrant, find $\cos x$ and $\tan x$.

726 Problem Let $x < -1$. Find $\sin \operatorname{arcsec} x$ as a function of x.

727 Problem Find $\cos \arctan(-\frac{1}{3})$.

728 Problem Find $\arctan(\tan(-6))$, $\operatorname{arccot}(\cot(-10))$.

729 Problem Give a sensible definition of the Principal Cotangent, Secant, and Cosecant functions, and their inverses. Graph each of these functions.

730 Problem Solve the following equations.

1. $\sec^2 x - \sec x - 2 = 0$
2. $\tan x + \cot x = 2$
3. $\tan 4x = 1$
4. $2\sec^2 x + \tan^2 x - 3 = 0$
5. $2\cos x - \sin x = 0$
6. $\tan(x + \frac{\pi}{3}) = 1$
7. $3\cot^2 x + 5\csc x + 1 = 0$
8. $2\sec^2 x = 5\tan x$
9. $\tan^2 x + \sec^2 x = 17$
10. $6\cos^2 x + \sin x - 5 = 0$

731 Problem Prove that

$$\tan x = \cot\left(\frac{\pi}{2} - x\right),$$

$$\cot x = \tan\left(\frac{\pi}{2} - x\right).$$

732 Problem Prove that if $x \in \mathbb{R}$ then

$$\arctan x + \operatorname{arccot} \frac{1}{x} = \frac{\pi}{2}\operatorname{sgn}(x),$$

where $\operatorname{sgn}(x) = -1$ if $x < 0$, $\operatorname{sgn}(x) = 1$ if $x > 0$, and $\operatorname{sgn}(0) = 0$.

733 Problem Graph $x \mapsto (\arctan \circ \tan)(x)$

734 Problem Let $x \in]0;1[$. Prove that

$$\arcsin x = \operatorname{arccot} \frac{\sqrt{1-x^2}}{x}.$$

735 Problem Let $x \in]0;1[$. Prove that

$$\arccos x = \arctan \frac{\sqrt{1-x^2}}{x} = \operatorname{arccot} \frac{x}{\sqrt{1-x^2}}.$$

736 Problem Let $x > 0$. Prove that

$$\arctan x = \arcsin \frac{x}{\sqrt{1+x^2}} = \arccos \frac{1}{\sqrt{1+x^2}}.$$

737 Problem Let $x > 0$. Prove that

$$\operatorname{arccot} x = \arcsin \frac{1}{\sqrt{1+x^2}} = \arccos \frac{x}{\sqrt{1+x^2}}.$$

738 Problem Prove the following identities. Assume, whenever necessary, that the given expressions are defined.

1. $$\sin x \tan x = \sec x - \cos x$$
2. $\tan^3 x + 1 = (\tan x + 1)(\sec^2 x - \tan x)$
3. $1 + \tan^2 x = \dfrac{1}{2 - 2\sin x} + \dfrac{1}{2 + 2\sin x}$
4. $\dfrac{\sec\alpha \sin\alpha}{\tan\alpha + \cot\alpha} = \sin^2\alpha$
5. $\dfrac{1 - \sin\alpha}{\cos\alpha} = \dfrac{\cos\alpha}{1 + \sin\alpha}$
6. $$7\sec^2 x - 6\tan^2 x + 9\cos^2 x = \frac{(1 + 3\cos^2 x)^2}{\cos^2 x}$$
7. $\dfrac{1 - \tan^2 t}{1 + \tan^2 t} = \cos^2 t - \sin^2 t$
8. $$\frac{1 + \tan B + \sec B}{1 + \tan B - \sec B} = (1 + \sec B)(1 + \csc B)$$

9.6 Addition Formulae

Why bother? In this section we prove several important formulæ .

We will now derive the following formulæ.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{9.21}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \tag{9.22}$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \tag{9.23}$$

We begin by proving

739 Theorem Let $(a,b) \in \mathbb{R}^2$. Then $\cos(a-b) = \cos a\cos b + \sin a\sin b$.

Proof: *Consider the points $A(\cos b, \sin b)$ and $B(\cos a, \sin a)$ in figure 9.34. Their distance is*

$$\begin{aligned}\sqrt{(\cos b - \cos a)^2 + (\sin b - \sin a)^2} &= \sqrt{\cos^2 b - 2\cos b\cos a + \cos^2 a + \sin^2 b - 2\sin b\sin a + \sin^2 a} \\ &= \sqrt{2 - 2(\cos a\cos b + \sin a\sin b)}.\end{aligned}$$

If we rotate A b radians clockwise to $A'(1,0)$, and B b radians clockwise to $B'(\cos(a-b), \sin(a-b))$ as in figure 9.35, the distance is preserved, that is, the distance of A' to B', which is

$$\sqrt{(\cos(a-b)-1)+\sin^2(a-b)} = \sqrt{1-2\cos(a-b)+\cos^2(a-b)+\sin^2(a-b)} = \sqrt{2-2\cos(a-b)},$$

then equals the distance of A to B. Therefore we have

$$\begin{aligned}\sqrt{2-2(\cos a\cos b + \sin a\sin b)} = \sqrt{2-2\cos(a-b)} &\implies 2-2(\cos a\cos b + \sin a\sin b) = 2-2\cos(a-b) \\ &\implies \cos(a-b) = \cos a\cos b + \sin a\sin b.\end{aligned}$$

❑

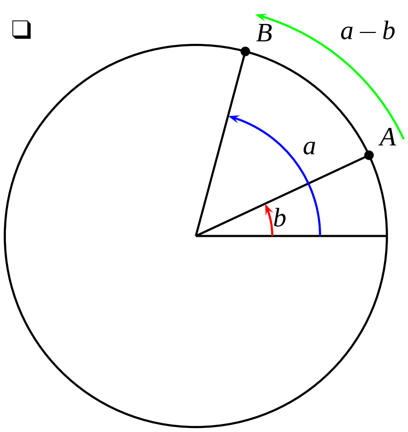


Figure 9.34: Theorem 739.

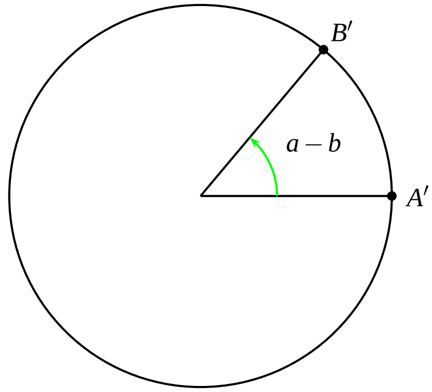


Figure 9.35: Theorem 739.

740 Corollary $\cos(a+b) = \cos a\cos b - \sin a\sin b$.

Proof: *This follows by replacing b by $-b$ in Theorem 739, using the fact that $x \mapsto \cos x$ is an even function and so $\cos(-b) = \cos b$, and that $x \mapsto \sin x$ is an odd function and so $\sin(-b) = -\sin b$:*

$$\cos(a+b) = \cos(a-(-b)) = \cos a\cos(-b) + \sin a\sin(-b) = \cos a\cos b - \sin a\sin b.$$

❑

741 Theorem Let $(a,b) \in \mathbb{R}^2$. Then $\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$.

Proof: *We use the fact that* $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ *and that* $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. *Thus*

$$\begin{aligned} \sin(a+b) &= \cos\left(\frac{\pi}{2} - (a+b)\right) \\ &= \cos\left(\left(\frac{\pi}{2} - a\right) - b\right) \\ &= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b \\ &= \sin a \cos b + \cos a \sin b, \end{aligned}$$

proving the addition formula. For the difference formula, we have

$$\sin(a-b) = \sin(a+(-b)) = \sin a \cos(-b) + \sin(-b)\cos a = \sin a \cos b - \sin b \cos a.$$

❑

742 Theorem Let $(a,b) \in \mathbb{R}^2$. Then $\tan(a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$.

Proof: *Using the formulæ derived above,*

$$\begin{aligned} \tan(a \pm b) &= \frac{\sin(a \pm b)}{\cos(a \pm b)} \\ &= \frac{\sin a \cos b \pm \sin b \cos a}{\cos a \cos b \mp \sin a \sin b}. \end{aligned}$$

Dividing numerator and denominator by $\cos a \cos b$ *we obtain the result.* ❑

By letting $a+b = A, a-b = B$ in the above results we obtain the following corollary.

743 Corollary

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{9.24}$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right) \tag{9.25}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \tag{9.26}$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right) \tag{9.27}$$

744 Example Given that $\cos a = -.1$ and $\pi < a < \frac{3\pi}{2}$, and that $\sin b = .2$ and $0 < b < \frac{\pi}{2}$, find $\cos(a+b)$.

Solution: Since $\mathscr{C}(a)$ is in the third quadrant, $\sin a = -\sqrt{1-(.1)^2} = -\sqrt{0.99}$. As $\mathscr{C}(b)$ is in the first quadrant, $\cos b = \sqrt{1-(.2)^2} = \sqrt{0.96}$. By the addition formula for the cosine

$$\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ &= (-.1)(\sqrt{0.96}) - (-\sqrt{0.99})(.2) \\ &= .2\sqrt{.99} - .1\sqrt{.96}. \end{aligned}$$

745 Example Write $\sin 5x \cos x$ as a sum of sines.

Solution: We have

$$\sin 6x = \sin(5x + x) = \sin 5x \cos x + \sin x \cos 5x$$
$$\sin 4x = \sin(5x - x) = \sin 5x \cos x - \sin x \cos 5x$$

Adding both equalities and dividing by 2, we gather,

$$\sin 5x \cos x = \frac{1}{2}\sin 6x + \frac{1}{2}\sin 4x.$$

746 Example Solve the equation

$$\sin 6x + \sin 4x = 0.$$

Solution: As $\sin 6x + \sin 4x = 2\sin 5x \cos x$ we must have either $\sin 5x = 0$ or $\cos x = 0$. Thus

$$x = \frac{\pi n}{5}, \quad x = \pm\frac{\pi}{2} + \pi n, n \in \mathbb{Z}.$$

747 Example Write $\sin x \sin 2x$ as a sum of cosines.

Solution: We have

$$\cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x,$$
$$\cos x = \cos(2x - x) = \cos 2x \cos x + \sin 2x \sin x.$$

Subtracting both equalities $\cos 3x - \cos x = -2\sin 2x \sin x$, whence

$$\sin 2x \sin x = -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x.$$

748 Example Find the exact value of $\cos\frac{7\pi}{12}$.

Solution: Observe that $\frac{7}{12} = \frac{1}{3} + \frac{1}{4}$. Using the addition formulæ

$$\begin{array}{rcl} \cos\frac{7\pi}{12} & = & \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ & = & \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\ & = & (\frac{1}{2})(\frac{\sqrt{2}}{2}) - (\frac{\sqrt{3}}{2})(\frac{\sqrt{2}}{2}) \\ & = & \frac{\sqrt{2}-\sqrt{6}}{4} \end{array}.$$

749 Example (i) Write $\sqrt{3}\cos x + \sin x$ in the form $A\cos(x - \theta)$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. (ii) Use the preceding identity in order to solve the equation

$$\sqrt{3}\cos x + \sin x = -1.$$

(iii) Find all the solutions in the interval $[0; 2\pi]$.

