Part I. Entire Functions of Finite Order

A function $f(z)$ analytic in the whole complex plane, i.e., a function represented by a power series of the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \ , \quad \lim_{n \to \infty} \sqrt[n]{|c_n|} = 0 \ ,$$

is called an entire function. This is the simplest class of analytic functions containing all polynomials. Polynomials are classified according to their degree, i.e., according to their growth as $|z| \to \infty$. An entire function can grow in various ways along different directions. For a general characterization of the growth, the function

$$M_f(r) = \max_{|z|=r} |f(z)|$$

is introduced. It follows from the Maximum Principle that this function increases monotonically.

The more roots a polynomial has, the faster it grows. This property is extended to entire functions, but it is much more complex. The relationship between the growth of an entire function and the distribution of its roots is the main subject matter of the theory of entire functions.

Here we would like to point out, without giving precise definitions, that there is a large cycle of theorems which state that if an entire function f "grows slowly enough" and its roots are "arranged very densely", then $f(z) \equiv 0$. These are uniqueness theorems similar to the simplest uniqueness theorem for polynomials (a polynomial of degree n having more than n roots is identically equal to zero). The solution of many completeness problems for various functional systems, in particular eigenfunctions of boundary value problems, reduces to such theorems.

Another cycle of questions involves the study of relationship between the growth (or decrease) of a function along different directions, and its global growth characterized by the function $M_f(r)$. A polynomial grows in all directions uniformly. The asymptotic behavior of an entire function as $z \to \infty$ is much more complicated. The main facts pertaining to this problem can be stated in the following way: an entire function having a "small" global growth cannot "decrease too fast in some direction", but must "grow on a large enough part of the complex plane". The simplest fact of this type, directly implied by the Liouville theorem, can be formulated in the following way: if $f(z) = O(1)$, $z \to \infty$, and $f(z_n) \to 0$ for some sequence $z_n \to \infty$, then the function $f(z)$ is identically equal to zero. More refined estimates are usually based on various versions of Phragmén-Lindelöf theorems. Theorems of this type are used in functional analysis (in particular, in the theory of nonselfadjoint operators and in the theory of Banach algebras), in harmonic analysis, and in some problems of mathematical physics.

Finally, some problems of expanding functions of a real or complex variable into special functional series (problems of bases) reduce to certain questions of the theory of interpolation by entire functions.

Thus, the theory of entire functions provides us with a powerful tool to solve many problems of classical and functional analysis.

This is the approach which will be used to present the theory of entire functions in this monograph.

LECTURE 1

Growth of Entire Functions

1.1. The growth scale for entire functions

We shall start by considering an important question: how fast can the function $M_f(r)$ grow?

THEOREM 1. *If, for a nonnegative* λ, *the equation*

$$\liminf_{r\to\infty} \frac{M_f(r)}{r^\lambda} = 0$$

holds, then $f(z)$ *is a polynomial whose degree does not exceed* λ.

PROOF. We shall use the Cauchy inequalities

$$|c_n| \le \frac{M_f(r)}{r^n} .$$

For $n > \lambda$ we obtain

$$|c_n| \le \liminf_{r\to\infty} \frac{M_f(r)}{r^n} = 0.$$

Thus, in order to classify entire functions according to their growth, we must construct a scale of monotonic functions that grow faster than any polynomial.

Can the function $M_f(r)$ grow arbitrarily fast?

PROBLEM 1. Let $\varphi(r)$ be a function growing as $r \to \infty$. Construct an entire function $f(z)$ to satisfy the inequality $M_f(r) > 1 + \varphi(r)$.

On the other hand, there exist entire functions with a slow rate of growth.

PROBLEM 2. Let $\psi(r)$ be an arbitrary function increasing unrestrictedly as $r \to \infty$. Construct an entire function $g(z)$ which is not a polynomial and satisfies the inequality $M_g(r) < 1 + r^{\psi(r)}$.

HINT. Look for a function in the form of a power series with positive coefficients.

1.2. Order and type of entire functions

Let us introduce the following notation. If an inequality $h(r) < \varphi(r)$ holds for sufficiently large values of r, we shall call it an asymptotic inequality and write $h(r) \overset{\text{as}}{<} \varphi(r)$. If the same inequality holds for some sequence of values $r_n \to \infty$, then we shall write $h(r) \overset{\text{n}}{<} \varphi(r)$.

An entire function $f(z)$ is called a *function of finite order* if $M_f(r) \overset{\text{as}}{<} \exp(r^k)$ for some $k > 0$. The *order* (or the *order of growth*) of an entire function f is the greatest lower bound of those values of k for which this asymptotic inequality is fulfilled. We shall usually denote the order of an entire function f by $\rho = \rho_f$. It follows from the definition of the order that

$$e^{r^{\rho-\varepsilon}} \overset{\text{n}}{<} M_f(r) \overset{\text{as}}{<} e^{r^{\rho+\varepsilon}} .$$

By taking the logarithm twice we obtain

$$\rho - \varepsilon \overset{\text{n}}{<} \frac{\log\log M_f(r)}{\log r} \overset{\text{as}}{<} \rho + \varepsilon ,$$

whence

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} .$$

Do entire functions of any nonnegative order exist? This question will be answered at the end of the lecture.

Note that among the functions of the same order there are functions growing in different ways. For instance, take $e^{r/\log r}$, e^r and $e^{r\log r}$. These functions are not entire, but it is not difficult to find entire functions for which $M(r)$ grows in the same way. Such functions are distinguished by using another characteristic, namely the type.

Let ρ be the order of an entire function f. The function is said to have a *finite type* if for some $A > 0$ the inequality

$$M_f(r) \overset{\text{as}}{<} e^{Ar^\rho}$$

is fulfilled.

The greatest lower bound for those values of A for which the latter asymptotic inequality is fulfilled is called the *type* $\sigma = \sigma_f$ *(with respect to the order* ρ*) of the function* f. It follows from the definition of the type that

$$e^{(\sigma-\varepsilon)r^\rho} \overset{\text{n}}{<} M_f(r) \overset{\text{as}}{<} e^{(\sigma+\varepsilon)r^\rho} .$$

Having taken the logarithm and divided by r^ρ, we obtain

$$\sigma - \varepsilon \overset{\text{n}}{<} \frac{\log M_f(r)}{r^\rho} \overset{\text{as}}{<} \sigma + \varepsilon,$$

and therefore

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho} .$$

PROBLEM 3. Prove the inequalities $\rho_{fg} \le \max(\rho_f, \rho_g)$, $\rho_{f+g} \le \max(\rho_f, \rho_g)$, $\sigma_{fg} \le \sigma_f + \sigma_g$, and $\sigma_{f+g} \le \max(\sigma_f, \sigma_g)$.

If, for a given $\rho > 0$, the type of a function is infinite, then the function is of *maximal type*; for $0 < \sigma_f < \infty$ the type is *normal* or *mean*; for $\sigma_f = 0$ the type is *minimal*. In the last case, for any $\varepsilon > 0$ the asymptotic inequality

$$M_f(r) \overset{\text{as}}{<} e^{\varepsilon r^\rho}$$

is fulfilled.

Entire functions of order $\rho = 1$ and normal type σ are called *entire functions of exponential type* σ.

EXAMPLES. Verify that $\sin Az$ is of order $\rho = 1$ and type $\sigma = |A|$, which means that it is an entire function of exponential type $|A|$; $\sin\sqrt{z}/\sqrt{z}$ is of order $1/2$ and type 1; $\exp\{a_0 z^n + \ldots + a_n\}$, $a_0 \neq 0$, is of order n and type $|a_0|$.

1.3. The relation between the growth of an entire function and the decrease of the coefficients of its power series expansion

Let

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1}$$

be an entire function.

LEMMA 1. *If the asymptotic inequality*

$$M_f(r) \overset{\text{as}}{<} e^{Ar^K} \tag{2}$$

is fulfilled, then

$$|c_n| \overset{\text{as}}{<} \left(\frac{eAK}{n}\right)^{n/K} . \tag{3}$$

PROOF. By the Cauchy inequality, it follows from (2) that

$$|c_n| \leq \frac{M(r)}{r^n} < e^{Ar^K - n\log r}, \quad r \geq r_0 . \tag{4}$$

Minimizing the exponent with respect to r, we obtain $KAr^{K-1} - n/r = 0$ and $r_n^K = n/(AK)$. For sufficiently large n we have $r_n \geq r_0$. After substituting r_n in (4) we obtain (3).

LEMMA 2. *If the asymptotic inequality* (3) *is fulfilled, then*

$$M_f(r) \overset{\text{as}}{<} e^{(A+\varepsilon)r^K}, \quad \forall\varepsilon > 0 . \tag{5}$$

PROOF. First, note that if an entire function f satisfies inequality (5), then so does the function $f + Q$, where Q is a polynomial. Therefore, we can assume that $c_0 = 0$ and (3) holds for all $n \geq 1$. Thus, we have

$$\begin{aligned} |f(z)| &\leq \sum_{n=1}^{\infty} |c_n| r^n \leq \sum_{n=1}^{\infty} \left(\frac{eAK}{n}\right)^{n/K} r^n \\ &= \sum_{n=1}^{\infty} \left(\frac{eAr^K}{n/K}\right)^{n/K}, \quad |z| = r . \end{aligned}$$

Set $m = [n/K]$. Then, for sufficiently large r, we have

$$\left(\frac{eAr^K}{n/K}\right)^{n/K} \leq \left(\frac{eAr^K}{m}\right)^{m+1} .$$

Hence

$$|f(z)| \leq \sum_{m=1}^{\infty} \left(\frac{eAr^K}{m}\right)^{m+1} .$$

By using the Stirling formula

$$m! \sim \left(\frac{m}{e}\right)^m \sqrt{2\pi m}, \quad m \to \infty$$

and the inequality

$$\sqrt{2\pi m} < C\left(\frac{A+\varepsilon}{A}\right)^{m+1}, \quad m \geq 1\,,$$

we obtain

$$\begin{aligned}|f(z)| &\leq C_1 \sum_{m=1}^{\infty} \frac{e^m}{m^m}\left(Ar^K\right)^{m+1} < C_1 \sum_{m=1}^{\infty} \frac{(A+\varepsilon/2)^{m+1} r^{K(m+1)}}{m!} \\ &= C_1 e^{(A+\varepsilon/2)r^K}\left(A+\frac{\varepsilon}{2}\right)er^K \overset{\text{as}}{<} C_2 e^{(A+\varepsilon)r^K}\,,\end{aligned}$$

where C, C_1, C_2 are constants. Thus,

$$|f(z)| \overset{\text{as}}{<} e^{(A+\varepsilon)r^K}\,.$$

The lemma is proved.

These lemmas enable us to express the order and the type of an entire function in terms of the rate of decrease of the coefficients of its power expansion. Indeed, the order ρ equals the greatest lower bound of those K for which (5) holds for any $A > 0$, in particular, for $A + \varepsilon = 1$. By Lemmas 1 and 2 we have

$$\left(\frac{e(\rho-\varepsilon)}{n}\right)^{\frac{n}{\rho-\varepsilon}} \overset{\text{n}}{<} |c_n| \overset{\text{as}}{<} \left(\frac{e(\rho+\varepsilon)}{n}\right)^{\frac{n}{\rho+\varepsilon}}$$

for each $\varepsilon > 0$. Having taken the logarithms, we obtain

$$\frac{n}{\rho-\varepsilon}[\log e(\rho-\varepsilon) - \log n] \overset{\text{n}}{<} \log|c_n| \overset{\text{as}}{<} \frac{n}{\rho+\varepsilon}[\log e(\rho-\varepsilon) - \log n],$$

or

$$\frac{n\log n}{\rho+\varepsilon}[1+o(1)] \overset{\text{as}}{<} \log\frac{1}{|c_n|} \overset{\text{n}}{<} \frac{n\log n}{\rho-\varepsilon}[1+o(1)]\,.$$

Thus we have proved

THEOREM 2. *The order of the entire function* (1) *is determined by the formula*

(6) $$\rho = \limsup_{n\to\infty} \frac{n\log n}{\log(1/|c_n|)}\,.$$

Likewise, the type σ equals the greatest lower bound of those A for which estimate (3) holds with $K = \rho$. From this we deduce

THEOREM 3. *The type of the entire function* (1) *is determined by the formula*

(7) $$\sigma = \frac{1}{\rho e}\limsup_{n\to\infty}\left(n\sqrt[n]{|c_n|^\rho}\right)\,.$$

EXAMPLES. Let $0 < \rho < \infty$, $0 < \sigma < \infty$. The entire function

$$f(z) = \sum_{n=1}^{\infty} \left(\frac{e\sigma\rho}{n}\right)^{n/\rho} z^n$$

is of order ρ and of type σ. The function

$$f(z) = \sum_{n=2}^{\infty} \left(\frac{e\sigma\rho}{n\log n}\right)^{n/\rho} z^n$$

is of order ρ and of minimal type, whereas the function

$$f(z) = \sum_{n=2}^{\infty} \left(\frac{e\rho\log n}{n}\right)^{n/\rho} z^n$$

is of order ρ and of maximal type. The entire function

$$f(z) = \sum_{n=2}^{\infty} \left(\frac{1}{\log n}\right)^{n} z^n$$

is of infinite order, and the function

$$f(z) = \sum_{n=0}^{\infty} e^{-n^2} z^n$$

is of zero order.

PROBLEM 4. Using formulas (6) and (7), prove that the order and type of an entire function do not change under differentiation.

PROBLEM 5. If $f(z)$ is an entire function and the numbers $f^{(n)}(0)$ are integers, then either $f(z)$ is a polynomial, or the type σ_f of this function with respect to the order $\rho = 1$ is at least 1.

LECTURE 2

Main Integral Formulas for Functions Analytic in a Disk

To investigate the relation between the growth of an entire function and its zeros we shall need several formulas.

2.1. The Poisson formula and the Schwarz formula

We assume that the reader knows the classical Poisson formula which represents a function harmonic in a disk $\{z : \ |z| < R\}$ and continuous in its closure:

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\psi})\frac{R^2 - r^2}{R^2 - 2Rr\cos(\psi - \theta) + r^2}\, d\psi\ , \quad z = re^{i\theta}\ .$$

The same formula may be written in the form

$$(1) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}\, d\psi = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\operatorname{Re}\frac{\zeta + z}{\zeta - z}\, d\psi\ ,$$

where $\zeta = Re^{i\psi}$.

To represent a function $f = u + iv$ holomorphic in a disk $\{z : \ |z| < R\}$, whose real part u is continuous in $\{z : \ |z| \le R\}$, we shall be using the Schwarz formula:

$$(2) \qquad f(z) = \frac{1}{2\pi}\int_0^{2\pi} u(\zeta)\frac{\zeta + z}{\zeta - z}\, d\psi + iv(0)\ .$$

The latter formula follows from (1). Indeed, by (1) the real parts of the functions on the left and right sides of (2) coincide in $\{z : \ |z| < R\}$. Hence, the functions differ by a purely imaginary constant, and for $z = 0$ they coincide: $u(0) + iv(0) = f(0)$.

2.2. The Poisson-Jensen formula

If $f(z) \neq 0$ in a disk $\{z : \ |z| \le R\}$, then $\log f(z)$ is a holomorphic function in the disk, and by formula (2) we have

$$(3) \qquad \log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|\frac{Re^{i\psi} + z}{Re^{i\psi} - z}\, d\psi + iC\ .$$

Formula (1), as well as formula (3), implies

$$(4) \qquad \log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|\frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \psi) + r^2}\, d\psi\ .$$

Now let $a_1, a_2, \dots, a_n$ be the zeros of $f(z)$ in $\{z: |z| < R\}$ arranged according to increasing modulus. We shall make a permanent convention to write down each zero as many times as its multiplicity. Let $f(z) \neq 0$ for $|z| = R$, and let

$$\varphi(z) = f(z) \prod_{m=1}^{n} \frac{R^2 - \bar{a}_m z}{R(z - a_m)} . \tag{5}$$

It is evident that $|\varphi(Re^{i\psi})| = |f(Re^{i\psi})|$ and $\varphi(z) \neq 0$ for $|z| \leq R$. Let us apply formulas (3) and (4) to φ. It follows from (3) that

$$\log \varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| \frac{Re^{i\psi} + z}{Re^{i\psi} - z} \, d\psi + iC$$

or

$$\log f(z) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| \frac{Re^{i\psi} + z}{Re^{i\psi} - z} \, d\psi + \sum_{|a_m|<R} \log \frac{R(z - a_m)}{R^2 - z\bar{a}_m} + iC . \tag{6}$$

For the function $\log \dfrac{R(z - a_m)}{R^2 - z\bar{a}_m}$ to be single-valued it is necessary to cut the complex plane along the rays $\{z = re^{i \arg a_m}, \ r \geq |a_m|\}$. If some cut meets the point z, we shall slightly deform it counterclockwise. After separating the imaginary parts in (6) we obtain

$$\begin{aligned} \log |f(z)| = {} & \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - \psi) + r^2} \, d\psi \\ & + \sum_{|a_m|<R} \log \left| \frac{R(z - a_m)}{R^2 - z\bar{a}_m} \right| , \quad z = re^{i\theta} . \end{aligned} \tag{7}$$

Formula (7) was derived by R. Nevanlinna who named it after Poisson and Jensen. It forms a foundation of the Nevanlinna theory of distribution of values of meromorphic functions.

2.3. The Jensen formula

Let us assume, first, that $f(0) \neq 0$. If we set $z = 0$ in equation (7) we obtain

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| \, d\psi + \sum_{|a_m|<R} \log \frac{|a_m|}{R} .$$

The second term on the right can be written as a Stieltjes integral. Denoting by $n(t)$ the number of points a_m satisfying the inequality $|a_m| \leq t$, we obtain a left continuous, monotonic, integer-valued and piecewise constant function. It is called a *counting function* of zeros. We have

$$\sum_{|a_m|<R} \log \frac{R}{|a_m|} = \int_0^R \log \frac{R}{t} \, dn(t) = n(t) \log \frac{R}{t} \Big|_0^R + \int_0^R \frac{n(t)}{t} \, dt ,$$

and, finally,

$$\int_0^R \frac{n(t)}{t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| \, d\psi - \log |f(0)| . \tag{8}$$

This is the famous Jensen formula.

If $f(0) = 0$ (and, of course, $f \not\equiv 0$) we denote by k the multiplicity of the root at $z = 0$. Then formula (8) takes on the form

$$\int_0^R \frac{n(t)-n(0)}{t}\,dt + n(0)\log R = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|\,d\psi - \log\left|\frac{f^{(k)}(0)}{k!}\right| .$$

To prove the latter formula it is sufficient to apply (8) to the function $f(z)/z^k$.

2.4. The Nevanlinna characteristics

If $f(z)$ is a meromorphic function in the disk $\{z : |z| \le R\}$ with neither zeros nor poles on the circumference $\{z : |z| = R\}$ and at $z = 0$, then the function $\varphi(z)$ must be chosen different from (5). Namely, let

$$\varphi(z) = f(z) \prod_{|b_m|<R} \frac{R(z-b_m)}{R^2-\bar{b}_m z} \left\{ \prod_{|a_m|<R} \frac{R(z-a_m)}{R^2-\bar{a}_m z} \right\}^{-1} ,$$

where $\{b_m\}$ are poles of a function f in the disk $\{z : |z| < R\}$. Then formula (6) is replaced by

$$\log f(z) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})| \frac{Re^{i\psi}+z}{Re^{i\psi}-z}\,d\psi + \sum_{|a_m|<R} \log\frac{R(z-a_m)}{R^2-z\bar{a}_m} - \sum_{|b_m|<R} \log\frac{R(z-b_m)}{R^2-z\bar{b}_m} + iC .$$

The Poisson-Jensen formula (7) takes the form

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})| \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\psi)+r^2}\,d\psi + \sum_{|a_m|<R} \log\left|\frac{R(z-a_m)}{R^2-z\bar{a}_m}\right| - \sum_{|b_m|<R} \log\left|\frac{R(z-b_m)}{R^2-z\bar{b}_m}\right| , \tag{9}$$

and the Jensen formula becomes

$$\int_0^R \frac{n(t,0)}{t}\,dt - \int_0^R \frac{n(t,\infty)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|\,d\psi - \log|f(0)| . \tag{10}$$

Here $n(t,0)$ is the counting function of zeros, and $n(t,\infty)$ is the counting function of poles of the function f . Following R. Nevanlinna, let us introduce

$$m(R,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\psi})|\,d\psi ,$$

where $a^+ = \max(a,0)$, and

$$N(R,f) = \int_0^R \frac{n(t,\infty)-n(0,\infty)}{t}\,dt + n(0,\infty)\log R ,$$
$$T(R,f) = m(R,f) + N(R,f) .$$

The function $T(R, f)$ is called the Nevanlinna characteristic. In this notation, the Jensen formula (10) becomes

$$T(R, f) = T(R, 1/f) + C \,. \tag{11}$$

Here C is a constant, and if $f(0) \neq 0$ and $f(0) \neq \infty$, then $C = \log|f(0)|$.

It is easy to see that

$$T(R, af + b) = T(R, f) + O(1) \,, \quad a \neq 0 \,,$$

and hence equation (11) implies

$$T\Big(R, \frac{af+b}{cf+d}\Big) = T(R, f) + O(1) \,, \quad ad - bc \neq 0 \,.$$

This relation is called the *First Main Theorem of Nevanlinna.*

PROBLEM 1. Prove the following estimates:

$$\begin{aligned} T\Big(R, \sum_{\nu=1}^{n} f_\nu\Big) &\le \sum_{\nu=1}^{n} T(R, f_\nu) + \log n \,, \\ T\Big(R, \prod_{\nu=1}^{n} f_\nu\Big) &\le \sum_{\nu=1}^{n} T(R, f_\nu) \,. \end{aligned} \tag{12}$$

Here f_ν, $1 \le \nu \le n$, are meromorphic functions.

PROBLEM 2. Prove the following statements (H. Cartan).

1. If $f(z)$ is a meromorphic function, and $f(z) = f_1(z)/f_2(z)$, where $f_1(z)$, $f_2(z)$ are entire functions without common zeros, then

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \max(\log|f_1(re^{i\psi})|, \log|f_2(re^{i\psi})|)\, d\psi + O(1) \,.$$

2. If $f(z)$ is a meromorphic function, then

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} N\Big(r, \frac{1}{f - e^{i\theta}}\Big)\, d\theta + C_f \,,$$

where the constant C_f does not depend on r.

HINT. Use the formula

$$\frac{1}{2\pi}\int_0^{2\pi} \log|w - e^{i\theta}|\, d\theta = \log^+|w| \,.$$

The function $T(R, f)$ plays an important role in the study of entire and meromorphic functions. If f is an entire function, then

$$T(R, f) = m(R, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\psi})|\, d\psi \le \log^+ M_f(R) \,.$$

On the other hand, using the Poisson-Jensen formula (7) we have

$$\begin{aligned} \log|f(z)| &\le \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})| \frac{R^2 - r^2}{|Re^{i\psi} - z|^2}\, d\psi \\ &\le \frac{R+r}{R-r}\, \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(Re^{i\psi})|\, d\psi \,, \quad r = |z| \,, \end{aligned}$$

and

$$\log M_f(r) \le \frac{R+r}{R-r} m(R, f) .$$

For $R = 2r$ we obtain

$$\log M_f(r) \le 3m(2r, f). \tag{13}$$

If, in particular, $m(r, f) \overset{\text{as}}{<} Ar^k$, then

$$\log M_f(r) \overset{\text{as}}{<} 3 \cdot 2^k A r^k.$$

Hence, in defining the order of an entire function one can use the Nevanlinna characteristic $T(r, f)$ instead of $\log M_f(r)$.

Let f_1, f_2 be entire functions such that the quotient $\varphi = f_1/f_2$ is an entire function. Then the First Main Theorem of Nevanlinna and estimates (13) and (12) imply

$$\begin{aligned} \log M_\varphi(r) &\le 3T(2r, \varphi) \le 3[T(2r, f_1) + T(2r, f_2) + O(1)] \\ &\le 3 \log M_{f_1}(2r) + 3 \log M_{f_2}(2r) + O(1) . \end{aligned}$$

A theorem on the growth of a quotient of entire functions now directly follows from the latter inequality.

THEOREM 1. *If the quotient of two entire functions of order not greater than ρ is an entire function, then its order is also at most ρ. If, in addition, the numerator and denominator are of mean type with respect to ρ, then the quotient is of mean type with respect to ρ.*

We refer the reader to the monographs by Nevanlinna [**102**], Hayman [**51**], Goldberg and Ostrovskii [**43**] where the Nevanlinna theory of meromorphic functions and its applications can be found.

2.5. Some corollaries of the Jensen formula

Let f be an entire function. Then it follows directly from the Jensen formula that

$$\log |f(0)| \le \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\psi})| \, d\psi . \tag{14}$$

If $|f(0)| = 1$, then for $r > 0$ we have

$$\log M_f(er) \ge \int_0^{er} \frac{n(t)}{t} \, dt \ge \int_r^{er} \frac{n(t)}{t} \, dt \ge n(r),$$

and hence

$$n(r) < \log M_f(er) . \tag{15}$$

The modulus of an entire function may decrease in some directions, and inequality (14) shows that "in the mean" it decreases not faster than it grows. The latter inequality shows that an entire function with an upper bound for $M_f(r)$ cannot have too many zeros. We remark that if $|f(0)| \ne 1$, then (15) must be replaced by $n(r) < \log M_f(er) + \text{const}$.

LECTURE 3

Some Applications of the Jensen Formula

3.1. A theorem on (I)-quasianalyticity

A class C of functions defined on some interval is called (I)-quasianalytic if each function $g \in C$ vanishing almost everywhere on an interval, no matter how small is its length, vanishes almost everywhere on its domain of definition. We shall use the Jensen formula to prove a theorem on (I)-quasianalyticity.