Solution: First observe that $A \neq 0$, since $\sqrt{3}\cos x + \sin x$ is not identically 0. We have

$$A\cos(x - \theta) = A\cos x \cos\theta + A\sin x \sin\theta.$$

If the expression on the dextral side of the above equality is to be equal to $\sqrt{3}\cos x + \sin x$ then $A\cos\theta = \sqrt{3}$ and $A\sin\theta = 1$. This entails that $\tan\theta = \frac{\sqrt{3}}{3}$ and so $\theta = \frac{\pi}{6}$. This in turn yields $A = 2$. Hence

$$\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right).$$

Now, if $2\cos\left(x - \frac{\pi}{6}\right) = -1$, then

$$x - \frac{\pi}{6} = \pm\arccos(-\frac{1}{2}) + 2n\pi, \; n \in \mathbb{Z},$$

$$x = \frac{\pi}{6} \pm \frac{2\pi}{3} + 2n\pi,\ n \in \mathbb{Z},$$

which is the same family as $x = \frac{5\pi}{6} + 2n\pi, x = -\frac{\pi}{2} + 2n\pi$ and the solutions in $[0; 2\pi]$ are clearly $x = \frac{5\pi}{6}$ and $x = \frac{3\pi}{2}$.

Aliter: Write the equation as $\sqrt{3}\cos x + 1 = -\sin x$ and square

$$3\cos^2 x + 2\sqrt{3}\cos x + 1 = \sin^2 x.$$

Using $\sin^2 x = 1 - \cos^2 x$ we obtain

$$3\cos^2 x + 2\sqrt{3}\cos x + 1 = 1 - \cos^2 x,$$

or

$$(\cos x)(4\cos x + 2\sqrt{3}) = 0.$$

This equation has solutions $x = \pm\frac{\pi}{2} + 2n\pi$ and $x = \pm\frac{5\pi}{6} + 2n\pi$. Testing $x = \frac{\pi}{2}$ in the original equation $\sqrt{3}\cos x + \sin x = -1$ we see that it is not a solution, hence the family $x = \frac{\pi}{2} + 2n\pi$ is not part of the solution set of the original equation. The same happens when we test $x = -\frac{5\pi}{6}$, so we must also discard this family. The two remaining families, $x = \frac{5\pi}{6} + 2n\pi$, $x = -\frac{\pi}{2} + 2n\pi$ agree with our previous solution.

750 Example Obtain a formula for $\cos(a+b+c)$ in terms of cosines and sines of a, b, and c.

Solution: Using the addition formula twice

$$\begin{aligned}\cos(a+b+c) &= \cos a\cos(b+c) - \sin a\sin(b+c)\\ &= \cos a(\cos b\cos c - \sin b\sin c) -\\ &\qquad -\sin a(\sin b\cos c + \sin c\cos b)\\ &= \cos a\cos b\cos c - \cos a\sin b\sin c -\\ &\qquad -\sin a\sin b\cos c - \sin a\cos b\sin c\end{aligned}$$

751 Example (Canadian Mathematical Olympiad 1984) Given any 7 real numbers, prove that there are two of them, say, x and y, such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$

Solution: Let the numbers be $a_k, k = 1, 2, \ldots, 7$. There exists b_k such $a_k = \tan b_k$, since $\begin{array}{ccc}]-\frac{\pi}{2};\frac{\pi}{2}[& \to & \mathbb{R} \\ x & \mapsto & \tan x \end{array}$ is a bijection. Divide the interval $]-\frac{\pi}{2};\frac{\pi}{2}[$ into six subintervals, each of length $\frac{\pi}{6}$. Since we have 7 b_k's and 6 subintervals, two of the b_k's, say b_s and b_t, must lie in the same subinterval. Assuming $b_s \geq b_t$ we then have $0 \leq b_s - b_t \leq \frac{\pi}{6}$. Since $x \mapsto \tan x$ is an increasing function,

$$\tan 0 \leq \tan(b_s - b_t) \leq \tan\frac{\pi}{6},$$

which is to say,

$$0 \leq \frac{\tan b_s - \tan b_t}{1 + \tan b_s \tan b_t} \leq \frac{1}{\sqrt{3}}.$$

This implies that

$$0 \leq \frac{a_s - a_t}{1 + a_s a_t} \leq \frac{1}{\sqrt{3}},$$

which completes the proof.

752 Example Prove that if

$$\frac{a-b}{1+ab} + \frac{b-c}{1+bc} + \frac{c-a}{1+ca} = 0,$$

for real numbers a, b, c, then at least two of the numbers a, b, c are equal.

Solution: $\exists u,v,w$ with $-\frac{\pi}{2} < u,v,w < \frac{\pi}{2}$ such that $a = \tan u, b = \tan v, c = \tan w$ (why?). The given equation becomes

$$\frac{\tan u - \tan v}{1+\tan u \tan v} + \frac{\tan v - \tan w}{1+\tan v \tan w} + \frac{\tan w - \tan u}{1+\tan w \tan u} = 0.$$

Using the addition for the tangents, the preceding relation is equivalent to

$$\tan(u-v) + \tan(v-w) + \tan(w-u) = 0.$$

Applying $\tan X + \tan Y = (\tan(X+Y))(1-\tan X \tan Y)$ with $X = u-v$ and $Y = v-w$, we obtain

$$(\tan(u-w))(1-\tan(u-v)\tan(v-w)) + \tan(w-u) = 0.$$

Factorising the above expression,

$$(\tan(u-w))(\tan(u-v))(\tan(v-w)) = 0.$$

This implies that one of the tangents in this product must be 0. Since

$$-\pi < u-w, u-v, v-w < \pi,$$

this means that one of these differences must be exactly 0, which in turn implies that two of the numbers a,b,c are equal.

753 Example Prove that

$$\arctan a + \arctan b = \begin{cases} \arctan \frac{a+b}{1-ab} & \text{if } ab < 1, \\ \frac{\pi}{2}(\operatorname{sgn}(a)) & \text{if } ab = 1, \\ \arctan \frac{a+b}{1-ab} + \frac{\pi}{2}(\operatorname{sgn}(a)) & \text{if } ab > 1. \end{cases}.$$

Solution: Put $x = \arctan a, y = \arctan b$. If $(x,y) \in]-\frac{\pi}{2}; \frac{\pi}{2}[^2$ and $x+y \neq \frac{(2n+1)\pi}{2}, n \in \mathbb{Z}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{a+b}{1-ab}.$$

Now, $-\pi < x+y < \pi$. Conditioning on x we have,

$$-\frac{\pi}{2} < x+y < \frac{\pi}{2} \iff \left| \begin{array}{l} x = 0 \\ \text{or } x > 0 \text{ and } y < \frac{\pi}{2} - x \\ \text{or } x < 0 \text{ and } y > -\frac{\pi}{2} - x \end{array} \right.$$

The above choices hold if and only if

$$\begin{array}{l} a = 0 \\ \text{or } a > 0 \text{ and } b < \frac{1}{a} \\ \text{or } a < 0 \text{ and } b > \frac{1}{a} \end{array}.$$

Hence, if $ab < 1$, then $x+y \in]-\frac{\pi}{2}; \frac{\pi}{2}[$ and thus

$$x+y = \arctan(\tan(x+y)) = \arctan \frac{a+b}{1-ab}.$$

If $ab > 1$ and $a > 0$ then $x+y \in]\frac{\pi}{2}; \pi[$ and thus

$$x+y = \arctan \frac{a+b}{1-ab} + \pi.$$

If $ab > 1$ and $a < 0$, then $x+y \in]-\pi; -\frac{\pi}{2}[$ and thus

$$x+y = \arctan \frac{a+b}{1-ab} - \pi.$$

The case $ab = 1$ is left as an exercise.

754 Example Solve the equation $\arccos x = \arcsin\frac{1}{3} + \arccos\frac{1}{4}$.

Solution: Observe that $\arccos x \in [0;\pi]$ and that since both $0 \le \arcsin\frac{1}{3} \le \frac{\pi}{2}$ and $0 \le \arccos\frac{1}{4} \le \frac{\pi}{2}$, we have $0 \le \arcsin\frac{1}{3} + \arccos\frac{1}{4} \le \pi$. Hence, we may take cosines on both sides of the equation and obtain

$$\begin{aligned} x &= \cos(\arccos x) \\ &= \cos(\arcsin\tfrac{1}{3} + \arccos\tfrac{1}{4}) \\ &= (\cos\arcsin\tfrac{1}{3})(\cos\arccos\tfrac{1}{4}) - (\sin\arcsin\tfrac{1}{3})(\sin\arccos\tfrac{1}{4}) \\ &= \tfrac{\sqrt{2}}{6} - \tfrac{\sqrt{15}}{12} \end{aligned}.$$

755 Example (Machin's Formula) Prove that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

Solution: Observe that

$$\begin{aligned} 4\arctan\tfrac{1}{5} &= 2\arctan\tfrac{1}{5} + 2\arctan\tfrac{1}{5} \\ &= 2\arctan\tfrac{\frac{1}{5}+\frac{1}{5}}{1-\frac{1}{5}\cdot\frac{1}{5}} \\ &= 2\arctan\tfrac{5}{12} \\ &= \arctan\tfrac{5}{12} + \arctan\tfrac{5}{12} \\ &= \arctan\tfrac{\frac{5}{12}+\frac{5}{12}}{1-\frac{5}{12}\cdot\frac{5}{12}} \\ &= \arctan\tfrac{120}{119}. \end{aligned}$$

Also

$$\begin{aligned} \arctan\tfrac{120}{119} - \arctan\tfrac{1}{239} &= \arctan\tfrac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}} \\ &= \arctan 1 \\ &= \tfrac{\pi}{4}. \end{aligned}$$

Upon assembling the equalities, we obtain the result.