THEOREM 1 (Pólya). *If a 2π-periodic function $f \in L^2[-\pi,\pi]$ is represented by a lacunary Fourier series*

$$f(t) \sim \sum_{k=-\infty}^{\infty} c_{n_k} e^{in_k t}\,, \qquad \lim_{k\to\pm\infty} \frac{n_k}{k} = +\infty$$

and $f(t) = 0$ almost everywhere on an arbitrarily small interval, then $f(t) = 0$ almost everywhere on $[-\pi,\pi]$.

PROOF. Making a shift of the periodic function $f(t)$, we obtain a function $\varphi(t) = f(t+h)$ equal to zero for $\pi - \delta < |t| \le \pi$ and represented by a series

$$\varphi(t) \sim \sum_{k=-\infty}^{\infty} c_{n_k} e^{in_k h} e^{in_k t} = \sum_{k=-\infty}^{\infty} d_{n_k} e^{in_k t}\,.$$

By a well-known formula we have

$$d_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t) e^{-int}\,dt\,.$$

According to the conditions of Theorem 1 "many" coefficients d_n are equal to zero. We shall prove that in this case all coefficients vanish.

Let $\varphi(t) \not\equiv 0$. We define the function

$$\Phi(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t) e^{-izt}\,dt = \frac{1}{2\pi}\int_{-\pi+\delta}^{\pi-\delta} \varphi(t) e^{-izt}\,dt\,.$$

It is easy to see that $\Phi(z)$ is an entire function, and that

$$|\Phi(x+iy)| \le \frac{1}{2\pi}\int_{-\pi+\delta}^{\pi-\delta} |\varphi(t)|\,dt\, e^{(\pi-\delta)|y|} \le \frac{1}{\sqrt{2\pi}} \|\varphi\|_{L^2[-\pi,\pi]} e^{(\pi-\delta)|y|}$$

or

$$\log|\Phi(re^{i\theta})| \le (\pi-\delta) r|\sin\theta| + C_\varphi\,. \tag{1}$$

Let $n(t)$ be the counting function for zeros of the function $\Phi(z)$. We have $\Phi(n) = 0$ for $n \neq n_k$, and hence

$$n(t) \geq 2[t] + 1 - n_1^+(t) - n_1^-(t) ,$$

where $n_1^+(t)$ and $n_1^-(t)$ are the numbers of points n_k located inside intervals $[0, t)$ and $[-t, 0)$ respectively. Let $n_0 < 0 \leq n_1$. Then, for $n_k \leq t < n_{k+1}$, $k > 0$, we have

$$n_1^+(t) = k = o(n_k) = o(t) , \quad t \to \infty .$$

In the same way, $n_1^-(t) = o(t)$, $t \to \infty$. Hence

$$n(t) \overset{\text{as}}{>} (2 - \varepsilon)t$$

for all $\varepsilon > 0$. Therefore,

$$\int_{r_0}^{r} \frac{n(t)}{t}\, dt \overset{\text{as}}{>} (2 - 2\varepsilon)r . \tag{2}$$

Applying the Jensen formula (equation (8), Section 2.3), and estimates (1) and (2), we obtain

$$N(r) \overset{\text{as}}{\leq} \frac{1}{2\pi} \int_0^{2\pi} \left(\pi - \frac{\delta}{2}\right) r |\sin\theta|\, d\theta = \frac{4}{2\pi}\left(\pi - \frac{\delta}{2}\right) r = (2 - \frac{\delta}{\pi})r ,$$

and so

$$2 - 2\varepsilon < 2 - \delta/\pi .$$

This is a contradiction! The function $\Phi(z)$ has too many roots for its insufficiently fast growth. Thus, $\Phi(z) \equiv 0$, all Fourier coefficients d_n are equal to zero, and $f(t) = 0$ almost everywhere. The theorem is proved.

PROBLEM 1. Let

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n t} , \quad \sum_{n=-\infty}^{\infty} |c_n| < \infty ,$$

where λ_n are real numbers, and let

$$\frac{\lambda_n}{n} \to +\infty , \quad |n| \to \infty .$$

If $f(t) = 0$ on an interval, no matter how small, then $f(t) \equiv 0$.

HINT. Use the identity

$$\int_0^{\infty} f(t)e^{-tz}\, dt = \sum_{n=-\infty}^{\infty} \frac{c_n}{z - i\lambda_n} , \quad \operatorname{Re} z > 0 ,$$

and apply the Jensen formula to the meromorphic function on the right-hand side of the latter formula.

The reader can find more sophisticated theorems related to the same field in the monographs by Levin [**82**] (Appendix 2), Levinson [**84**], Koosis [**72**].

3.2. The convergence exponent and the upper density of the sequence of zeros

DEFINITION. Given a sequence $a_1, a_2, \dots, a_n, \dots$, $a_n \neq 0$, $\lim_{n\to\infty} a_n = \infty$, the greatest lower bound of λ's such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda}$$

converge is called the *convergence exponent.*

Let $n(r)$ be the counting function of a sequence $\{a_n\}$. We denote by ρ_1 its order; i.e.,

$$\rho_1 = \limsup_{r\to\infty} \frac{\log n(r)}{\log r} .$$

The number

$$\overline{\Delta} = \limsup_{r\to\infty} \frac{n(r)}{r^{\rho_1}}$$

is called the upper density of the sequence $\{a_n\}$ with respect to the given order ρ_1. It is evident that

$$r^{\rho_1-\varepsilon} \overset{\text{n}}{<} n(r) \overset{\text{as}}{<} r^{\rho_1+\varepsilon}$$

and

$$(\overline{\Delta} - \varepsilon) r^{\rho_1} \overset{\text{n}}{<} n(r) \overset{\text{as}}{<} (\overline{\Delta} + \varepsilon) r^{\rho_1}$$

for every $\varepsilon > 0$. The number

$$\underline{\Delta} = \liminf_{r\to\infty} \frac{n(r)}{r^{\rho_1}}$$

is called the lower density of the sequence $\{a_n\}$ with respect to the given order ρ_1.

PROBLEM 2. Prove the identities

$$\overline{\Delta} = \limsup_{n\to\infty} \frac{n}{|a_n|^\rho},$$

$$\underline{\Delta} = \liminf_{n\to\infty} \frac{n}{|a_n|^\rho} .$$

LEMMA 1. *Let a series*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda}$$

be convergent for some $\lambda > 0$. Then the integral

$$\int_0^\infty \frac{n(t)}{t^{\lambda+1}}\, dt$$

converges, and

$$\lim_{t\to+\infty} \frac{n(t)}{t^\lambda} = 0 .$$

PROOF. Since

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^\lambda} = \int_0^\infty \frac{dn(t)}{t^\lambda}, \tag{3}$$

upon integrating by parts we find

$$\int_0^r \frac{dn(t)}{t^\lambda} = \frac{n(r)}{r^\lambda} + \lambda \int_0^r \frac{n(t)}{t^{\lambda+1}}\, dt. \tag{4}$$

The convergence of the series in (3) implies that both summands on the right-hand side of (4) are bounded from above. The second summand does not decrease, and therefore tends to the finite limit, which together with the inequality

$$\frac{n(r)}{r^\lambda} \le \lambda \int_r^\infty \frac{n(t)}{t^{\lambda+1}}\, dt$$

proves Lemma 1.

LEMMA 2. *The convergence exponent of the sequence $\{a_n\}$ is equal to the order ρ_1 of its counting function.*

PROOF. Let K be the convergence exponent, and let $\lambda > K$. Then the series in (3) converges, and by Lemma 1 we have $\lim\limits_{r\to\infty} \dfrac{n(r)}{r^\lambda} = 0$. Hence $\rho_1 \le \lambda$ and $\rho_1 \le K$. On the other hand,

$$n(t) \overset{\text{as}}{<} t^{\rho_1+\varepsilon/2}, \quad \varepsilon > 0.$$

Hence for $\lambda = \rho_1 + \varepsilon$ the integral

$$\int_0^\infty \frac{n(t)}{t^{\lambda+1}}\, dt$$

converges and $n(t)/t^\lambda \to 0$, $t \to \infty$. It follows from (3) and (4) that the series in (3) converges, and therefore $K \le \rho_1$, proving Lemma 2.

PROBLEM 3. Prove that the convergence of the series in (3) is equivalent to the convergence of the integral on the right-hand side of (4).

PROBLEM 4. Prove that if the terms of a converging series form a decreasing sequence $a_1 \ge a_2 \ge a_3 \ge \dots$, then $na_n \to 0$ (E. Borel).

THEOREM 2 (Hadamard). *The convergence exponent of zeros of an entire function does not exceed its growth order.*

PROOF. According to a corollary to the Jensen formula, Section 2.5, we have

$$n(r) \le \log M_f(er) + O(1).$$

It follows that

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} \le \limsup_{r\to\infty} \frac{\log\log M_f(er)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r},$$

i.e., $\rho_1 \le \rho$. The theorem is proved.

We shall give another corollary of the Jensen formula, namely, a uniqueness theorem that does not permit an entire function to vanish on a "dense set".

THEOREM 3. *Let $f(z)$ be an entire function of type not greater than σ with respect to an order ρ . If $f(z)$ vanishes on a set Λ and at least one of the inequalities*

$$\overline{\Delta}(\Lambda) > e\rho\sigma \ , \tag{5}$$

$$\underline{\Delta}(\Lambda) > \rho\sigma \tag{6}$$

holds, where $\overline{\Delta}(\Lambda)$ and $\underline{\Delta}(\Lambda)$ are the upper and lower density of the sequence Λ with respect to the order ρ, then $f(z) \equiv 0$.

PROOF. Let us assume, for example, that (5) holds. We denote by $n_\Lambda(r)$ the counting function of the sequence Λ and set $n(r) = n_f(r)$.

For every $\lambda > 1$, we have

$$n_\Lambda(r) \le n(r) \le \frac{1}{\log \lambda} \int_r^{\lambda r} \frac{n(t)}{t}\, dt \le \frac{1}{\log \lambda} N(\lambda r) \ .$$

By the Jensen formula, with $f \not\equiv 0$,

$$N(\lambda r) \le \log M_f(\lambda r) + O(1) \overset{\text{as}}{<} (\sigma + \varepsilon)\lambda^\rho r^\rho \ .$$

Hence

$$\overline{\Delta}(\Lambda) \le \frac{\sigma \lambda^\rho}{\log \lambda}.$$

Minimizing with respect to λ, we obtain $\overline{\Delta}(\Lambda) \le e\rho\sigma$, a contradiction.

If $\underline{\Delta}(\Lambda) > \rho\sigma$, then for some $\varepsilon > 0$ we have $n(r) \ge n_\Lambda(r) \overset{\text{as}}{>} (\rho\sigma + 2\varepsilon) r^\rho$. Hence $N(r) > \dfrac{1}{\rho}(\rho\sigma + \varepsilon) r^\rho$, and using the Jensen formula we obtain

$$\log M_f(r) \overset{\text{as}}{>} \frac{1}{\rho}(\rho\sigma + \varepsilon) r^\rho \ , \quad \sigma > \frac{1}{\rho}(\rho\sigma + \varepsilon) \ ,$$

a contradiction again.

3.3. Completeness of a system of exponential functions

DEFINITION. A system $\{x_k\}$ of elements of a linear topological space E is said to be complete if the closure of its linear hull coincides with E. In other words, each element $x \in E$ may be approximated by finite linear combinations of elements of a complete system $\{x_k\}$.

If a system is not complete, then the closure of its linear hull is a proper subspace $L \subset E$. If E is a locally convex space, then by the Hahn-Banach theorem there exists a nonzero linear functional $f \in E^*$ such that $f(x) = 0$ for every element $x \in \{x_k\}$. The existence of such a functional is a necessary and sufficient condition of the noncompleteness.

Let a system $\{e^{i\lambda_k t}\}$ be given with real exponents λ_k.

THEOREM 4. *Let $n(t)$ be the counting function of a sequence $\{\lambda_k\} = \Lambda$. If*

$$\liminf_{t \to \infty} \frac{n(t)}{t} > 2 \ , \tag{7}$$

then the system $\{e^{i\lambda_k t}\}$ is complete in the space of continuous functions $C[-\pi, \pi]$.

PROOF. If the completeness fails, then by the F. Riesz theorem on the form of a linear functional on the space of continuous functions there exists a nonconstant function $\sigma(t)$ of bounded variation such that

$$\int_{-\pi}^{\pi} e^{i\lambda_k t}\, d\sigma(t) = 0\,, \quad \lambda_k \in \Lambda\,.$$

The function

$$\Phi(\lambda) = \int_{-\pi}^{\pi} e^{i\lambda t}\, d\sigma(t)$$

is entire, not identically equal to zero and satisfies the inequality

$$|\Phi(s+i\tau)| \le (\mathrm{Var}\ \sigma) e^{\pi|\tau|}\,.$$

Since $\Phi(\lambda_k) = 0$, we obtain, by the Jensen formula, Section 2.3,

$$\int_{r_0}^{r} \frac{n(t)}{t}\, dt \le \frac{\pi r}{2\pi} \int_0^{2\pi} |\sin\theta|\, d\theta + O(1) = 2r + O(1)\,, \quad r \to \infty\,.$$

Since for some $\varepsilon > 0$ we have

$$\frac{n(t)}{t} \overset{\text{as}}{>} 2 + \varepsilon\,,$$

we conclude that

$$(2+\varepsilon) r + O(1) \le 2r + O(1)\,, \quad r \to \infty\,.$$

This is a contradiction. The theorem is proved.

PROBLEM 5. Let λ_0 be an arbitrary real number and let $\lambda_n = n - \delta_n$, $\lambda_{-n} = -n + \delta_n$ be pairwise distinct real numbers with $|n| > \delta_n \ge \delta > 0$, $n = \pm 1, \pm 2, \dots$. Prove that the system of exponential functions $\{e^{i\lambda_n t}\}$ is complete in the space $C[-\pi, \pi]$.

Some more sofisticated theorems on the completeness of a family $\{e^{i\lambda_n t}\}$ will be proved in the second part of the monograph. In particular, it will be proved that the assertion of Theorem 4 remains in force if the *lower* limit in (7) is changed to the *upper* limit.

3.4. Completeness of a special system of functions in countably normed spaces

Let us consider the space $A(D)$ of all analytic functions in a simply connected domain $D \subset \mathbb{C}$. Let us choose an expanding sequence $G_1, G_2, \dots$ of compact sets which exhaust D from the inside and are such that every G_m is compactly imbedded in D:

$$G_1 \Subset G_2 \Subset \cdots \Subset G_m \Subset \cdots\,; \quad G_1 \cup G_2 \cup \cdots \cup G_m \cup \cdots = D\,,$$

and let us introduce the system of norms

$$\|f\|_m = \sup_{z \in G_m} |f(z)|\,. \tag{8}$$

The space $A(D)$ endowed with the system of norms (8) is countably normed. The following proposition describes the general form of a linear functional on the space $A(D)$.

THEOREM 5. *For every linear functional $F \in A^*(D)$ there exists a unique function $\varphi(\zeta)$ analytic on a closed simply connected set $\mathbb{C}\backslash D'$, $D' \Subset D$, equal to zero at infinity and such that the value of F at a function $f \in A(D)$ is determined by the identity*

$$F[f] = \frac{1}{2\pi i}\int_l f(\zeta)\varphi(\zeta)\,d\zeta\,. \tag{9}$$

Here l is a simple closed curve lying inside D such that φ is analytic on l and outside l.[3]

PROOF. It follows from the definition of topology by norms (8) that a homogeneous and additive functional $F[f]$ is continuous if and only if there exist a number $m \geq 1$ and a constant C such that the inequality

$$|F[f]| \leq C\|f\|_m \tag{10}$$

holds. Let $F[f]$ be some linear functional on the space $A(D)$, and let $G_m \Subset D$ be a domain corresponding to the norm in (10). Let us choose an intermediate domain D', $G_m \Subset D' \Subset D$. By (10), the functional F can be extended to a linear functional on the space $C(G_m)$ of functions continuous on G_m with the norm (8).

Now let $\zeta \in \mathbb{C}\backslash D'$. Noting that $\dfrac{1}{\zeta - z} \in C(G_m)$, we define the function

$$\varphi(\zeta) = F\Big[\frac{1}{\zeta - z}\Big], \tag{11}$$

where the functional F is applied with respect to the variable z. It is natural to call the function φ the Cauchy-Stieltjes transform of F. By (10), the function $\varphi(\zeta)$ extends to an analytic function on the closed set $\mathbb{C}\backslash D'$. If $\zeta \to \infty$, then $\left\|\dfrac{1}{\zeta - z}\right\|_m \to 0$, and it follows from (10) that $\varphi(\infty) = 0$. Finally, let a simple curve l encircling G_m in the domain D' be chosen close enough to the boundary $\partial D'$ for the function φ to be analytic on l and in the component of the set $\mathbb{C}\backslash l$ containing the infinite point. Then it follows from (11) that

$$\begin{aligned}\frac{1}{2\pi i}\int_l \varphi(\zeta)f(\zeta)\,d\zeta &= \frac{1}{2\pi i}\int_l F\Big[\frac{1}{\zeta - z}\Big]f(\zeta)\,d\zeta \\ &= F\Big[\frac{1}{2\pi i}\int_l \frac{f(\zeta)}{\zeta - z}\,d\zeta\Big] = F[f]\end{aligned}$$

for every function $f \in A(D)$, which proves (9).

Let φ_1 and φ_2 be two functions determining the functional $F[f]$ according to (9). We may assume that the same curve l corresponds to each of them. If $\psi = \varphi_1 - \varphi_2$, then

$$\int_l \psi(\zeta)\zeta^k\,d\zeta = 0\,, \quad k = 0, 1, 2, \dots\,.$$

Since the function ψ is analytic in the exterior of l and $\psi(\infty) = 0$, it follows that $\psi \equiv 0$. Thus, every linear functional is representable by equation (9). The converse statement is obvious. The theorem is proved.

[3] In what follows we shall say that the bounded domain whose boundary is l contains all singularities of φ.

Let us apply Theorem 5 to study the completeness of the system of functions $\varphi_n(z) = F(\lambda_n z)$, where $F(z)$ is an entire function and $\{\lambda_n\}$ is a sequence of complex numbers. First results on completeness of a system of functions $\{F(\lambda_n z)\}$ were obtained by A. O. Gelfond in 1937. A. I. Markushevich obtained more complete results using a method close to that described in this section.

THEOREM 6. *Let an entire function of order ρ*

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

with all coefficients a_n nonzero, be of type not exceeding σ, and let $\Lambda = \{\lambda_n\}$ be a sequence of complex numbers. Then the system $\varphi_n(z) = F(\lambda_n z)$ is complete in the disk $\{z : |z| < R\}$, where

$$R^\rho = \frac{1}{\rho\sigma} \max\left(\frac{\overline{\Delta}(\Lambda)}{e}, \underline{\Delta}(\Lambda)\right), \tag{12}$$

and $\overline{\Delta}(\Lambda)$ and $\underline{\Delta}(\Lambda)$ are the upper and lower densities of the sequence Λ with respect to the order ρ.

PROOF. Suppose that the system $\{F(\lambda_n z)\}$ is not complete in the space $A(D)$, $D = \{z : |z| < R\}$. Then by Theorem 5 there exists a function $\psi(z)$ that is analytic outside a disk $\{z : |z| < r\}$, $r < R$, vanishes at infinity, is not equal to zero identically and satisfies

$$\int_{|z|=r} F(\lambda_n z)\psi(z)\, dz = 0\ , \quad n = 0, 1, 2, \dots\ , \quad r < R\ .$$

Let us consider the function

$$\Phi(\lambda) = \int_{|z|=r} F(\lambda z)\psi(z)\, dz\ .$$

This is an entire function vanishing at the points of Λ. Let us estimate its growth. Using

$$|F(\lambda z)| \overset{\text{as}}{\leq} \exp\{(\sigma + \varepsilon)|\lambda|^\rho r^\rho\}\ , \quad \varepsilon > 0\ , \quad r = |z|\ ,$$

we obtain

$$\begin{aligned} |\Phi(\lambda)| &\leq \int_{|z|=r} |F(\lambda z)||\psi(z)||dz| \\ &\overset{\text{as}}{\leq} 2\pi r M \exp\{(\sigma + \varepsilon)|\lambda|^\rho r^\rho\}\ , \quad M = \max_{|z|=r} |\psi(z)|\ , \quad \varepsilon > 0\ . \end{aligned}$$

Hence the type of the function Φ does not exceed σr^ρ. If the sequence Λ satisfies $\lim_{n\to\infty} |\lambda_n| = \infty$, then by the inequality $r < R$ and equation (12) it follows from Theorem 3 that $\Phi \equiv 0$. The same statement is a trivial corollary of the uniqueness theorem if Λ has a finite condensation point. Indeed, if

$$\psi(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}}\ ,$$

then

$$\Phi(\lambda) = \int_{|z|=r} F(\lambda z)\psi(z)\, dz = \sum_{n=0}^{\infty} a_n b_n \lambda^n \equiv 0\ .$$

Since all numbers a_n are different from zero, all numbers b_n must be equal to zero implying $\psi(z) \equiv 0$. This contradiction proves Theorem 6.

REMARK. If $\overline{\Delta}(\Lambda) = \infty$, then the system $\{F(\lambda_n z)\}$ is complete in $A(\mathbb{C})$.

LECTURE 4

Factorization of Entire Functions of Finite Order

4.1. The Weierstrass canonical product

In our study of infinite products we shall assume that an infinite product of entire functions

$$\prod_{n=1}^{\infty} g_n(z)$$

converges at a point z_0 if for some N there exists the limit

$$\lim_{M\to\infty} \prod_{n=N}^{M} g_n(z_0) ,$$

not equal to either zero or infinity. The same infinite product is said to converge uniformly on a set K if for some N the products $\prod_{n=N}^{M} g_n(z)$ tend to a function $h_N(z)$ uniformly with respect to $z \in K$ as $M \to \infty$.

It follows directly from the definition of convergence of infinite products that a general term of a convergent product tends to unity. Hence it is possible after omitting a finite number of factors to define $\log g_n(z)$ using the principal value of logarithm. It follows that (uniform) convergence is equivalent to (uniform) convergence of the series

$$\sum_{n=N}^{\infty} \log g_n(z) .$$

A product is said to be absolutely convergent if the latter series converges absolutely for some N.

Let $\{a_n\}$ be a sequence of complex numbers not equal to zero and such that for some nonnegative integer p the series

$$\sum_n |a_n|^{-p-1} < \infty .$$

Let us introduce the infinite product

$$\Pi(z) = \prod_n G(z/a_n, p) ,$$

where

$$G(u,p) = \begin{cases} 1-u, & p=0, \\ (1-u)\exp\left[u + \dfrac{u^2}{2} + \cdots + \dfrac{u^p}{p}\right], & p>0 . \end{cases}$$

The functions $G(u,p)$ are called the *Weierstrass primary factors*.

The inequality

$$|\log G(u,p)| \le \sum_{k=p+1}^{\infty} \frac{|u|^k}{k} \le 2|u|^{p+1},$$

evident for $|u| \le 1/2$, implies that the infinite product $\Pi(z)$ converges absolutely and uniformly in every disk $\{z : |z| \le R < \infty\}$. This product is called the *Weierstrass canonical product of genus p.*

4.2. The Hadamard theorem

One of the main theorems of the theory of entire functions is

THEOREM 1 (Hadamard). *An entire function f of finite order ρ may be represented in the form*

$$f(z) = z^m e^{P_q(z)} \prod_{n=1}^{\infty} G\Big(\frac{z}{a_n}, p\Big), \tag{1}$$

where $a_1, a_2, \dots$ are all nonzero roots of the function $f(z)$, $p \le \rho$, $P_q(z)$ is a polynomial in z of degree $q \le \rho$, and m is the multiplicity of the root at the origin.

PROOF. Let us use formula (6) from Lecture 2:

$$\log f(z) = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\psi})| \frac{Re^{i\psi}+z}{Re^{i\psi}-z}\, d\psi + \sum_{|a_n|<R} \log \frac{R(z-a_n)}{R^2 - z\bar{a}_n} + iC\,.$$

Without loss of generality we assume here that $m = 0$, i.e., $f(0) \ne 0$. Differentiating this formula $p+1$ times with $p = [\rho]$, we obtain

$$\begin{aligned}[\log f(z)]^{(p+1)} &= \frac{(p+1)!}{2\pi} \int_0^{2\pi} \log|f(Re^{i\psi}| \frac{2Re^{i\psi}}{(Re^{i\psi}-z)^{p+2}}\, d\psi \\ &\quad + \sum_{|a_n|<R} \frac{p!\bar{a}_n^{p+1}}{(R^2-\bar{a}_n z)^{p+1}} - \sum_{|a_n|<R} \frac{p!}{(a_n - z)^{p+1}}\,.\end{aligned}$$

It follows that

$$\begin{aligned}&\Big|[\log f(z)]^{p+1} + \sum_{|a_n|<R} \frac{p!}{(a_n-z)^{p+1}}\Big| \\ &\qquad \le \frac{(p+1)!}{2\pi} \log M_f(R) \frac{4\pi R}{(R-r)^{p+2}} + \frac{p!n(R)}{(R-r)^{p+1}}\,, \qquad r = |z|\,.\end{aligned}$$

The estimates

$$\begin{aligned}\log M_f(R) &\overset{\text{as}}{<} R^{\rho+\varepsilon}\,, \\ n(R) \le \log M_f(eR) &\overset{\text{as}}{<} R^{\rho+\varepsilon}\end{aligned}$$

yield, after passing to the limit as $R \to \infty$,

$$[\log f(z)]^{(p+1)} = -p! \sum_{n=1}^{\infty} \frac{1}{(a_n - z)^{p+1}}\,.$$

Integrating both sides of this indentity along any path joining the points 0 and z and not intersecting cuts from the points $a_1, a_2, \dots$ to infinity, we obtain

$$\log f(z) - P_q(z) = \sum_{n=1}^{\infty} \left[\log\left(1 - \frac{z}{a_n}\right) + \frac{z}{a_n} + \cdots + \frac{z^p}{p a_n^p} \right], \quad q \le \rho .$$

Now formula (1) follows, which proves Theorem 1.