Homework

756 Problem Demonstrate the identity

$$\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b = \cos^2 b - \cos^2 a$$

757 Problem Prove that for all real numbers x,

$$\cos\left(2x - \frac{4\pi}{3}\right) + \cos 2x + \cos\left(2x + \frac{4\pi}{3}\right) = 0.$$

758 Problem Using the fact that $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, find the exact value of the following.

1. $\cos\pi/12$
2. $\sin\pi/12$

759 Problem Write $\cot(a+b)$ in terms of $\cot a$ and $\cot b$.

760 Problem Write $\sin x \sin 2x$ as a sum of cosines.

761 Problem Write $\cos x \cos 4x$ as a sum of cosines.

762 Problem Write using only one arcsine: $\arccos\frac{4}{5} - \arccos\frac{1}{4}$.

763 Problem Write using only one arctangent: $\arctan\frac{1}{3} - \arctan\frac{1}{4}$.

764 Problem Write using only one arctangent: $\operatorname{arccot}(-2) - \arctan(-\frac{2}{3})$.

765 Problem Write $\sin x \cos 2x$ as a sum of sines.

766 Problem Write $\sin x \sin 2x \sin 3x$ as a sum of sines.

767 Problem Given real numbers a, b with $0 < a < \pi/2$ and $\pi < b < 3\pi/2$ and given that $\sin a = 1/3$ and $\cos b = -1/2$, find $\cos(a-b)$.

768 Problem Solve the equation $\cos x + \cos 3x = 0.$.

769 Problem Solve the equation

$$\arcsin(\tan x) = x.$$

770 Problem Solve the equation

$$\arccos x = \arcsin(1-x).$$

771 Problem Solve the equation

$$\arctan x + \arctan 2x = \frac{\pi}{4}.$$

772 Problem Prove the identity

$$\cos^4 x = \frac{1}{8}(\cos 4x + 4\cos 2x + 3).$$

773 Problem Prove the identities

$$\tan a + \tan b = \frac{\sin(a+b)}{(\cos a)(\cos b)},$$

$$\cot a + \cot b = \frac{\sin(a+b)}{(\sin a)(\sin b)}.$$

774 Problem Given that $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$ and satisfy $\sin\alpha = 12/13, \cos\beta = 8/17, \sin\gamma = 4/5$, find the value of $\sin(\alpha+\beta-\gamma)$ and $\cos(\alpha-\beta+2\gamma)$.

775 Problem Establish the identity

$$\frac{\sin(a-b)\sin(a+b)}{1-\tan^2 a \cot^2 b} = -\cos^2 a \sin^2 b.$$

776 Problem Find real constants a, b, c such that

$$\sin 3x - \sqrt{3}\cos 3x = a\sin(bx+c).$$

Use this to solve the equation

$$\sin 3x - \sqrt{3}\cos 3x = -\sqrt{2}.$$

777 Problem Solve the equation

$$\sin 2x + \cos 2x = -1$$

778 Problem Simplify: $\sin(\operatorname{arcsec}\frac{17}{8} - \arctan(-\frac{2}{3}))$.

779 Problem Shew that if $\cot(a+b) = 0$ then $\sin(a+2b) = \sin a$.

780 Problem Let $a+b+c = \frac{\pi}{2}$. Write $\cos a \cos b \cos c$ as a sum of sines.

781 Problem Shew that the amplitude of $x \mapsto a\sin Ax + b\cos Ax$ is $\sqrt{a^2+b^2}$.

782 Problem Solve the equation

$$\cos x - \sin x = 1.$$

783 Problem Let $a+b+c = \pi$. Simplify

$$\sin^2 a + \sin^2 b + \sin^2 c - 2\cos a \cos b \cos c.$$

784 Problem Prove that if

$$\cot a + \csc a \cos b \sec c = \cot b + \cos a \csc b \sec c,$$

then either $a - b = k\pi$, or $a+b+c = \pi + 2m\pi$ or $a+b-c = \pi + 2n\pi$ for some integers k, m, n.

785 Problem Prove that if

$$\tan a + \tan b + \tan c = \tan a \tan b \tan c,$$

then $a+b+c = k\pi$ for some integer k.

786 Problem Prove that if any of $a+b+c$, $a+b-c$, $a-b+c$ or $a-b-c$ is equal to an odd multiple of π, then

$$\cos^2 a + \cos^2 b + \cos^2 c + 2\cos a \cos b \cos c = 1,$$

and that the converse is also true.

9.7 Polar Co-ördinates

We now consider an alternative system of co-ördinates. This system will be useful in parametrising figures having central symmetry. We associate the Cartesian point (x, y) with the polar point $(r; \theta)$ through the equations

$$x = r\cos\theta, \quad y = r\sin\theta, x^2 + y^2 = r^2.$$

Observe that the polar co-ördinates of a point is not unique, since sin and cos are periodic with period 2π.
We now consider several examples.

787 Example The Cartesian co-ördinates of the polar point $(2\sqrt{3}; \frac{2\pi}{3})$ are $x = 2\sqrt{3}\cos\frac{2\pi}{3} = -\sqrt{3}$ and $y = 2\sqrt{3}\sin\frac{2\pi}{3} = 3$.

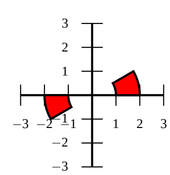


Figure 9.36: Example 788.

788 Example The region

$$\{(r;\theta) : 1 \leq |r| \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{6}\}$$

appears in figure 9.36.

We can also graph in polar coordinates. Just like there are families of curves which everyone must know in Cartesian co-ördinates (e.g., quadratic curves, power functions, exponential functions, trigonometric functions, etc.), there are families of curves in polar co-ördinates that are so common that familiarity with them is desired.
We make the following remarks.

1. Quite often a polar equation has the form $r = f(\theta)$.
2. If the change $\theta \longleftrightarrow -\theta$ leaves the equation unchanged, then the graph has symmetry with respect $\theta = 0$, that is, the x-axis.
3. If the change $\theta \longleftrightarrow \pi - \theta$ leaves the equation unchanged, then the graph has symmetry with respect $\theta = \frac{\pi}{2}$, that is, the y-axis.
4. If the change $r \longleftrightarrow -r$ leaves the equation unchanged, then the graph is symmetric about the origin.

Some examples appear below.

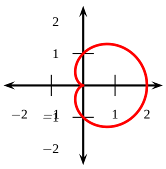


Figure 9.37: $r = 1 + \cos\theta$

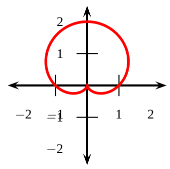


Figure 9.38: $r = 1 + \sin\theta$

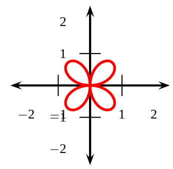


Figure 9.39: $r = \cos 2\theta \sin\theta$

An important family is that of the *limaçons*, with equation

$$r = a + b\cos\theta \quad \text{or} \quad r = a + b\sin\theta$$

If $a > b$, the limaçon is loopless, if $a = b$, then the limaçon has a cusp (and we call it a *cardiod*) and if $a < b$ then the limaçon has a loop.

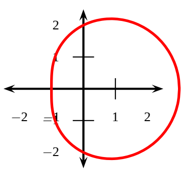


Figure 9.40: $r = 2 + \cos\theta$

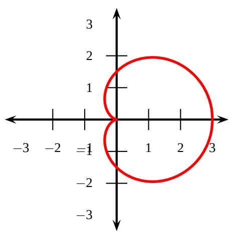


Figure 9.41: $r = 1.5 + 1.5\cos\theta$

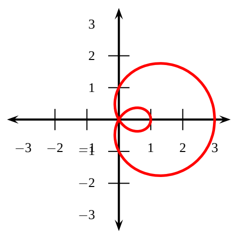


Figure 9.42: $r = 1 + 2\cos\theta$

Here are they are again with the cosine replaced by a sine.

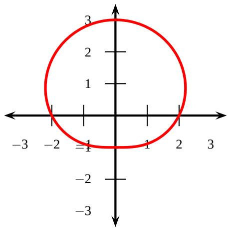


Figure 9.43: $r = 2 + \sin\theta$

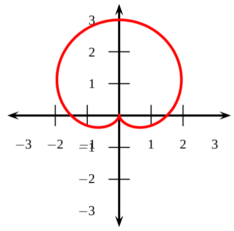


Figure 9.44: $r = 1.5 + 1.5\sin\theta$

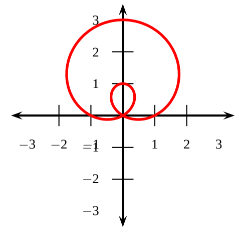


Figure 9.45: $r = 1 + 2\sin\theta$

Another important family are the roses. Here are some of the form

$$r = \cos n\theta,$$

with n even.

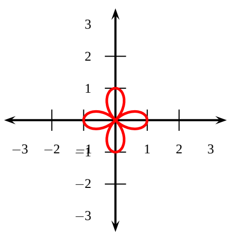


Figure 9.46: $r = \cos 2\theta$

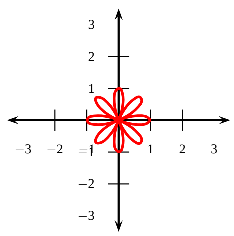


Figure 9.47: $r = \cos 4\theta$

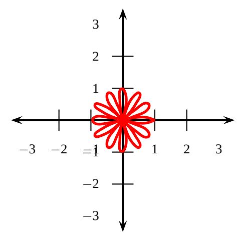


Figure 9.48: $r = \cos 6\theta$

Another important family are the roses. Here are some of the form

$$r = \cos n\theta,$$

with n odd.

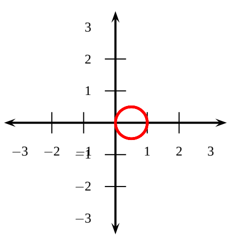


Figure 9.49: $r = \cos\theta$

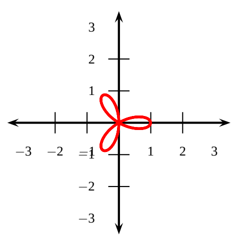


Figure 9.50: $r = \cos 3\theta$

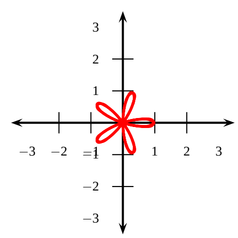


Figure 9.51: $r = \cos 5\theta$

Here are some of the form

$$r = \sin n\theta,$$

with n even.