Let us remark that the Hadamard theorem was proved with $p = [\rho]$, $q \le [\rho]$. Representation (1) is possible, generally speaking, with different integers p and q. In what follows, p will denote the smallest integer for which the series $\sum_1^\infty |a_n|^{-p-1}$ converges. With this convention the number q is determined uniquely.

The integer $g = \max(p, q)$ is called the genus of an entire function f. It follows from the Hadamard theorem that the genus of an entire function does not exceed its order.

An entire function of order zero has the form

$$f(z) = C z^m \prod_{n=1}^{\omega} \left(1 - \frac{z}{a_n}\right), \quad \omega \le \infty$$

with

$$\sum_{n=1}^{\omega} \frac{1}{|a_n|} < \infty .$$

For $\rho < 1$, by the Hadamard theorem a function $f(z)$ of order ρ has exactly the same form.

An entire function of genus one has the form

$$f(z) = C z^m e^{az+b} \prod_{n=1}^{\omega} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} .$$

EXAMPLE. The function $f(z) = \dfrac{\sin \pi\sqrt{z}}{\pi\sqrt{z}}$ is entire and of order $\rho = 1/2$. Its zeros are $a_n = n^2$, $n = 1, 2, \dots$. According to the Hadamard theorem,

$$f(z) = C \prod_{n=1}^{\infty} \left(1 - \frac{z}{n^2}\right),$$

and since $f(0) = 1$, we have $C = 1$. Substituting z^2 instead of z, we obtain

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

or

$$\sin \pi z = \pi z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n}\right) e^{z/n}.$$

As usual, the prime here means that the factor corresponding to $n = 0$ is omitted.

PROBLEM 1. Let $-\infty < a < b < \infty$. Show that the Fourier transform $F(z)$ of a function $f \in L^2_{[a,b]}$ has an infinite set of zeros (nonreal, generally speaking). The same is true if

$$F(z) = \int_a^b e^{itz} d\sigma(t) , \quad -\infty < a < b < \infty ,$$

where $\sigma(t)$ is a function of bounded variation which is not a step function with a single jump.

PROBLEM 2 (Laguerre). Let $f(z) = e^{-\alpha z^2} g(z)$, where $\alpha \geq 0$ and $g(z)$ is a real entire function of genus $p \leq 1$ with real zeros. Prove that the zeros of the derivative $f'(z)$ are also real and interlace with the zeros of $f(z)$.

HINT. Use the Hadamard theorem and study $f'(z)/f(z)$.

PROBLEM 3. Let $f(z)$ and $g(z)$ be entire functions of order $\rho < 2$ such that

$$f^2(z) + g^2(z) \equiv 1 .$$

Prove that $f(z) = \cos(\alpha z + \beta)$, $g(z) = \sin(\alpha z + \beta)$, $\alpha, \beta \in \mathbb{C}$.

4.3. Estimates for canonical products

Let $\{a_n\}$ be a sequence of complex numbers, $\lim_{n\to\infty} a_n = \infty$, and let $n(r)$ be its counting function. Suppose that for some integer $p \geq 0$ the series $\sum_1^\infty |a_n|^{-p-1}$ converges, and denote

$$\Pi(z) = \prod_{n=1}^{\infty} G\Big(\frac{z}{a_n}, p\Big) .$$

As we proved earlier using the Jensen formula, Section 2.3, the estimate $n(r) \leq \log M_\Pi(er)$ is valid. Now we shall estimate $\log M_\Pi(r)$ from above using $n(r)$. To this end we shall obtain an estimate for the Weierstrass primary factor $G(u,p)$.

LEMMA 1 (the Borel estimate). *For $u \in \mathbb{C}$ the estimates*

$$\log |G(u,p)| \leq A_p \frac{|u|^{p+1}}{1+|u|} , \quad p > 0 , \quad A_p = 3e(2 + \log p) ,$$
$$\log |G(u,0)| \leq \log(1+|u|)$$

are valid.

PROOF. The latter inequality is evident. Let $p > 0$. If $|u| < p/(p+1)$, then expanding $\log(1-u)$ we obtain

$$\log |G(u,p)| \leq \sum_{n=p+1}^{\infty} \frac{|u|^n}{n} \leq \frac{|u|^{p+1}}{(p+1)(1-|u|)} \leq |u|^{p+1} .$$

If, on the other hand, $|u| > p/(p+1)$, then the inequality $\log(1+|u|) < |u|$ yields

$$\begin{aligned}\log|G(u,p)| &\le 2|u| + \frac{|u|^2}{2} + \cdots + \frac{|u|^p}{p} \\ &= |u|^p\Big(\frac{1}{p} + \frac{1}{p-1}\frac{1}{|u|} + \cdots + \frac{1}{2}\frac{1}{|u|^{p-2}} + 2\frac{1}{|u|^{p-1}}\Big) \\ &\le |u|^p\Big(\frac{p+1}{p}\Big)^{p-1}\Big(2 + \frac{1}{2} + \cdots + \frac{1}{p}\Big) < e(2+\log p)|u|^p\frac{1+|u|}{1+|u|} \\ &= e(2+\log p)\Big(1+\frac{1}{|u|}\Big)\frac{|u|^{p+1}}{1+|u|} < A_p\frac{|u|^{p+1}}{1+|u|}\,,\end{aligned}$$

proving Lemma 1.

THEOREM 2. *Let $\{a_n\}$ be a sequence of complex numbers. If the series*

$$\sum_{n=1}^{\infty}\frac{1}{|a_n|^{p+1}} \tag{2}$$

converges, then the product

$$\Pi(z) = \prod_{n=1}^{\infty} G\Big(\frac{z}{a_n}, p\Big)$$

converges uniformly on every compact set and satisfies the estimate

$$\log|\Pi(z)| \le K_p r^p\Big\{\int_0^r \frac{n(t)}{t^{p+1}}\,dt + r\int_r^{\infty}\frac{n(t)}{t^{p+2}}\,dt\Big\}\,,$$

where $K_p = (p+1)A_p$, $r = |z|$.

PROOF. First, let $\rho \ge 1$. It follows from the Borel estimate that

$$\begin{aligned}\log|\Pi(z)| &\le A_p\sum_{n=1}^{\infty}\frac{r^{p+1}}{|a_n|^p(r+|a_n|)} = A_p r^{p+1}\int_0^{\infty}\frac{dn(t)}{t^p(t+r)} \\ &= A_p r^{p+1}\frac{n(t)}{t^p(t+r)}\Big|_0^{\infty} + A_p r^{p+1}\int_0^{\infty}\Big[\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}\Big]n(t)\,dt\,.\end{aligned} \tag{3}$$

Since the series (2) converges, by Lemma 1 from the preceding lecture we have

$$\frac{n(t)}{t^{p+1}} \to 0\,, \quad t\to\infty\,,$$

and

$$\begin{aligned}\log|\Pi(z)| &\le A_p r^{p+1}\Big\{\int_0^r + \int_r^{\infty}\Big\}\Big[\frac{p}{t^{p+1}(t+r)} + \frac{1}{t^p(t+r)^2}\Big]n(t)\,dt \\ &\le K_p r^p\Big\{\int_0^r\frac{n(t)}{t^{p+1}}\,dt + r\int_r^{\infty}\frac{n(t)}{t^{p+2}}\,dt\Big\}\,, \quad r = |z|\,.\end{aligned}$$

For $p = 0$ the estimate of the canonical product is simplified:

$$\log|\Pi(z)| \le \sum_{n=1}^{\infty} \log\left(1 + \frac{r}{|a_n|}\right) = \int_0^\infty \log\left(1 + \frac{r}{t}\right) dn(t)$$
$$= r\int_0^\infty \frac{n(t)}{t(t+r)}\,dt \le \int_0^r \frac{n(t)}{t}\,dt + r\int_r^\infty \frac{n(t)}{t^2}\,dt\,.$$

The theorem is proved.

THEOREM 3 (Borel). *The growth order ρ of a canonical product is equal to the convergence exponent of the sequence of its zeros.*

PROOF. Let p be the smallest integer such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$$

converges, where $\{a_n\}$ is the sequence of zeros of the canonical product $\Pi(z)$, and let ρ_1 be the convergence exponent of the sequence $\{a_n\}$. Then $p \le \rho_1 \le p+1$.

First, let $\rho_1 < p+1$. We choose $\varepsilon > 0$ such that $\rho_1 + \varepsilon < p+1$. Then

$$n(t) \overset{\text{as}}{<} t^{\rho_1+\varepsilon}\,.$$

It follows from the preceding theorem that

$$\begin{aligned}\log M_\Pi(r) &\le K_p r^p\Big\{O(1) + \int_0^r t^{\rho_1+\varepsilon-p-1}\,dt + r\int_r^\infty t^{\rho_1+\varepsilon-p-2}\,dt\Big\}\\ &\le K_p r^p\Big\{O(1) + \frac{r^{\rho_1+\varepsilon-p}}{\rho_1+\varepsilon-p} + \frac{r^{\rho_1+\varepsilon-p}}{p+1-\rho_1-\varepsilon}\Big\} \overset{\text{as}}{<} r^{\rho_1+2\varepsilon}.\end{aligned} \tag{4}$$

Now consider the case $\rho_1 = p+1$. We proved in Lemma 1, Section 3.2, that

$$\frac{n(r)}{r^{p+1}}\,, \quad \int_r^\infty \frac{n(t)}{t^{p+2}}\,dt$$

tend to zero as $r \to \infty$. Hence it follows from Theorem 2 that

$$\log M_\Pi(r) \overset{\text{as}}{<} \varepsilon r^{p+1} = \varepsilon r^{\rho_1} \tag{5}$$

for any $\varepsilon > 0$.[4]

Thus, in both cases $\rho \le \rho_1$. Comparing this inequality with the corollary to the Jensen formula derived in Section 2.5, we obtain $\rho = \rho_1$. Theorem 3 is proved.

PROBLEM 4. Find a necessary and sufficient conditions for a sequence of complex numbers $\{\alpha_k\}$ to be such that the infinite product

$$\prod_k \frac{\sin \alpha_k z}{\alpha_k z}$$

converge to an entire function. Under what conditions imposed on $\{\alpha_k\}$ this is a function of exponential type?

[4]This is the Poincaré theorem.

LECTURE 5

The Connection between the Growth of Entire Functions and the Distribution of their Zeros

5.1. Functions of noninteger order

THEOREM 1. *The convergence exponent of the zero set of an entire function f of noninteger order is equal to the order of growth of f.*

PROOF. Let f be an entire function of noninteger order ρ, let ρ_1 be the convergence exponent of its zeros , and let $\Pi(z)$ be the canonical product corresponding to the set of zeros of f. According to the Hadamard representation (Theorem 1, Section 4.2), we have

$$f(z) = z^m e^{P_q(z)} \Pi(z), \quad \deg P_q = q \, . \tag{1}$$

Using the Borel theorem (Theorem 3, Section 4.3), we obtain

$$\log M_f(r) \overset{\text{as}}{<} c_1 r^q + r^{\rho_1 + \varepsilon} \, , \quad \varepsilon > 0.$$

Hence

$$\log M_f(r) \overset{\text{as}}{<} r^{\lambda + 2\varepsilon} \, , \quad \lambda = \max(\rho_1, q) \, ,$$

and $\rho \le \lambda$. The opposite inequality is true, since by virtue of Theorem 2, Section 3.2, we have $\rho_1 \le \rho$, and by virtue of the Hadamard theorem $q \le \rho$. The theorem is proved.

THEOREM 2. *If the order ρ of an entire function $f(z)$ is not an integer, then its type σ_f and the upper density of zeros $\overline{\Delta}_f$ simultaneously are equal either to zero, or to infinity, or to positive numbers.*

PROOF. According to Theorem 3, Section 3.2, we have $\overline{\Delta}_f \le e\rho\sigma_f$. To estimate σ_f from above via $\overline{\Delta}_f$ we shall use the bound of a canonical product of genus p proved in Theorem 2, Section 4.3. The inequality

$$n(t) \overset{\text{as}}{<} (\overline{\Delta}_f + \varepsilon) t^\rho \, , \quad \varepsilon > 0 \, ,$$

yields

$$\log M_\Pi(r) \le K_p r^p \Big\{ O(1) + (\overline{\Delta}_f + \varepsilon) \int_0^r t^{\rho - p - 1} \, dt + (\overline{\Delta}_f + \varepsilon) r \int_r^\infty t^{\rho - p - 2} \, dt \Big\} \, .$$

Since $p < \rho < p + 1$, we have

$$\log M_\Pi(r) \overset{\text{as}}{<} C_\rho (\overline{\Delta}_f + \varepsilon) r^\rho \, ,$$

and by Hadamard's representation

$$\log M_f(r) \overset{\text{as}}{<} a_0 r^q + C_\rho(\overline{\Delta}_f + \varepsilon) r^\rho \overset{\text{as}}{<} C_\rho(\overline{\Delta}_f + r\varepsilon) r^\rho$$

or

$$\sigma_f \le C_\rho \overline{\Delta}_f , \tag{2}$$

which proves Theorem 2.