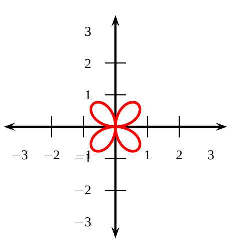


Figure 9.52: $r = \sin 2\theta$

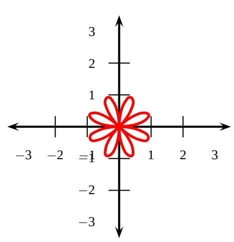


Figure 9.53: $r = \sin 4\theta$

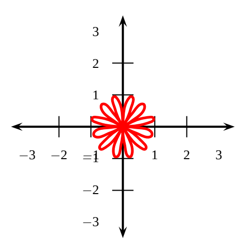


Figure 9.54: $r = \sin 6\theta$

Here are some of the form

$$r = \sin n\theta,$$

with n odd.

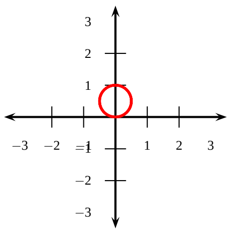


Figure 9.55: $r = \sin\theta$

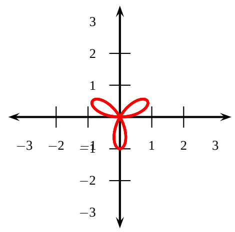


Figure 9.56: $r = \sin 3\theta$

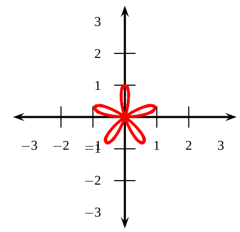


Figure 9.57: $r = \sin 5\theta$

The last example we will consider are the *lemniscates.*

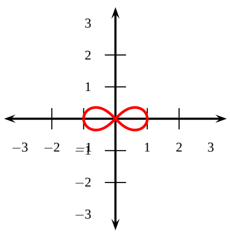


Figure 9.58: $r^2 = \cos 2\theta$

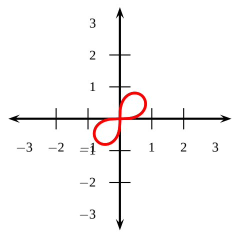


Figure 9.59: $r^2 = \sin 2\theta$

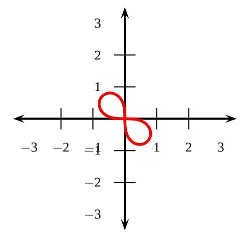


Figure 9.60: $r^2 = -\sin 2\theta$

Answers

587 F; T; F; F; T; T

588
1. $\frac{3\pi}{5}$, quadrant II ;
2. $\frac{7\pi}{5}$, quadrant III ;
3. $\frac{7\pi}{5}$, quadrant III;
4. $\frac{8\pi}{57}$, quadrant I;
5. $\frac{9\pi}{8}$, quadrant III;
6. $\frac{6\pi}{79}$, quadrant I;
7. $\frac{6\pi}{7}$, quadrant II;
8. 1, quadrant I;
9. 2, quadrant II;
10. 3, quadrant II;
11. 4, quadrant III; (xii) 5, quadrant IV;
12. 6, quadrant IV;
13. $100-30\pi$, quadrant IV;
14. $2\pi-3.14$, quadrant III;
15. $2\pi-3.15$, quadrant II

589 (i) $\frac{3\pi}{20}$, $\frac{7\pi}{20}$, $\frac{11\pi}{20}$, $\frac{3\pi}{4}$, $\frac{19\pi}{20}$; (ii) $-\frac{17\pi}{20}$, $-\frac{13\pi}{20}$, $-\frac{9\pi}{20}$, $-\frac{\pi}{4}$, $\frac{-\pi}{20}$.

590 Yes; No.

621 F; F; T; T; F; T; F; F; T; F; T; T; F; T; F; F

622 $\cos t = 0.6$

623 $\sin u = \sqrt{.19}$

624 $\cos t = \dfrac{3\sqrt{2}}{5}$

625 $\sin u = -\dfrac{\sqrt{3}}{4}$

626 $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$, $\sin\frac{5\pi}{6} = \frac{1}{2}$

627 $\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

628 $\sin(\frac{31\pi}{6}) = -\frac{1}{2}$ and $\cos(\frac{31\pi}{6}) = -\frac{\sqrt{3}}{2}$

629 $\sin(\frac{20\pi}{3}) = \frac{\sqrt{3}}{2}$ and $\cos(\frac{20\pi}{3}) = -\frac{1}{2}$

630 $\sin(\frac{17\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{17\pi}{4}) = \frac{\sqrt{2}}{2}$

631 $\sin(\frac{-15\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{-15\pi}{4}) = \frac{\sqrt{2}}{2}$

632 $\sin(\frac{202\pi}{3}) = -\frac{\sqrt{3}}{2}$ and $\cos(\frac{202\pi}{3}) = -\frac{1}{2}$

633 $\sin(\frac{171\pi}{4}) = \frac{\sqrt{2}}{2}$ and $\cos(\frac{171\pi}{4}) = -\frac{\sqrt{2}}{2}$

646 Hint: Use the Arithmetic-Geometric-Mean Inequality $\frac{a+b}{2} \geq \sqrt{ab}$, for non-negative real numbers a, b.

660 F; F; F; F

684 F; F; T; T; F; T; T; F; T; F

685 $\{-\frac{5\pi}{6}, -\frac{\pi}{6}\}$

686 $\{\frac{\pi}{12} + \frac{n\pi}{3}, n \in \mathbb{Z}\}$

687 $\{\pm\frac{2\pi}{3} + 2\pi n, 2\pi n, n \in \mathbb{Z}\}$.

688 $\{(-1)^{n+1}\frac{\pi}{6} + \frac{n\pi}{3}, n \in \mathbb{Z}\}$; $\{(-1)^{n+1}\frac{\pi}{6} + \frac{n\pi}{3}, n = 295, 296, 297, 298, 299, 300\}$

689 $\{(2n+1)\pi, n \in \mathbb{Z}\}$

690 $\{\frac{n\pi}{2},\ n \in \mathbb{Z}\}$

691 $\emptyset$

692 $\{-\frac{\pi}{6} + n\pi, \frac{2\pi}{3} + n\pi\}$

693 (1) $\{-\frac{\pi}{3}, \frac{\pi}{3}\}$; (2) $\{-\frac{\pi}{6}, \frac{\pi}{6}\}$; (3) No solutions in this interval; (4) All the solutions belong to this interval $\{\frac{6}{(-1)^n\pi + 2n\pi},\ n \in \mathbb{Z}\}$; (5) $\{-\frac{\pi}{2}, \frac{\pi}{2}\}$

694 $\frac{2\sqrt{2}}{3}$

695 $\frac{\sqrt{5}}{3}$

696 $\frac{\sqrt{5}}{3}$

697 $5-2\pi$; $4\pi-10$

722 F; F; F; F

723 $\sin x = -\frac{2}{3}, \cos x = \frac{\sqrt{5}}{3}, \tan x = -\frac{-2\sqrt{5}}{5}$.

724 $\sin x = -\frac{2\sqrt{5}}{5}, \cos x = -\frac{\sqrt{5}}{5}$

725 $\cos x = -\sqrt{1-t^4}, \tan x = -\frac{t^2}{\sqrt{1-t^4}}$

726 $\sin \operatorname{arcsec} x = -\sqrt{1-\frac{1}{x^2}}$

727 $\frac{3\sqrt{10}}{10}$

728 $2\pi-6$; $4\pi-10$

758 $\cos(\pi/12) = \frac{\sqrt{2}}{4}(\sqrt{3}+1)$, $\sin(\pi/12) = \frac{\sqrt{2}}{4}(\sqrt{3}-1)$.

759 $\frac{\cot a \cot b - 1}{\cot a + \cot b}$

760 $\frac{1}{2}\cos x - \frac{1}{2}\cos 3x$

761 $\frac{1}{2}\cos 3x + \frac{1}{2}\cos 5x$

762 $-\arcsin\frac{4\sqrt{15}-3}{20}$

763 $\arctan\frac{1}{13}$.

764 $\pi + \arctan\frac{1}{8}$

765 $\frac{1}{2}\sin 3x - \frac{1}{2}\sin x$

766 $\frac{1}{4}\sin 2x + \frac{1}{4}\sin 4x - \frac{1}{4}\sin 6x$

767 $-(\frac{\sqrt{2}}{6} + \frac{\sqrt{3}}{6})$

768 $x = \pm\frac{\pi}{4} + n\pi,\ x = \pm\frac{\pi}{2} + 2n\pi,\ n \in \mathbb{Z}$

769 $x = 0$.

770 $x = 0$ or $x = 1$.

771 $x = \frac{\sqrt{17}-3}{4}$

Appendix **A**

Complex Numbers

A.1 Arithmetic of Complex Numbers

One uses the symbol i to denote the *imaginary unit* $i = \sqrt{-1}$. Then $i^2 = -1$.

789 Example Find $\sqrt{-25}$.

Solution: $\sqrt{-25} = 5i$.
Since $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$, etc., the powers of i repeat themselves cyclically in a cycle of period 4.

790 Example Find i^{1934}.

Solution: Observe that $1934 = 4(483) + 2$ and so $i^{1934} = i^2 = -1$.

791 Example For any integral α one has

$$i^{\alpha} + i^{\alpha+1} + i^{\alpha+2} + i^{\alpha+3} = i^{\alpha}(1 + i + i^2 + i^3) = i^{\alpha}(1 + i - 1 - i) = 0.$$

If a, b are real numbers then the object $a + bi$ is called a *complex number*. One uses the symbol $\mathbb{C}$ to denote the set of all complex numbers. If $a, b, c, d \in \mathbb{R}$, then the sum of the complex numbers $a + bi$ and $c + di$ is naturally defined as

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{A.1}$$

The product of $a + bi$ and $c + di$ is obtained by multiplying the binomials:

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i \tag{A.2}$$

792 Example Find the sum $(4 + 3i) + (5 - 2i)$ and the product $(4 + 3i)(5 - 2i)$.

Solution: One has

$$(4 + 3i) + (5 - 2i) = 9 + i$$

and

$$(4 + 3i)(5 - 2i) = 20 - 8i + 15i - 6i^2 = 20 + 7i + 6 = 26 + 7i.$$

793 Definition Let $z \in \mathbb{C}, (a, b) \in \mathbb{R}^2$ with $z = a + bi$. The *conjugate* $\overline{z}$ of z is defined by

$$\overline{z} = \overline{a + bi} = a - bi \tag{A.3}$$

794 Example The conjugate of $5+3i$ is $\overline{5+3i} = 5-3i$. The conjugate of $2-4i$ is $\overline{2-4i} = 2+4i$.

The conjugate of a real number is itself, that is, if $a \in \mathbb{R}$, then $\overline{a} = a$. Also, the conjugate of the conjugate of a number is the number, that is, $\overline{\overline{z}} = z$.

795 Theorem The function $z : \mathbb{C} \to \mathbb{C}$, $z \mapsto \overline{z}$ is multiplicative, that is, if z_1, z_2 are complex numbers, then

$$\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2} \tag{A.4}$$

Proof: *Let $z_1 = a+bi, z_2 = c+di$ where a, b, c, d are real numbers. Then*

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a+bi)(c+di)} \\ &= \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i \end{aligned}$$

Also,

$$\begin{aligned} \overline{z_1} \cdot \overline{z_2} &= (\overline{a+bi})(\overline{c+di}) \\ &= (a-bi)(c-di) \\ &= ac-adi-bci+bdi^2 \\ &= (ac-bd)-(ad+bc)i, \end{aligned}$$

which establishes the equality between the two quantities. ❑

796 Example Express the quotient $\dfrac{2+3i}{3-5i}$ in the form $a+bi$.

Solution: One has

$$\frac{2+3i}{3-5i} = \frac{2+3i}{3-5i} \cdot \frac{3+5i}{3+5i} = \frac{-9+19i}{34} = \frac{-9}{34} + \frac{19i}{34}$$

797 Definition The *modulus* $|a+bi|$ of $a+bi$ is defined by

$$|a+bi| = \sqrt{(a+bi)(\overline{a+bi})} = \sqrt{a^2+b^2} \tag{A.5}$$

Observe that $z \mapsto |z|$ is a function mapping $\mathbb{C}$ to $[0; +\infty[$.

798 Example Find $|7+3i|$.

Solution: $|7+3i| = \sqrt{(7+3i)(7-3i)} = \sqrt{7^2+3^2} = \sqrt{58}$.

799 Example Find $|\sqrt{7}+3i|$.

Solution: $|\sqrt{7}+3i| = \sqrt{(\sqrt{7}+3i)(\sqrt{7}-3i)} = \sqrt{7+3^2} = 4$.

800 Theorem The function $z \mapsto |z|$, $\mathbb{C} \to \mathbb{R}_+$ is multiplicative. That is, if z_1, z_2 are complex numbers then

$$|z_1 z_2| = |z_1||z_2| \tag{A.6}$$

Proof: *By Theorem 795, conjugation is multiplicative, hence*

$$\begin{aligned} |z_1z_2| &= \sqrt{z_1z_2\overline{z_1z_2}} \\ &= \sqrt{z_1z_2\overline{z_1}\cdot\overline{z_2}} \\ &= \sqrt{z_1\overline{z_1}z_2\overline{z_2}} \\ &= \sqrt{z_1\overline{z_1}}\sqrt{z_2\overline{z_2}} \\ &= |z_1||z_2| \end{aligned}$$

whence the assertion follows. ❑

801 Example Write $(2^2+3^2)(5^2+7^2)$ as the sum of two squares.

Solution: The idea is to write $2^2+3^2 = |2+3i|^2$, $5^2+7^2 = |5+7i|^2$ and use the multiplicativity of the modulus. Now

$$\begin{aligned} (2^2+3^2)(5^2+7^2) &= |2+3i|^2|5+7i|^2 \\ &= |(2+3i)(5+7i)|^2 \\ &= |-11+29i|^2 \\ &= 11^2+29^2 \end{aligned}$$

A.2 Equations involving Complex Numbers

Recall that if $ux^2+vx+w=0$ with $u\neq 0$, then the roots of this equation are given by the *Quadratic Formula*

$$x = -\frac{v}{2u} \pm \frac{\sqrt{v^2-4uw}}{2u} \tag{A.7}$$

The quantity v^2-4uw under the square root is called the *discriminant* of the quadratic equation $ux^2+vx+w=0$. If u,v,w are real numbers and this discriminant is negative, one obtains complex roots.
Complex numbers thus occur naturally in the solution of quadratic equations. Since $i^2=-1$, one sees that $x=i$ is a root of the equation $x^2+1=0$. Similary, $x=-i$ is also a root of x^2+1.

802 Example Solve $2x^2+6x+5=0$

Solution: Using the quadratic formula

$$x = -\frac{6}{4} \pm \frac{\sqrt{-4}}{4} = -\frac{3}{2} \pm i\frac{1}{2}$$

In solving the problems that follow, the student might profit from the following identities.

$$s^2-t^2 = (s-t)(s+t) \tag{A.8}$$

$$s^{2k}-t^{2k} = (s^k-t^k)(s^k+t^k),\ k\in\mathbb{N} \tag{A.9}$$

$$s^3-t^3 = (s-t)(s^2+st+t^2) \tag{A.10}$$

$$s^3+t^3 = (s+t)(s^2-st+t^2) \tag{A.11}$$

803 Example Solve the equation $x^4-16=0$.

Solution: One has $x^4-16=(x^2-4)(x^2+4)=(x-2)(x+2)(x^2+4)$. Thus either $x=-2, x=2$ or $x^2+4=0$. This last equation has roots $\pm 2i$. The four roots of $x^4-16=0$ are thus $x=-2, x=2, x=-2i, x=2i$.

804 Example Find the roots of $x^3 - 1 = 0$.

Solution: $x^3 - 1 = (x-1)(x^2+x+1)$. If $x \neq 1$, the two solutions to $x^2+x+1 = 0$ can be obtained using the quadratic formula, getting $x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

805 Example Find the roots of $x^3 + 8 = 0$.

Solution: $x^3 + 8 = (x+2)(x^2-2x+4)$. Thus either $x = -2$ or $x^2 - 2x + 4 = 0$. Using the quadratic formula, one sees that the solutions of this last equation are $x = 1 \pm i\sqrt{3}$.

806 Example Solve the equation $x^4 + 9x^2 + 20 = 0$.

Solution: One sees that

$$x^4 + 9x^2 + 20 = (x^2+4)(x^2+5) = 0$$

Thus either $x^2 + 4 = 0$, in which case $x = \pm 2i$ or $x^2 + 5 = 0$ in which case $x = \pm i\sqrt{5}$. The four roots are $x = \pm 2i, \pm i\sqrt{5}$

Homework

807 Problem Perform the following operations. Write your result in the form $a + bi$, with $(a,b) \in \mathbb{R}^2$.

1. $\sqrt{36} + \sqrt{-36}$
2. $(4+8i) - (9-3i) + 5(2+i) - 8i$
3. $4 + 5i + 6i^2 + 7i^3$
4. $i(1+i) + 2i^2(3-4i)$
5. $(8-9i)(10+11i)$
6. $i^{1990} + i^{1991} + i^{1992} + i^{1993}$
7. $\dfrac{2-i}{2+i}$
8. $\dfrac{1-i}{1+2i} + \dfrac{1+i}{1+2i}$
9. $(5+2i)^2 + (5-2i)^2$
10. $(1+i)^3$

808 Problem Find real numbers a, b such that

$$(a-2) + (5b+3)i = 4 - 2i$$

809 Problem Write $(2^2+3^2)(3^2+7^2)$ as the sum of two squares.

810 Problem Prove that $(1+i)^2 = 2i$ and that $(1-i)^2 = -2i$. Use this to write

$$\frac{(1+i)^{2004}}{(1-i)^{2000}}$$

in the form $a + bi$, $(a,b) \in \mathbb{R}^2$.

811 Problem Prove that $(1+i\sqrt{3})^3 = 8$. Use this to prove that

$$(1+i\sqrt{3})^{30} = 2^{30}.$$

812 Problem Find $|5+7i|$, $|\sqrt{5}+7i|$, $|5+i\sqrt{7}|$ and $|\sqrt{5}+i\sqrt{7}|$.

813 Problem Prove that if k is an integer then

$$(4k+1)i^{4k} + (4k+2)i^{4k+1} + (4k+3)i^{4k+2} + (4k+4)i^{4k+3} = -2-2i.$$

Use this to prove that

$$1 + 2i + 3i^2 + 4i^3 + \cdots + 1995i^{1994} + 1996i^{1995} = -998 - 998i.$$

814 Problem If z and z' are complex numbers with either $|z| = 1$ or $|z'| = 1$, prove that

$$\left|\frac{z - z'}{1 - \overline{z}z'}\right| = 1.$$

815 Problem Prove that if z, z' and w are complex numbers with $|z| = |z'| = |w| = 1$, then

$$|zz' + zw + z'w| = |z + z' + w|$$

816 Problem Prove that if n is an integer which is not a multiple of 4 then

$$1^n + i^n + i^{2n} + i^{3n} = 0.$$

Now let

$$f(x) = (1+x+x^2)^{1000} = a_0 + a_1x + \cdots + a_{2000}x^{2000}.$$

By considering $f(1) + f(i) + f(i^2) + f(i^3)$, find

$$a_0 + a_4 + a_8 + \cdots + a_{2000}.$$

817 Problem Find all the roots of the following equations.

1. $x^2 + 8 = 0$
2. $x^2 + 49 = 0$
3. $x^2 - 4x + 5 = 0$
4. $x^2 - 3x + 6 = 0$
5. $x^4 - 1 = 0$
6. $x^4 + 2x^2 - 3 = 0$
7. $x^3 - 27 = 0$
8. $x^6 - 1 = 0$
9. $x^6 - 64 = 0$

Appendix B

Binomial Theorem

B.1 Pascal's Triangle

It is well known that

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{B.1}$$

Multiplying this last equality by $a+b$ one obtains

$$(a+b)^3 = (a+b)^2(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

Again, multiplying

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \tag{B.2}$$

by $a+b$ one obtains

$$(a+b)^4 = (a+b)^3(a+b) = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Dropping the variables, a pattern with the coefficients emerges, a pattern called *Pascal's Triangle.*

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
....
....

Notice that each entry different from 1 is the sum of the two neighbours just above it.
Pascal's Triangle can be used to expand binomials to various powers, as the following examples shew.