5.2. Functions of integer order

An entire function of integer order ρ may not have zeros at all. It is possible that $\rho = q$, q being the degree of the polynomial in Hadamard's representation, and the order of the canonical product ρ_1 is less than ρ. But another feature of entire functions of integer order is more essential. It turns out that, for an integer order, inequality (2) may fail even for a canonical product. The upper density of the zero set may be finite while the canonical product may be of maximal type.

Consider, for example, the entire functions

$$\sin \frac{\pi}{2} z = \frac{\pi}{2} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{4n^2}\right) = \frac{\pi}{2} z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{2n}\right) e^{z/2n}$$

and

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} ,$$

where γ is the Euler constant. It is evident that, for both functions, $n(t) \sim t$ and the convergence exponent of zeros is $\rho_1 = 1$. Since the functions differ inessentially from the canonical products, they are of order one. For the former function we have

$$\frac{1}{2}(e^{\frac{\pi}{2}|y|} - e^{-\frac{\pi}{2}|y|}) \le \left|\sin \frac{\pi}{2} z\right| < e^{\frac{\pi}{2}|y|} ,$$
$$\frac{1}{2}(e^{\frac{\pi}{2}r} - e^{-\frac{\pi}{2}r}) \le M(r) < e^{\frac{\pi}{2}r} .$$

Hence, $\log M(r) \sim (\pi/2)r$ and $\sigma = \pi/2$. For the latter function, according to the Stirling formula,

$$\log \frac{1}{\Gamma(z)} = -\left(z - \frac{1}{2}\right) \log z + z - \frac{1}{2} \log 2\pi + O\left(\frac{1}{|z|}\right) = -z(1 + o(1)) \log z ,$$

where the plane is assumed to be cut along the negative real axis, and $|\arg z| < \pi$. Hence

$$\log \frac{1}{|\Gamma(z)|} = -(1 + o(1))(\cos\varphi) r \log r , \quad z = re^{i\varphi} , \quad \frac{\pi}{2} < |\varphi| < \pi ,$$

and therefore $\log M(r) \ge Cr \log r$. This means that $f = 1/\Gamma$ is of maximal type.

We shall see that the "root of all evil" is the presence of a symmetry in the distribution of zeros of the first function and its absence for the second function.

We remind the reader that p denotes the smallest integer for which the series

$$\sum \frac{1}{|a_n|^{p+1}}$$

converges. By virtue of the Hadamard and Borel theorems from the previous lecture we have $p \le \rho \le p+1$. Two cases are possible if ρ is an integer: either $\rho = p+1$ and the series $\sum |a_n|^{-\rho}$ converges, or $\rho = p$ and the same series diverges.

In what follows we denote by a_ρ the coefficient of z^ρ of the polynomial $P(z)$ in Hadamard's representation.

THEOREM 3 (Lindelöf). *If $\rho = p+1$, then $f(z)$ is an entire function of minimal type for $a_\rho = 0$ and of mean type for $a_\rho \neq 0$.*[5]

PROOF. According to the Hadamard theorem, we have

$$\log|f(z)| \le \mathrm{Re}(a_\rho z^\rho) + \log|\Pi(z)| + O(|z|^{\rho-1}) , \quad z \to \infty . \tag{3}$$

It follows from inequality (5), Section 4.3, that if $\rho = p+1$, then

$$\log M_\Pi(r) \overset{\text{as}}{<} \varepsilon r^\rho , \quad \varepsilon > 0 , \tag{4}$$

and inequality (3) yields

$$\log M_f(r) \overset{\text{as}}{<} (|a_\rho| + 3\varepsilon) r^\rho. \tag{5}$$

To estimate $\log M_f(r)$ from below we start from the evident relation

$$m(r, \exp(a_\rho z^\rho + \cdots + a_0)) = m\Big(r, \frac{f}{\Pi}\Big) , \tag{6}$$

where m is the Nevanlinna proximity function (see Section 2.4). It follows that

$$\frac{|a_\rho|}{2\pi} r^\rho \overset{\text{as}}{<} m(r, \exp(a_\rho z^\rho + \cdots + a_0)) < m(r,f) + m\Big(r, \frac{1}{\Pi}\Big) + \log 2 . \tag{7}$$

By virtue of Jensen's formula

$$N(r,0) = m(r,\Pi) - m\Big(r, \frac{1}{\Pi}\Big) - \log|\Pi(0)| ,$$

whence

$$m\Big(r, \frac{1}{\Pi}\Big) \le m(r,\Pi) . \tag{8}$$

Taking into account (7), (8) and (4) we obtain

$$\frac{|a_\rho|}{2\pi} r^\rho \overset{\text{as}}{<} m(r,f) + m(r,\Pi) + O(1) \overset{\text{as}}{<} \log M_f(r) + 3\varepsilon r^\rho , \quad \varepsilon > 0 . \tag{9}$$

The statement of Theorem 3 follows from (9) and (5).

THEOREM 4 (Lindelöf). *Let $\rho = p$. Set*

$$\delta_f(r) = \Big| a_\rho + \frac{1}{\rho} \sum_{|a_n|<r} a_n^{-\rho} \Big| , \quad \overline{\delta}_f = \limsup_{r\to\infty} \delta_f(r) ,$$

and $\gamma_f = \max(\overline{\Delta}_f, \overline{\delta}_f)$. Then σ_f and γ_f simultaneously are equal either to zero, or to infinity, or to positive numbers.

[5] A stronger statement $\sigma_f = |a_\rho|$ is true.

PROOF. We shall use the formula

$$[\log f(z)]^{(\rho)} = \frac{\rho!}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})| \frac{2Re^{i\psi}}{(Re^{i\psi}-z)^{\rho+1}}\,d\psi + \sum_{|a_n|<R} \frac{(\rho-1)!\overline{a}_n^{\rho}}{(R^2-\overline{a}_n z)^{\rho}} - \sum_{|a_n|<R} \frac{(\rho-1)!}{(a_n-z)^{\rho}}$$

proved in Section 4.2. If we set $z = 0$, we obtain

$$[\log f(z)]^{(\rho)}_{z=0} + (\rho-1)! \sum_{|a_n|<R} a_n^{-\rho} = \frac{2\rho!}{2\pi R^{\rho}}\int_0^{2\pi} \log|f(Re^{i\psi})| e^{-i\rho\psi}\,d\psi + \sum_{|a_n|<R} \frac{(\rho-1)!\overline{a}_n^{\rho}}{R^{2\rho}}$$

or

$$\left|[\log f(z)]^{(\rho)}_{z=0} + (\rho-1)! \sum_{|a_n|<R} a_n^{-\rho}\right| \le \frac{2\rho!}{R^{\rho}}\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(Re^{i\psi})|\big|\,d\psi + (\rho-1)!\frac{n(R)}{R^{\rho}}\,. \tag{10}$$

It is easily seen that the logarithm of a primary factor has a root at $z = 0$ of multiplicity $\rho+1$. Hence,

$$[\log(\Pi(z)]^{(\rho)}_{z=0} = 0$$

and

$$[\log f(z)]^{(\rho)}_{z=0} = \rho! a_{\rho}\,. \tag{11}$$

In addition,

$$\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(Re^{i\psi})|\big|\,d\psi = m(R,f) + m\Big(R,\frac{1}{f}\Big)\,.$$

The Jensen formula yields

$$m\Big(R,\frac{1}{f}\Big) = m(R,f) - N(R,0) + O(1) < m(R,f) + O(1)\,,$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(Re^{i\psi})|\big|\,d\psi \le 2m(R,f) + O(1) \le 2\log M_f(R) + O(1)\,. \tag{12}$$

The Jensen formula yields the estimate

$$n(R) \le \log M_f(eR) + O(1)\,. \tag{13}$$

Substituting (11)–(13) in (10) we obtain

$$\left|\rho! a_{\rho} + (\rho-1)! \sum_{|a_n|<R} a_n^{-\rho}\right| \le 4\frac{\rho!}{R^{\rho}}\log M_f(R) + (\rho-1)!\,\frac{\log M_f(eR)}{R^{\rho}} + O\Big(\frac{1}{R^{\rho}}\Big)$$

or

$$\delta_f(R) \overset{\text{as}}{<} C\,\frac{\log M_f(eR)}{R^\rho}\;,$$

with a constant C independent of the function f. It follows from this inequality that $\overline{\delta}_f \le Ce^\rho\sigma_f$. Since by virtue of (13) we have $\overline{\Delta}_f \le e^\rho\sigma_f$, we obtain finally

$$\gamma_f = \max(\overline{\delta}_f, \overline{\Delta}_f) \le C_1\sigma_f\;. \tag{14}$$

To estimate σ_f from above via γ_f we shall write the Hadamard representation of $f(z)$ in the form

$$f(z) = \exp\Big[\Big(a_\rho + \frac{1}{\rho}\sum_{|a_n|<r} a_n^{-\rho}\Big)z^\rho\Big]\exp P_{\rho-1}(z)$$
$$\times \prod_{|a_n|<r} G\Big(\frac{z}{a_n},\rho-1\Big)\prod_{|a_n|\ge r} G\Big(\frac{z}{a_n},\rho\Big)\,,\quad r=|z|\,,$$

where $P_{\rho-1}$ is a polynomial of degree at most $\rho-1$. Using the estimate of the primary factor $G(u,p)$ given by Lemma 1, Section 4.3, we obtain

$$\log|f(z)| \le \delta_f(r)r^\rho + A_\rho\Big[r^\rho\int_0^r \frac{dn(t)}{t^{\rho-1}(t+r)} + r^{\rho+1}\int_r^\infty \frac{dn(t)}{t^\rho(t+r)}\Big] + o(r^\rho)\;.$$

Much as in the proof of Theorem 2, Section 4.3, integration by parts yields

$$\log M_f(r) \le \delta_f(r)r^\rho + K_\rho\Big\{r^{\rho-1}\int_0^r \frac{n(t)}{t^\rho}\,dt + r^{\rho+1}\int_r^\infty \frac{n(t)}{t^{\rho+2}}\,dt\Big\} + o(r^\rho)\;.$$

Now we apply the inequality $n(t) \overset{\text{as}}{<} (\overline{\Delta}+\varepsilon)t^\rho$, $\varepsilon>0$, and obtain

$$\log M_f(r) \le \delta_f(r)r^\rho + 2K_\rho(\overline{\Delta}_f+\varepsilon)r^\rho\;.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log M_f(r)}{r^\rho} \le \overline{\delta}_f + 2K_\rho\overline{\Delta}_f \le C_\rho\gamma_f\;,$$

with a constant C_ρ, which proves $\sigma_f \le C_\rho\gamma_f$. Together with (14) this proves Theorem 4.

PROBLEM 1 (Valiron). Prove the following statement.

If ρ is not an integer, then convergence of the integral

$$\int^\infty \frac{\log M_f(r)}{r^{\rho+1}}dr \tag{15}$$

is equivalent to convergence of the series

$$\sum_{n=1}^\infty \frac{1}{|a_n|^\rho}\;. \tag{16}$$

If ρ is an integer, then convergence of the integral (15) is equivalent to convergence of both the series (16) and the integral

$$\int^\infty \frac{\delta_f(r)dr}{r}\;.$$

LECTURE 6

Theorems of Phragmén and Lindelöf

Let $f(z)$ be an analytic function in a bounded domain D, let ζ be a point of the boundary Γ of this domain, and let $U_\delta(\zeta)$ be the δ-neighborhood of the point ζ. Set

$$\limsup_{z\to\zeta} |f(z)| = \lim_{\delta\to 0} \sup_{z\in U_\delta(\zeta)\cap D} |f(z)| \ .$$

If the inequality

$$\limsup_{z\to\zeta} |f(z)| \le M$$

holds at all points of Γ, then we shall say that $|f(z)| \le M$ on the boundary of the domain D. The *Maximum Principle* for functions analytic in a bounded domain may be stated as follows:

If $|f(z)| \le M$ on the boundary of a domain M, then $|f(z)| \le M$ in D.

This statement easily follows from the Maximum Principle in its standard form and compactness of the boundary of D.