818 Example

$$\begin{aligned}(4x+5)^3 &= (4x)^3 + 3(4x)^2(5) + 3(4x)(5)^2 + 5^3 \\ &= 64x^3 + 240x^2 + 300x + 125\end{aligned}$$

819 Example

$$\begin{aligned}(2x-y^2)^4 &= (2x)^4 + 4(2x)^3(-y^2) + 6(2x)^2(-y^2)^2 + \\ &\quad +4(2x)(-y^2)^3 + (-y^2)^4 \\ &= 16x^4 - 32x^3y^2 + 24x^2y^4 - 8xy^6 + y^8\end{aligned}$$

820 Example

$$\begin{aligned}(2+i)^5 &= 2^5+5(2)^4(i)+10(2)^3(i)^2+\\ &\quad +10(2)^2(i)^3+5(2)(i)^4+i^5\\ &= 32+80i-80-40i+10+i\\ &= -38+39i\end{aligned}$$

821 Example

$$\begin{aligned}(\sqrt{3}+\sqrt{5})^4 &= (\sqrt{3})^4+4(\sqrt{3})^3(\sqrt{5})\\ &\quad +6(\sqrt{3})^2(\sqrt{5})^2+4(\sqrt{3})(\sqrt{5})^3+(\sqrt{5})^4\\ &= 9+12\sqrt{15}+90+20\sqrt{15}+25\\ &= 124+32\sqrt{15}\end{aligned}$$

822 Example Given that $a-b=2, ab=3$ find a^3-b^3.

Solution: One has

$$\begin{aligned}8 &= 2^3\\ &= (a-b)^3\\ &= a^3-3a^2b+3ab^2-b^3 = a^3-b^3-3ab(a-b)\\ &= a^3-b^3-18,\end{aligned}$$

whence $a^3-b^3=26$.
Aliter: Observe that $4=2^2=(a-b)^2=a^2+b^2-2ab=a^2-b^2-6$, whence $a^2+b^2=10$. This entails that

$$a^3-b^3=(a-b)(a^2+ab+b^2)=(2)(10+3)=26,$$

as before.

B.2 Homework

823 Problem Expand

1. $(x-4y)^3$
2. $(x^3+y^2)^4$
3. $(2+3x)^3$
4. $(2i-3)^4$
5. $(2i+3)^4+(2i-3)^4$
6. $(2i+3)^4-(2i-3)^4$
7. $(\sqrt{3}-\sqrt{2})^3$
8. $(\sqrt{3}+\sqrt{2})^3+(\sqrt{3}-\sqrt{2})^3$
9. $(\sqrt{3}+\sqrt{2})^3-(\sqrt{3}-\sqrt{2})^3$

824 Problem Prove that

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

Prove that

$$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd)$$

Generalise.

825 Problem Compute $(x+2y+3z)^2$.

826 Problem Given that $a+2b=-8,\ ab=4$, find (i) a^2+4b^2, (ii) a^3+8b^3, (iii) $\dfrac{1}{a}+\dfrac{1}{2b}$.

827 Problem The sum of the squares of three consecutive positive integers is 21170. Find the sum of the cubes of those three consecutive positive integers.

828 Problem What is the coefficient of x^4y^6 in

$$(x\sqrt{2}-y)^{10}?$$

Answer: 840.

829 Problem Expand and simplify

$$(\sqrt{1-x^2}+1)^7-(\sqrt{1-x^2}-1)^7.$$

Appendix C

Sequences and Series

C.1 Sequences

830 Definition A sequence of real numbers is a function whose domain is the set of natural numbers and whose output is a subset of the real numbers. We usually denote a sequence by one of the notations

$$a_0, a_1, a_2, \ldots,$$

or

$$\{a_n\}_{n=0}^{+\infty}.$$

Sometimes we may not start at $n = 0$. In that case we may write

$$a_m, a_{m+1}, a_{m+2}, \ldots,$$

or

$$\{a_n\}_{n=m}^{+\infty},$$

where m is a non-negative integer.

We will be mostly interested in two types of sequences: sequences that have an explicit formula for their n-th term and sequences that are defined recursively.

831 Example Let $a_n = 1 - \frac{1}{2^n}, n = 0, 1, \ldots$. Then $\{a_n\}_{n=0}^{+\infty}$ is a sequence for which we have an explicit formula for the n-th term. The first five terms are

$$\begin{array}{lllll} a_0 & = & 1 - \frac{1}{2^0} & = & 0, \\ a_1 & = & 1 - \frac{1}{2^1} & = & \frac{1}{2}, \\ a_2 & = & 1 - \frac{1}{2^2} & = & \frac{3}{4}, \\ a_3 & = & 1 - \frac{1}{2^3} & = & \frac{7}{8}, \\ a_4 & = & 1 - \frac{1}{2^4} & = & \frac{15}{16}. \end{array}$$

832 Example Let

$$x_0 = 1, \quad x_n = \left(1 + \frac{1}{n}\right) x_{n-1}, \quad n = 1, 2, \ldots.$$

Then $\{x_n\}_{n=0}^{+\infty}$ is a sequence recursively defined. The terms $x_1, x_2, \ldots, x_5$ are

$$\begin{array}{lllll} x_1 & = & \left(1+\frac{1}{1}\right)x_0 & = & 2, \\ x_2 & = & \left(1+\frac{1}{2}\right)x_1 & = & 3, \\ x_3 & = & \left(1+\frac{1}{3}\right)x_2 & = & 4, \\ x_4 & = & \left(1+\frac{1}{4}\right)x_3 & = & 5, \\ x_5 & = & \left(1+\frac{1}{5}\right)x_4 & = & 6. \end{array}$$

You might conjecture that an explicit formula for x_n is $x_n = n+1$, and you would be right!

833 Definition A sequence $\{a_n\}_{n=0}^{+\infty}$ is said to be *increasing* if $a_n \leq a_{n+1}$ $\forall n \in \mathbb{N}$[1] and *strictly increasing* if $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$[2]

Similarly, a sequence $\{a_n\}_{n=0}^{+\infty}$ is said to be *decreasing* if $a_n \geq a_{n+1}$ $\forall n \in \mathbb{N}$[3] and *strictly decreasing* if $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$[4]

A sequence is *monotonic* if is either increasing, strictly increasing, decreasing, or strictly decreasing.

834 Example Recall that $0! = 1$, $1! = 1$, $2! = 1\cdot 2 = 2$, $3! = 1\cdot 2\cdot 3 = 6$, etc. Prove that the sequence $x_n = n!, n = 0,1,2,\ldots$ is strictly increasing for $n \geq 1$.

Solution: For $n > 1$ we have

$$x_n = n! = n(n-1)! = nx_{n-1} > x_{n-1},$$

since $n > 1$. This proves that the sequence is strictly increasing.

835 Example Prove that the sequence $x_n = 2 + \dfrac{1}{2^n}$, $n = 0,1,2,\ldots$ is strictly decreasing.

Solution: We have

$$\begin{array}{lll} x_{n+1} - x_n & = & \left(2+\dfrac{1}{2^{n+1}}\right) - \left(2+\dfrac{1}{2^n}\right) \\ & = & \dfrac{1}{2^{n+1}} - \dfrac{1}{2^n} \\ & = & -\dfrac{1}{2^{n+1}} \\ & < & 0, \end{array}$$

whence

$$x_{n+1} - x_n < 0 \implies x_{n+1} < x_n,$$

i.e., the sequence is strictly decreasing.

836 Example Prove that the sequence $x_n = \dfrac{n^2+1}{n}$, $n = 1,2,\ldots$ is strictly increasing.

Solution: First notice that $\dfrac{n^2+1}{n} = n + \dfrac{1}{n}$. Now,

$$\begin{array}{lll} x_{n+1} - x_n & = & \left(n+1+\dfrac{1}{n+1}\right) - \left(n+\dfrac{1}{n}\right) \\ & = & 1 + \dfrac{1}{n+1} - \dfrac{1}{n} \\ & = & 1 - \dfrac{1}{n(n+1)} \\ & > & 0, \end{array}$$

[1] Some people call these sequences *non-decreasing*.
[2] Some people call these sequences *increasing*.
[3] Some people call these sequences *non-increasing*.
[4] Some people call these sequences *decreasing*.

since from 1 we are subtracting a proper fraction less than 1. Hence

$$x_{n+1} - x_n > 0 \implies x_{n+1} > x_n,$$

i.e., the sequence is strictly increasing.

837 Definition A sequence $\{x_n\}_{n=0}^{+\infty}$ is said to be *bounded* if eventually the absolute value of every term is smaller than a certain positive constant. The sequence is *unbounded* if given an arbitrarily large positive real number we can always find a term whose absolute value is greater than this real number.

838 Example Prove that the sequence $x_n = n!, n = 0, 1, 2, \ldots$ is unbounded.

Solution: Let $M > 0$ be a large real number. Then its integral part $\lfloor M \rfloor$ satisfies the inequality $M - 1 < \lfloor M \rfloor \leq M$ and so $\lfloor M \rfloor + 1 > M$. We have

$$x_{\lfloor M \rfloor + 1} = (\lfloor M \rfloor + 1)! = (\lfloor M \rfloor + 1)(\lfloor M \rfloor)(\lfloor M \rfloor - 1) \cdots 2 \cdot 1 > M,$$

since the first factor is greater than M and the remaining factors are positive integers.

839 Example Prove that the sequence $a_n = \dfrac{n+1}{n}$, $n = 1, 2, \ldots,$ is bounded.

Solution: Observe that $a_n = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$. Since $\dfrac{1}{n}$ strictly decreases, each term of a_n becomes smaller and smaller. This means that each term is smaller that $a_1 = 1 + \dfrac{1}{2}$. Thus $a_n < 2$ for $n \geq 2$ and the sequence is bounded.

Homework

840 Problem Find the first five terms of the following sequences.

1. $x_n = 1 + (-2)^n, n = 0, 1, 2, \ldots$
2. $x_n = 1 + (-\frac{1}{2})^n, n = 0, 1, 2, \ldots$
3. $x_n = n! + 1, n = 0, 1, 2, \ldots$
4. $x_n = \dfrac{1}{n! + (-1)^n}, n = 2, 3, 4, \ldots$
5. $x_n = \left(1 + \dfrac{1}{n}\right)^n, n = 1, 2, \ldots,$

841 Problem Decide whether the following sequences are eventually monotonic or non-monotonic. Determine whether they are bounded or unbounded.

1. $x_n = n$, $n = 0, 1, 2, \ldots$
2. $x_n = (-1)^n n$, $n = 0, 1, 2, \ldots$
3. $x_n = \dfrac{1}{n!}$, $n = 0, 1, 2, \ldots$
4. $x_n = \dfrac{n}{n+1}$, $n = 0, 1, 2, \ldots$
5. $x_n = n^2 - n$, $n = 0, 1, 2, \ldots$
6. $x_n = (-1)^n$, $n = 0, 1, 2, \ldots$
7. $x_n = 1 - \dfrac{1}{2^n}$, $n = 0, 1, 2, \ldots$
8. $x_n = 1 + \dfrac{1}{2^n}$, $n = 0, 1, 2, \ldots$

C.2 Convergence and Divergence

We are primarily interested in the behaviour that a sequence $\{a_n\}_{n=0}^{+\infty}$ exhibits as n gets larger and larger. First some shorthand.

842 Definition The notation $n \to +\infty$ means that the natural number n increases or tends towards $+\infty$, that is, that it becomes bigger and bigger.

843 Definition We say that the sequence $\{x_n\}_{n=0}^{+\infty}$ *converges*[5] to a limit L, written $x_n \to L$ as $n \to +\infty$, if eventually all terms after a certain term are closer to L by any preassigned distance. A sequence which does not converge is said to *diverge*.

To illustrate the above definition, some examples are in order.

[5]This definition is necessarily imprecise, as we want to keep matters simple. A more precise definition is the following: we say that a sequence $c_n, n = 0, 1, 2, \ldots$ *converges* to L (written $c_n \to L$) as $n \to +\infty$, if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $|c_n - L| < \varepsilon\ \forall n > N$. We say that a sequence $d_n, n = 0, 1, 2, \ldots$ diverges to $+\infty$ (written $d_n \to +\infty$) as $n \to +\infty$, if $\forall M > 0\ \exists N \in \mathbb{N}$ such that $d_n > M\ \forall n > N$. A sequence $f_n, n = 0, 1, 2, \ldots$ diverges to $-\infty$ if the sequence $-f_n, n = 0, 1, 2, \ldots$ converges to $+\infty$.

844 Example The constant sequence

$$1, 1, 1, 1, \ldots$$

converges to 1.

845 Example Consider the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots,$$

We claim that $\frac{1}{n} \to 0$ as $n \to +\infty$. Suppose we wanted terms that get closer to 0 by at least $.00001 = \frac{1}{10^5}$. We only need to look at the 100000-term of the sequence: $\frac{1}{100000} = \frac{1}{10^5}$. Since the terms of the sequence get smaller and smaller, any term after this one will be within .00001 of 0. We had to wait a long time—till after the 100000-th term—but the sequence eventually did get closer than .00001 to 0. The same argument works for any distance, no matter how small, so we can eventually get arbitrarily close to 0.[6].

846 Example The sequence

$$0, 1, 4, 9, 16, \ldots, n^2, \ldots$$

diverges to $+\infty$, as the sequence gets arbitrarily large.[7]

847 Example The sequence

$$1, -1, 1, -1, 1, -1, \ldots, (-1)^n, \ldots$$

has no limit (diverges), as it bounces back and forth from -1 to $+1$ infinitely many times.

848 Example The sequence

$$0, -1, 2, -3, 4, -5, \ldots, (-1)^n n, \ldots,$$

has no limit (diverges), as it is unbounded and alternates back and forth positive and negative values..

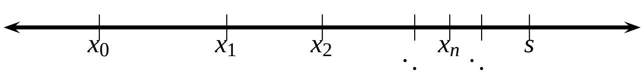


Figure C.1: Theorem 849.

When is it guaranteed that a sequence of real numbers has a limit? We have the following result.

[6]A rigorous proof is as follows. If $\varepsilon > 0$ is no matter how small, we need only to look at the terms after $N = \lfloor \frac{1}{\varepsilon} + 1 \rfloor$ to see that, indeed, if $n > N$, then

$$s_n = \frac{1}{n} < \frac{1}{N} = \frac{1}{\lfloor \frac{1}{\varepsilon} + 1 \rfloor} < \varepsilon.$$

Here we have used the inequality

$$t - 1 < \lfloor t \rfloor \le t, \quad \forall t \in \mathbb{R}.$$

[7]A rigorous proof is as follows. If $M > 0$ is no matter how large, then the terms after $N = \lfloor \sqrt{M} \rfloor + 1$ satisfy $(n > N)$

$$t_n = n^2 > N^2 = (\lfloor \sqrt{M} \rfloor + 1)^2 > M.$$

849 Theorem Every bounded increasing sequence $\{a_n\}_{n=0}^{+\infty}$ of real numbers converges to its supremum. Similarly, every bounded decreasing sequence of real numbers converges to its infimum.

Proof: *The idea of the proof is sketched in figure C.1. By virtue of Axiom* **??**, *the sequence has a supremum s. Every term of the sequence satisfies $a_n \leq s$. We claim that eventually all the terms of the sequence are closer to s than a preassigned small distance $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound for the sequence, there must be a term of the sequence, say a_{n_0} with $s - \varepsilon \leq a_{n_0}$ by virtue of the Approximation Property Theorem* **??**. *Since the sequence is increasing, we then have*

$$s - \varepsilon \leq a_{n_0} \leq a_{n_0+1} \leq a_{n_0+2} \leq a_{n_0+2} \leq \ldots \leq s,$$

which means that after the n_0-th term, we get to within ε of s.

To obtain the second half of the theorem, we simply apply the first half to the sequence $\{-a_n\}_{n=0}^{+\infty}$. ❑

Homework

850 Problem Give plausible arguments to convince yourself that

❶ $\frac{1}{2^n} \to 0$ as $n \to +\infty$

❷ $2^n \to +\infty$ as $n \to +\infty$

❸ $\frac{1}{n!} \to 0$ as $n \to +\infty$

❹ $\frac{n+1}{n} \to 1$ as $n \to +\infty$

❺ $(\frac{2}{3})^n \to 0$ as $n \to +\infty$

❻ $(\frac{3}{2})^n \to +\infty$ as $n \to +\infty$

❼ the sequence $(-2)^n, n = 0, 1, \ldots$ diverges as $n \to +\infty$

❽ $\frac{n}{2^n} \to 0$ as $n \to +\infty$

❾ $\frac{2^n}{n} \to +\infty$ as $n \to +\infty$

❿ the sequence $1 + (-1)^n, n = 0, 1, \ldots$ diverges as $n \to +\infty$

C.3 Finite Geometric Series

851 Definition A *geometric sequence* or *progression* is a sequence of the form

$$a,\ ar, ar^2,\ ar^3,\ ar^4, \ldots,$$

that is, every term is produced from the preceding one by multiplying a fixed number. The number r is called the *common ratio*.

❶ *Trivially, if $a = 0$, then every term of the progression is 0, a rather uninteresting case.*

❷ *If $ar \neq 0$, then the common ratio can be found by dividing any term by that which immediately precedes it.*

❸ *The n-th term of the progression*

$$a,\ ar, ar^2,\ ar^3,\ ar^4, \ldots,$$

is ar^{n-1}.

852 Example Find the 35-th term of the geometric progression

$$\frac{1}{\sqrt{2}},\ -2,\ \frac{8}{\sqrt{2}}, \ldots.$$

Solution: The common ratio is $-2 \div \frac{1}{\sqrt{2}} = -2\sqrt{2}$. Hence the 35-th term is $\frac{1}{\sqrt{2}}(-2\sqrt{2})^{34} = \frac{2^{51}}{\sqrt{2}} = 1125899906842624\sqrt{2}$.

853 Example The fourth term of a geometric progression is 24 and its seventh term is 192. Find its second term.

Solution: We are given that $ar^3 = 24$ and $ar^6 = 192$, for some a and r. Clearly, $ar \neq 0$, and so we find

$$\frac{ar^6}{ar^3} = r^3 = \frac{192}{24} = 8,$$

whence $r = 2$. Now, $a(2)^3 = 24$, giving $a = 3$. The second term is thus $ar = 6$.

854 Example Find the sum

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{64}.$$

Estimate (without a calculator!) how big this sum is.

Solution: Let

$$S = 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{64}.$$

Observe that the common ratio is 2. We multiply S by 2 and notice that every term, with the exception of the last, appearing on this new sum also appears on the first sum. We subtract S from $2S$:

$$\begin{array}{lclcccccccccccc}
S & = & 2 & + & 2^2 & + & 2^3 & + & 2^4 & + & \cdots & + & 2^{64} & & \\
2S & = & & & 2^2 & + & 2^3 & + & 2^4 & + & \cdots & + & 2^{64} & + & 2^{65} \\
\hline
2S - S & = & -2 + 2^{65} & & & & & & & & & & & &
\end{array}$$

Thus $S = 2^{65} - 2$. To estimate this sum observe that $2^{10} = 1024 \approx 10^3$. Therefore

$$2^{65} = (2^{10})^6 \cdot (2^5) = 32(2^{10})^6 \approx 32(10^3)^6 = 32 \times 10^{18} = 3.2 \times 10^{19}.$$

The exact answer (obtained via Maple ®), is

$$2^{65} - 2 = 36893488147419103230.$$

My pocket calculator yields $3.689348815 \times 10^{19}$. Our estimate gives the right order of decimal places.

1. *If a chess player is paid* \$2 *for the first square of a chess board,* \$4 *for the second square,* \$8 *for the third square, etc., after reaching the* 64*-th square he would be paid* \$36893488147419103230*. Query: After which square is his total more than* \$1000000*?*
2. *From the above example, the sum of a geometric progression with positive terms and common ratio* $r > 1$ *grows rather fast rather quickly.*

855 Example Sum

$$\frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^{99}}.$$

Solution: Put

$$S = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots + \frac{2}{3^{99}}.$$

Then

$$\frac{1}{3}S = \frac{2}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \cdots + \frac{2}{3^{100}}.$$

Subtracting,

$$S - \frac{1}{3}S = \frac{2}{3}S = \frac{2}{3} - \frac{2}{3^{100}}.$$

It follows that

$$S = \frac{3}{2}\left(\frac{2}{3} - \frac{2}{3^{100}}\right) = 1 - \frac{1}{3^{99}}.$$

The sum of the first two terms of the series in example 855 is $\frac{2}{3}+\frac{2}{3^2}=\frac{8}{9}$, which, though close to 1 is not as close as the sum of the first 99 terms. A geometric progression with positive terms and common ratio $0<r<1$ has a sum that grows rather slowly.

To close this section we remark that the approximation $2^{10}\approx 1000$ is a useful one. It is nowadays used in computer lingo, where a kilobyte is 1024 bytes—"kilo" is a Greek prefix meaning "thousand."

856 Example Without using a calculator, determine which number is larger: 2^{900} or 3^{500}.

Solution: The idea is to find a power of 2 close to a power of 3. One readily sees that $2^3=8<9=3^2$. Now, raising both sides to the 250-th power,

$$2^{750}=(2^3)^{250}<(3^2)^{250}=3^{500}.$$

The inequality just obtained is completely useless, it does not answer the question addressed in the problem. However, we may go around this with a similar idea. Observe that $9<8\sqrt{2}$: for, if $9\geq 8\sqrt{2}$, squaring both sides we would obtain $81>128$, a contradiction. Raising $9<8\sqrt{2}$ to the 250-th power we obtain

$$3^{500}=(3^2)^{250}<(8\sqrt{2})^{250}=2^{875}<2^{900},$$

whence 2^{900} is greater.