6.1. Functions analytic inside an angle

For a function $f(z)$ analytic inside an angle $D = \{z : \alpha < \arg z < \beta\}$ we set

$$M_f(r) = \sup\{|f(re^{i\theta})| : \alpha < \theta < \beta\} \ .$$

THEOREM 1. *Let D be an angle of opening π/λ, and let $f(z)$ be a function analytic in D satisfying an asymptotic estimate*

$$\log M_f(r) \overset{\text{as}}{<} r^\rho \ , \tag{1}$$

where $\rho < \lambda$. If $f(z)$ is bounded by a constant M on the sides of D, then $|f(z)| \le M$ for $z \in D$.

PROOF. Without loss of generality we can assume that $D = \{re^{i\theta} : |\theta| < \alpha\}$, $\alpha = \pi/2\lambda$. Let us choose a number ρ_1 such that $\rho < \rho_1 < \lambda$, and set

$$\varphi_\delta(z) = f(z)e^{-\delta z^{\rho_1}} \ , \quad \delta > 0 \ .$$

The asymptotic inequality

$$|\varphi_\delta(z)| \overset{\text{as}}{<} e^{|z|^\rho - \delta|z|^{\rho_1}\cos\rho_1\alpha}$$

holds inside the whole angle D. Since $\rho < \rho_1$ and $\cos\rho_1\alpha > 0$, the inequality

$$|\varphi_\delta(Re^{i\theta})| \le M \ , \quad -\alpha < \theta < \alpha,$$

holds for $|z| = R > R_\delta$. Applying the Maximum Principle to the function $\varphi_\delta(z)$ inside the sector $D_R = \{re^{i\theta} : r < R, |\theta| < \alpha\}$, we find that $|\varphi_\delta(z)| \le M$ at an arbitrary point. In other words,

$$|f(z)| \le Me^{\delta|z|^{\rho_1}} .$$

As R tends to infinity we see that this inequality is fullfilled everywhere inside the angle D.

Since $\delta > 0$ is an arbitrary number, we obtain $|f(z)| \le M$ in D, completing the proof of Theorem 1.

PROBLEM 1. Prove that if, for some $\rho < \lambda$, the function $f(z)$ in Theorem 1 satisfies the condition

$$\liminf_{r\to\infty} \frac{\log M_f(r)}{r^\rho} = 0,$$

then its conclusion remains valid.

THEOREM 2. *If a function $f(z)$ analytic inside an angle*

$$D = \left\{z : |\arg z| < \alpha = \frac{\pi}{2\rho}\right\}$$

satisfies the asymptotic inequalities

$$\log M_f(r) \overset{\text{as}}{<} (\sigma + \varepsilon)r^\rho$$

for all $\varepsilon > 0$, and $f(z)$ is bounded on the sides of D by a constant M, then

$$|f(re^{i\theta})| \le Me^{\sigma r^\rho \cos \rho\theta} , \quad re^{i\theta} \in D .$$

PROOF. The function

$$\varphi_\varepsilon(z) = f(z)e^{-(\sigma+\varepsilon)z^\rho}$$

is bounded on a positive ray and on the boundary of D. According to the previous theorem, it is bounded by a constant in each angle $D_+ = \{z : 0 < \arg z < \pi/2\rho\}$, $D_- = \{z : -\pi/2\rho < \arg z < 0\}$. Applying the previous theorem once more, we obtain $|\varphi(z)| \le M$ for $z \in D$, or

$$|f(re^{i\theta})| \le Me^{(\sigma+\varepsilon)r^\rho \cos \rho\theta} , \quad re^{i\theta} \in D , \quad \varepsilon > 0 .$$

The statement of Theorem 2 follows when $\varepsilon \to 0$.

The following corollary from Theorem 2 is frequently used.

THEOREM 3. *If $f(z)$, $z = x + iy$, is an analytic function in the half-plane $\{z : \operatorname{Im} z > 0\}$ such that, for all $\varepsilon > 0$,*

$$M_f(r) \overset{\text{as}}{<} e^{(\sigma+\varepsilon)r},$$

and $|f(x)| \le M$ on the real axis, then

$$|f(x + iy)| \le Me^{\sigma y} . \tag{2}$$

PROOF. If we take $\alpha = \pi/2$ and $\rho = 1$ in Theorem 2 and apply this theorem to $f(-iz)$, we obtain Theorem 3.

REMARK 1. The estimate given by (2) is sharp. It is attained for functions of the form $f(z) = M\gamma e^{-i\sigma z}$, $|\gamma| = 1$. On the other hand, it is easy to verify, introducing the function $f(z)e^{i\sigma z}$, that if the equality is attained in (2) at least at one point, then $f(z) = M\gamma e^{-i\sigma z}$, $|\gamma| = 1$.

REMARK 2. If $f(z)$ is an entire function of exponential type σ, and $|f(x)| \le M$, $-\infty < x < \infty$, then

$$|f(x+iy)| \le Me^{\sigma|y|} \tag{3}$$

in the whole plane.

REMARK 3. If the growth of an entire function $f(z)$ is not higher than of first order and minimal type, and if $|f(z)| \le M$ on the real axis, then $f(z) = \text{const}$.

PROBLEM 2. If $f(z)$ is an entire function of exponential type $\sigma > 0$, and $f(x) \to 0$ as $|x| \to \infty$, then $f(x+iy)e^{-\sigma|y|} \rightrightarrows 0$ as $|x| \to \infty$, uniformly with respect to y.

PROBLEM 3. If $f(z)$ is an entire function of exponential type σ, and

$$|f(x)| \le C(1 + |x|^n),$$

then

$$f(z)e^{-\sigma|y|} \le C_1(1 + |z|^n).$$

If, in addition, $\sigma = 0$, then $f(z)$ is a polynomial.

PROBLEM 4. Prove the following statements.

1. A nonconstant entire function $f(z)$ satisfying the condition

$$\lim_{r\to\infty} \frac{\log M_f(r)}{\sqrt{r}} = 0$$

cannot be bounded on any ray emanating from the origin.

2. If $f(z)$ is a nonconstant entire function of minimal type with respect to the order $1/2$, then the function

$$\mu_f(r) = \min\{|f(z)| : |z| = r\}$$

cannot be bounded as $r \to \infty$ (Wiman).

HINT. If $f(z)$ is as stated with zeros $a_1, a_2, \dots$, $f(0) = 1$, and

$$\varphi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{|a_n|}\right),$$

then $\mu_f(r) \ge |\varphi(r)|$. Now apply statement 1.

There are many results on the connection between the growth rate of an entire function and the rate of its decrease. We shall mention here the $\cos \pi\rho$-theorem due to Wiman and Valiron stating that

$$\limsup_{r\to\infty} \frac{\log \mu_f(r)}{\log M_f(r)} \ge \cos \pi\rho$$

for an entire function $f(z)$ of order $\rho \leq 1$, and the Beurling theorem stating that, for every entire function $f(z)$,

$$\limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{\log M_f(r)} \geq -1$$

for every $\theta \in [0, 2\pi]$. Proofs and further results can be found in Beurling [**13**], Kjellberg [**70**], Hayman and Kjellberg [**55**] and in the monograph Essén [**34**].

PROBLEM 5. Prove the following statements.

1. Let $f(z)$ be a bounded analytic function in the right half-plane. If

$$\lim_{x\to+\infty} \frac{\log|f(x)|}{x} = -\infty ,$$

then $f \equiv 0$.

2. Let $f(z)$ be an analytic function in the right half-plane. If

$$|f(z)| \leq Me^{-c|z|} , \quad \operatorname{Re} z \geq 0,$$

for some $c > 0$, then $f \equiv 0$.

HINT. Introduce an auxiliary analytic function $F(z) = f(z)\exp(-\varepsilon z \log z)$, $\operatorname{Re} z > 0$, and apply the first statement.

The Phragmén and Lindelöf theorems proved above treated functions analytic inside an angle.

PROBLEM 6. If $f(z)$ is an analytic function in the strip $\{z : |\operatorname{Im} z| < b\}$, $|f(x \pm ib)| \leq M$, and

$$\limsup_{|x|\to\infty} \frac{\log\log\max\{|f(x+iy)| : |y| \leq b\}}{|x|} < \frac{\pi}{2b} ,$$

then $|f(z)| \leq M$ in the whole strip.

Theorems of a similar type are often used for functions analytic in some other unbounded domains. Such theorems can be found in the monographs by Pólya and Szegö [**111**] (Sect. III, Chap. 5, §6), Evgrafov [**36**] and Tsuji [**123**].

Many theorems of the theory of entire functions remain valid for more general classes of functions.

6.2. Entire functions with values in Banach algebras

A function $\varphi : G \to E$, where G is a domain in $\mathbb{C}$ and E is a Banach space, is called analytic if for all $\lambda \in G$ there exists the derivative

$$\varphi'(\lambda) = \lim_{h\to 0} \frac{\varphi(\lambda+h)-\varphi(\lambda)}{h} , \tag{4}$$

where the limit is considered with respect to the norm in E. It is evident that, for every linear functional $f \in E^*$, the function $f[\varphi(\lambda)]$ is analytic. This remark permits theorems on complex-valued analytic functions to be extended to E-valued analytic functions.

For example, let φ be an entire function with values in E such that $\|\varphi(\lambda)\| \leq C$ for all $\lambda \in \mathbb{C}$. For every linear function $f \in E^*$, according to the Liouville theorem, we have $f[\varphi(\lambda)] \equiv \text{const}$. By the Hahn-Banach theorem it follows that $\varphi(\lambda) \equiv \text{const}$. Thus, we have proved the Liouville theorem for E-valued functions.

The growth characteristics for E-valued entire functions are defined in the same way as for complex-valued functions, with the norm in place of the modulus. For example,

$$M_\varphi(r) = \max\{\|\varphi(\lambda)\| : |\lambda| \le r\}$$

for an E-valued entire function $\varphi(z)$. The Phragmén and Lindelöf theorems proved in the previous section remain valid for E-valued functions as well. Indeed, if an abstract function $\varphi(z)$ is analytic in $D = \{z : |\arg z| < \alpha\}$ and satisfies the inequalities

$$\|\varphi(re^{\pm i\alpha})\| \le M ,$$
$$\|\varphi(z)\| \overset{\text{as}}{<} \exp|z|^\rho , \quad \rho < \frac{\pi}{2\alpha} ,$$

and if $f \in B^*$ is a normalized linear functional, then for the scalar analytic function $\Psi(z) = f[\varphi(z)]$ we have

$$|\Psi(re^{\pm i\alpha})| \le M , \quad |\Psi(z)| \overset{\text{as}}{<} \exp|z|^\rho .$$

Applying Theorem 1, we obtain $|f[\varphi(z)]| \le M$ for $z \in D$, and

$$\|\varphi(z)\| = \sup\{|f[\varphi(z)]| : f \in B^*, \ \|f\| = 1\} \le M, \quad z \in D .$$

The same method can be applied for extending other theorems on scalar analytic functions to abstract analytic functions.

Let us also remark that the formulas expressing the order and type of an entire function via the coefficients of its power expansion

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

remain valid if $|c_n|$ is replaced by $\|c_n\|$.

Various theorems on analytic and entire E-valued functions are often used in the theory of Banach algebras (see Gelfand, Raikov, and Shilov [**38**], Bourbaki [**19**], Rudin [**118**], Brudnyi and Gorin [**21**]). We shall describe the simplest examples.

Let B be a Banach algebra with unity e. This means that B is a Banach space and for each pair (x, y) of its elements the product xy is defined which is a bilinear function of x and y, and the inequality $\|xy\| \le \|x\|\|y\|$ holds. In what follows we assume that B is an associative algebra, i.e., $x(yz) = (xy)z$ for all $x, y, z \in B$.

An element $e \in B$ is called the unity of the algebra if $xe = ex = e$ for all $x \in B$.

We remind the reader that the spectrum $\text{spec}(x)$ of an element $x \in B$ is the set of all $\lambda \in \mathbb{C}$ such that $\lambda e - x$ is not invertible in B.

THEOREM 4. *The spectrum of an arbitrary element $x \in B$ is not void.*

PROOF. If the spectrum is void, then the resolvent $(\lambda e - x)^{-1}$ is a B-valued entire function. Its norm tends to 0 as $\lambda \to \infty$. Using the Liouville theorem we conclude that $(\lambda e - x)^{-1} \equiv 0$ which is a contradiction to the identity $\lim_{\lambda\to\infty} \lambda(\lambda e - x)^{-1} = e$.

THEOREM 5 (Le Page). *If $\|xy\| \le C\|yx\|$ for every pair of elements $x, y \in B$ with some constant C, then B is a commutative algebra.*

PROOF. Let us consider the entire function $\varphi(\lambda) = e^{-\lambda x} y e^{\lambda x}$ with

$$e^{\lambda x} = \sum_{k=0}^{\infty} \frac{\lambda^k x^k}{k!} . \tag{5}$$

According to the hypothesis we have

$$\|\varphi(\lambda)\| = \|(e^{-\lambda x} y) e^{\lambda x}\| \leq C \|e^{\lambda x}(e^{-\lambda x} y)\| = C\|y\| ,$$

and by the Liouville theorem $\varphi(\lambda) \equiv \text{const}$. Hence $\varphi'(0) = -xy + yx = 0$, and B is a commutative algebra.