You couldn't solve example 856 using most pockets calculators and the mathematical tools you have at your disposal (unless you were really *clever!). Later in this chapter we will see how to solve this problem using logarithms.*

Homework

857 Problem Find the 17-th term of the geometric sequence

$$-\frac{2}{3^{17}},\frac{2}{3^{16}},-\frac{2}{3^{15}},\cdots.$$

858 Problem The 6-th term of a geometric progression is 20 and the 10-th is 320. Find the absolute value of its third term.

859 Problem Find the sum of the following geometric series.

1. $$1+3+3^2+3^3+\cdots+3^{49}.$$
2. If $y\neq 1$, $$1+y+y^2+y^3+\cdots+y^{100}.$$
3. If $y\neq 1$, $$1-y+y^2-y^3+y^4-y^5+\cdots-y^{99}+y^{100}.$$
4. If $y\neq 1$, $$1+y^2+y^4+y^6+\cdots+y^{100}.$$

860 Problem A colony of amoebas[8] is put in a glass at 2 : 00 PM. One second later each amoeba divides in two. The next second, the present generation divides in two again, etc.. After one minute, the glass is full. When was the glass half-full?

861 Problem Without using a calculator: which number is greater 2^{30} or 30^2?

862 Problem In this problem you may use a calculator. Legend says that the inventor of the game of chess asked the Emperor of China to place a grain of wheat on the first square of the chessboard, 2 on the second square, 4 on the third square, 8 on the fourth square, etc.. (1) How many grains of wheat are to be put on the last (64-th) square?, (2) How many grains, total, are needed in order to satisfy the greedy inventor?, (3) Given that 15 grains of wheat weigh approximately one gramme, what is the approximate weight, in kg, of wheat needed?, (4) Given that the annual production of wheat is 350 million tonnes, how many years, approximately, are needed in order to satisfy the inventor (assume that production of wheat stays constant)[9].

863 Problem Prove that

$$1+2\cdot 5+3\cdot 5^2+4\cdot 5^3+\cdots+99\cdot 5^{100}=\frac{99\cdot 5^{101}}{4}-\frac{5^{101}-1}{16}.$$

864 Problem Shew that

$$1+x+x^2+\cdots+x^{1023}=(1+x)(1+x^2)(1+x^4)\cdots(1+x^{256})(1+x^{512}).$$

865 Problem Prove that

$$1+x+x^2+\cdots+x^{80}=(x^{54}+x^{27}+1)(x^{18}+x^9+1)(x^6+x^3+1)(x^2+x+1).$$

[8] Why are amoebas bad mathematicians? Because they divide to multiply!

[9] Depending on your ethnic preference, the ruler in this problem might be an Indian maharajah or a Persian shah, but never an American businessman!!!

C.4 Infinite Geometric Series

866 Definition Let

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

be the sequence of partial sums of a geometric progression. We say that the *infinite* geometric sum

$$a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots$$

converges to a finite number s if $|s_n - s| \to 0$ as $n \to +\infty$. We say that infinite sum

$$a + ar + ar^2 + \cdots + ar^{n-1} + ar^n + \cdots$$

diverges if there is no finite number to which the sequence of partial sums converges.

867 Lemma If $0 < a < 1$ then $a^n \to 0$ as $n \to 0$.

Proof: *Observe that by multiplying through by a we obtain*

$$0 < a < 1 \implies 0 < a^2 < a \implies 0 < a^3 < a^2 \implies \ldots$$

and so

$$0 < \ldots < a^n < a^{n-1} < \ldots < a^3 < a^2 < a < 1,$$

that is, the sequence is decreasing and bounded. By Theorem 849 the sequence converges to its infimum $\inf_{n \geq 0} a^n = 0$. ❑

868 Theorem Let $a,\ ar,\ ar^2,\ \ldots$ with $|r| \neq 1$, be a geometric progression. Then

1. The sum of its first n terms is
$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r},$$
2. If $|r| < 1$, the infinite sum converges to
$$a + ar + ar^2 + \cdots = \frac{a}{1 - r},$$
3. If $|r| > 1$, the infinite sum diverges.

Proof: *Put*

$$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Then

$$rS = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Subtracting,

$$S - rS = S(1 - r) = a - ar^n.$$

Since $|r| \neq 1$ *we may divide both sides of the preceding equality in order to obtain*

$$S = \frac{a - ar^n}{1 - r},$$

proving the first statement of the theorem.

Now, if $|r| < 1$, *then* $|r|^n \to 0$ *as* $n \to +\infty$ *by virtue of Lemma 867, and if* $|r| > 1$, *then* $|r|^n \to +\infty$ *as* $n \to +\infty$. *The second and third statements of the theorem follow from this.* ❑

We have thus created a dichotomy amongst infinite geometric sums. If their common ratio is smaller than 1 *in absolute value, the infinite geometric sum converges. Otherwise, the sum diverges.*

869 Example Find the sum of the infinite geometric series

$$\frac{3}{5^3} - \frac{3}{5^4} + \frac{3}{5^5} - \frac{3}{5^6} + \cdots.$$

Solution: We have $a = \frac{3}{5^3}, r = -\frac{1}{5}$ in Theorem 868. Therefore

$$\frac{3}{5^3} - \frac{3}{5^4} + \frac{3}{5^5} - \frac{3}{5^6} + \cdots = \frac{\frac{3}{5^3}}{1 - \left(-\frac{1}{5}\right)} = \frac{1}{50}.$$

870 Example Find the rational number which is equivalent to the repeating decimal $0.23\overline{45}$.

Solution:

$$0.23\overline{45} = \frac{23}{10^2} + \frac{45}{10^4} + \frac{45}{10^6} + \cdots = \frac{23}{10^2} + \frac{\frac{45}{10^4}}{1 - \frac{1}{10^2}} = \frac{23}{100} + \frac{1}{220} = \frac{129}{550}.$$

Note: The geometric series above did not start till the second term of the sum.

871 Example A celestial camel is originally at the point $(0,0)$ on the Cartesian Plane. The camel is told by a Djinn that if it wanders 1 unit right, $1/2$ unit up, $1/4$ unit left, $1/8$ unit down, $1/16$ unit right, and so, ad infinitum, then it will find a houris. What are the coordinate points of the houris?

Solution: Let the coordinates of the houris be (X,Y). Then

$$X = \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots = \frac{1}{1 - \left(-\frac{1}{4}\right)} = \frac{4}{5},$$

and

$$Y = \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2^5} - \frac{1}{2^7} \cdots = \frac{\frac{1}{2}}{1 - \left(-\frac{1}{4}\right)} = \frac{2}{5}.$$

872 Example What is wrong with the statement

$$1 + 2 + 2^2 + 2^3 + \cdots = \frac{1}{1-2} = -1?$$

Notice that the sinistral side is positive and the dextral side is negative.

Solution: The geometric sum diverges, as the common ratio 2 is > 1, so we may not apply the formula for an infinite geometric sum. There is an interpretation (called *convergence in the sense of Abel*), where statements like the one above do make sense.

Homework

873 Problem Find the sum of the given infinite geometric series.

1. $$\frac{8}{5} + 1 + \frac{5}{8} + \cdots$$

2. $$0.9 + 0.03 + 0.001 + \cdots$$

3. $$\frac{3+2\sqrt{2}}{3-2\sqrt{2}} + 1 + \frac{3-2\sqrt{2}}{3+2\sqrt{2}} + \cdots$$

4. $$\frac{\sqrt{3}}{\sqrt{2}} + \frac{\sqrt{2}}{3} + \frac{2\sqrt{2}}{9\sqrt{3}} + \cdots$$

5. $$1 + \frac{\sqrt{5}-1}{2} + \left(\frac{\sqrt{5}-1}{2}\right)^2 + \cdots$$

6. $$1 + 10 + 10^2 + 10^3 + \cdots$$

7. $$1 - x + x^2 - x^3 + \cdots, \quad |x| < 1.$$

8. $$\frac{\sqrt{3}}{\sqrt{3}+1}+\frac{\sqrt{3}}{\sqrt{3}+3}+\cdots$$

9. $$x-y+\frac{y^2}{x}-\frac{y^3}{x^2}+\frac{y^4}{x^3}-\frac{y^5}{x^4}+\cdots,$$

   with $|y|<|x|$.

874 Problem Give rational numbers (that is, the quotient of two integers), equivalent to the repeating decimals below.

1. $0.\overline{3}$
2. $0.\overline{6}$
3. $0.2\overline{5}$
4. $2.12\overline{35}$
5. $0.\overline{428571}$

875 Problem Give an example of an infinite series with all positive terms, adding to 666.

Answers

840 (1) 2, −1, 5, −7, 17; (2) 2, 1/2, 5/4, 7/8, 17/16; (3) 2, 2, 3, 7, 25; (4) 1/3, 1/5, 1/25, 1/119, 1/721; (5) 2, 9/4, 64/27, 625/256, 7776/3125

841 (1) Strictly increasing, unbounded (2) non-monotonic, unbounded (3) strictly decreasing, bounded (4) strictly increasing, bounded (5) strictly increasing, unbounded, (6) non-monotonic, bounded, (7) strictly increasing, bounded, (8) strictly decreasing, bounded

857 $-\frac{2}{3}$

858 One is given that $ar^5=20$ and $ar^9=320$. Hence $|ar^2|=\frac{5}{2}$

859 (1) $\frac{3^{50}-1}{2}=358948993845926294385124$, (2) $\frac{1-y^{101}}{1-y}$, (3) $\frac{1+y^{101}}{1+y}$, (4) $\frac{1-y^{102}}{1-y^2}$

860 At 2 : 00 : 59 PM (the second just before 2 : 01 PM.)

861 2^{30}

862 (1) $2^{63}=9223372036854775808$, (2) $2^{64}-1=18446744073709551614$, (3) 1.2×10^{15} kg, or 1200 billion tonnes (4) 3500 years

873 (1) $\frac{64}{15}$, (2) $\frac{27}{29}$, (3) $\frac{140+99\sqrt{2}}{8}$, (4) $\frac{27\sqrt{6}+18\sqrt{2}}{46}$, (5) $\frac{3+\sqrt{5}}{2}$, (6) diverges, (7) $\frac{1}{1+x}$, (8) $\frac{3}{2}$, (9) $\frac{x^2}{x-y}$

874 (1) $\frac{1}{3}$, (2) $\frac{2}{3}$, (3) $\frac{23}{90}$, (4) $\frac{21023}{9900}$, (5) $\frac{3}{7}$

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