DEFINITION. The value

$$\rho(x) = \sup\{|\lambda| : \lambda \in \operatorname{spec}(x)\}$$

is called the spectral radius of $x \in B$.

THEOREM 6 (I. Gelfand). *The identity*

$$\rho(x) = \lim_{k\to\infty} \|x^k\|^{1/k} \tag{6}$$

holds.

PROOF. Let us consider the analytic B-valued function

$$r_x(\lambda) = \sum_{k=0}^{\infty} \frac{x^k}{\lambda^{k+1}} . \tag{7}$$

The disk of convergence (centered at infinity) of this series coincides with the set

$$\{\lambda : |\lambda| > \limsup_{k\to\infty} \|x^k\|^{1/k}\} .$$

The series converges uniformly inside the disk, and multiplying it by $(x - \lambda e)$ we find $(x - \lambda e) r_x(\lambda) = r_x(x - \lambda e) = e$. Therefore, the function $r_x(\lambda)$ is the resolvent $(\lambda e - x)^{-1}$.

Hence, the convergence disk of series (7) coincides with the largest disk contained in $\overline{\mathbb{C}} \setminus \operatorname{spec}(x)$, which implies that $\rho(x) = \limsup_{k\to\infty} \|x^k\|^{1/k}$. In particular, we find $\rho(x) \leq \|x\|$.

Now let $\lambda \in \operatorname{spec}(x)$. Since $\lambda^n e - x^n = (\lambda e - x) y$, the invertibility of the left-hand side of this identity would imply $(\lambda e - x) y (\lambda^n e - x^n)^{-1} = e$ contradicting the assumption $\lambda \in \operatorname{spec}(x)$. Therefore, $\lambda^n \in \operatorname{spec}(x^n)$ and the inequality $\rho(x^n) \leq \|x^n\|$ implies $|\lambda| \leq \|x^n\|^{1/n}$. Hence,

$$\rho(x) \leq \liminf_{n\to\infty} \|x^n\|^{1/n} ,$$

proving the Gelfand formula (6).

COROLLARY. *The type σ_x of an entire function $e_x(\lambda) = e^{\lambda x}$ is equal to $\rho(x)$.*

THEOREM 7. *Let elements $x, y \in B$ be such that $xy = yx$. Then $\rho(x + y) \leq \rho(x) + \rho(y)$.*

PROOF. Since x and y are commuting, we have $e^{\lambda(x+y)} = e^{\lambda x}e^{\lambda y}$ which follows from the power representation of $e^{\lambda(x+y)}$. It remains to use the inequality for the type of the product of entire functions, which yields

$$\rho(x+y) = \sigma_{e^{\lambda x}e^{\lambda y}} \leq \sigma_{e^{\lambda x}} + \sigma_{e^{\lambda y}} = \rho(x) + \rho(y) \ ,$$

proving Theorem 7.

6.3. Applications of the Phragmén and Lindelöf theorems to Banach algebras

An element x of a Banach algebra B is called *real* if its spectrum $\operatorname{spec}(x)$ is a real set.

THEOREM 8. *Let every element of a Banach algebra B be representable in the form $w = x + iy$, where x and y are real, and, for every triple x, y, z of real elements, the identity*

$$\|xyz\| = \|yzx\|$$

hold. Then the algebra B is commutative.

PROOF. Let $\mathcal{L}$ be a curve surrounding the spectrum of x. Then

$$e^{\lambda x} = \frac{1}{2\pi i}\int_{\mathcal{L}} e^{\lambda\zeta}(\zeta e - x)^{-1}\, d\zeta \tag{8}$$

and

$$\|e^{\lambda x}\| \leq C \exp\left(\max_{\zeta\in\mathcal{L}} \operatorname{Re}(\lambda\zeta)\right)$$

with a constant C independent of λ. If x is a real element, then, however small $\varepsilon > 0$ be given, we may choose as $\mathcal{L}$ the boundary of a rectangle $\{(\zeta,\eta) : a \leq \zeta \leq b,\ |\eta| \leq \varepsilon\}$. Then for purely imaginary $\lambda = i\mu$ we obtain

$$\|e^{i\mu x}\| \leq C_\varepsilon e^{\varepsilon|\mu|} \ .$$

As in the proof of the Le Page theorem, let us consider the entire function $\varphi(\lambda) = e^{\lambda x}ye^{-\lambda x}$. Let x be a real element, and λ a real, while μ a nonreal number. Then, using the power expansion (5), it is easy to verify that, with $\mathcal{L}$ sufficiently close to the real axis, the element

$$\frac{1}{2\pi i}\int_{\mathcal{L}} (e^{\lambda\zeta} - \mu)^{-1}(\zeta e - x)^{-1}\, d\zeta$$

is inverse to $e^{\lambda x} - \mu$. Hence, $e^{\lambda x}$ is real for real λ. If y is also a real element, then $\|\varphi(\lambda)\| = \|e^{\lambda x}ye^{-\lambda x}\| = \|y\|$.

If $\lambda = i\mu$, then $\|\varphi(\lambda)\| \leq C'_\varepsilon e^{2\varepsilon|\lambda|}$. Hence, the entire function of exponential type $\varphi(\lambda)e^{2i\varepsilon\lambda}$ is bounded on the imaginary half-axis $\{\lambda : \lambda = i\mu,\ \mu > 0\}$ and on the real axis. By the Phragmén-Lindelöf theorem it is bounded in the entire upper half-plane $\{\lambda : \operatorname{Im}\lambda > 0\}$, and similarly, the function $\varphi(\lambda)e^{-2\varepsilon i\lambda}$ is bounded in $\{\lambda : \operatorname{Im}\lambda < 0\}$. Therefore, $\|\varphi(\lambda)\| \leq C''_\varepsilon e^{2\varepsilon|\lambda|}$, $\varepsilon > 0$, which means that $\varphi(\lambda)$ is of minimal type with respect to the order 1. On the real line we have $\|\varphi(\lambda)\| = \|y\|$. With account taken of Remark 2 of the previous section we obtain $\varphi(\lambda) = \varphi(0) = y$ and $e^{\lambda x}y = ye^{\lambda x}$. Comparing the coefficients at λ in the power expansion we find $xy = yx$.

To complete the proof we notice that the commutativity of real elements implies the commutativity of arbitrary elements. The theorem is proved.

Another application of the Phragmén-Lindelöf theorem is related to commutative Banach algebras. Let us denote by $\mathfrak{M}$ the space of multiplicative linear functionals on a Banach algebra B endowed with the weak topology. Let $x(\cdot)$ be the Gelfand transform of an element x, i.e., the continuous function on $\mathfrak{M}$ defined by $x(M) = M(x)$, $m \in \mathfrak{M}$. Then $\text{spec}(x) = \{z \in \mathbb{C} : z = x(M), M \in \mathfrak{M}\}$, cf. Gelfand, Raikov, and Shilov [**38**], Rudin [**118**].

THEOREM 9 (Gelfand). *The unity e of a commutative Banach algebra B is an extreme point of the unit sphere.*

PROOF. [6] Assuming the contrary, there exist elements u, v of the unit sphere such that $2e = u + v$. If $u = e + x$, $x \in B$, then $v = e - x$, and

$$\|e + tx\| = \|e - tx\| = 1 , \quad -1 \leq t \leq 1 .$$

For any nontrivial multiplicative functional $M \in \mathfrak{M}$ we have $M(e) = 1$ and $||M|| = 1$, and the previous equations with $t = 1$ yield

$$|1 + x(M)| \leq 1 , \quad |1 - x(M)| \leq 1.$$

Hence $x(M) = 0$ for every $M \in \mathfrak{M}$. Therefore, $\text{spec}(x) = 0$, and according to the Gelfand formula,

$$\lim_{k\to\infty} \|x^k\|^{1/k} = \rho(x) = 0 .$$

Let us consider the entire function $e^{\lambda x}$. By the corollary to Theorem 6, its type with respect to the order 1 equals 0. We shall show that the function is bounded on the real line.

It follows from equation (8) that

$$e^{\lambda x} = \lim_{n\to\infty} \left(e + \frac{\lambda x}{n}\right)^n , \quad n \to \infty .$$

Hence,

$$\|e^{\lambda x}\| = \lim_{n\to\infty} \left\|\left(e + \frac{\lambda x}{n}\right)^n\right\| .$$

For real λ and $n > |\lambda|$ we have $\|e + \lambda x/n\| = 1$. Hence, $\|(e + \lambda x/n)^n\| \leq 1$ and, at last, $\|e^{\lambda x}\| \leq 1$.

By the Phragmén-Lindelöf theorem (Remark 3) we obtain $e^{\lambda x} \equiv \text{const}$, which is possible only if $x = 0$. The theorem is proved.

[6]The proof which follows was given by M. Krein.

LECTURE 7

Subharmonic Functions

7.1. Definition and basic properties

A real function $u(z) < +\infty$ is called *subharmonic* in a domain D if at each point $z_0 \in D$ it satisfies two conditions:

a) upper semicontinuity

$$u(z_0) = \lim_{\delta \to 0} \sup_{|z-z_0|<\delta} u(z) \ ;$$

b) the mean-value property

$$u(z_0) \le \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta \ , \quad r < \delta(z_0) \ .$$

It is easy to check that the logarithm of modulus of an analytic function is subharmonic. In fact, many properties of the function $\log |f(z)|$ are extendable to the wider class of subharmonic functions.

Many properties of subharmonic functions follow directly from the definition (see, for example, Ronkin [**116**], Hayman and Kennedy [**54**]). Some of them are listed below.

1. *If $\varphi(t)$ is an increasing convex function and $u(z)$ is a subharmonic function, then $\varphi(u(z))$ is subharmonic as well.* In particular, $e^{u(z)}$ is subharmonic. Thus *if $f(z)$ is an analytic function, then for each $\lambda > 0$ the function $|f(z)|^\lambda$ is subharmonic.*

2. *Let $u_1, \dots , u_n$ be subharmonic functions in D. Then the upper envelope*

$$u(z) = \max \big(u_1(z), \dots , u_n(z) \big)$$

is subharmonic in D. In the case of an infinite family of subharmonic functions $\{u_\alpha(z)\}$, locally uniformly bounded from above, the upper envelope need not be upper semicontinuous. However, its upper semicontinuous regularization

$$u^*(z) = \lim_{\delta \to 0} \sup_{|z-\zeta|<\delta} u(\zeta)$$

is subharmonic.

3. *The limit of a decreasing or a uniformly convergent sequence of subharmonic functions is a subharmonic function.*

4. *The sum of finitely many subharmonic functions is a subharmonic function. Moreover, integration with respect to a parameter preserves subharmonicity.*

Namely, let $u(z,p)$, $(z,p) \in D \times G$, be a subharmonic function in D for every $p \in G$ and an upper semicontinuous function in $D \times G$, and let μ_p be a nonnegative measure in G. Then the function

$$u(z) = \int u(z,p)\, d\mu_p$$

is subharmonic in D.

5. The Maximum Principle is valid for subharmonic functions. It may be formulated as follows:

If a subharmonic function $u(z)$ in a domain D attains its maximum value at an interior point $z_0 \in D$, then $u(z) \equiv \text{const}$.

PROOF. If $M = \sup_{z\in D} u(z)$ and $u(z_0) = M$, then by the mean-value property b) we have

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta$$

for small enough $r > 0$. The upper semicontinuity of the function $u(z)$ and the estimate $u(z) \le M$ yield $u(z_0 + re^{i\theta}) = M$ for $0 \le \theta \le 2\pi$, $0 < r < \delta$. Thus, the set of points where $u(z) = M$ is open in D. On the other hand, the upper semicontinuity implies that this set is closed. Hence, $u(z) \equiv M$ for $z \in D$.

This theorem, combined with the lemma on finite covering, yields the following statement.

For $\zeta \in \partial D$ define

$$u(\zeta) = \limsup_{z\to\zeta,\, z\in D} u(z)\ .$$

Then the inequality

$$u(z) \le \sup_{\zeta\in\partial D} u(\zeta)$$

holds everywhere in D, with the equality valid only if $u(z)$ is constant in D.

PROBLEM 1. Prove the *principle of harmonic majorant*: in order that an upper semicontinuous function $u(z)$, $z \in D$, be subharmonic it is necessary and sufficient that for every subdomain $G \subset D$ and every harmonic function $h(z)$, $z \in G$, satisfying the inequality $u(z) \le h(z)$ for $z \in \partial G$, the same inequality hold everywhere on G.

6. *Let us define an average*

$$\mathfrak{N}(r, z; u) = \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})\, d\theta, \quad r < \operatorname{dist}(z, \partial D)\ ;$$

then

α) *$\mathfrak{N}(r, z; u)$ does not decrease as r increases;*
β) *$\lim_{r\to 0} \mathfrak{N}(r, z; u) = u(z)$.*

To prove α), notice that according to the property 4 the function

$$\mathfrak{N}(\zeta, z; u) = \frac{1}{2\pi}\int_0^{2\pi} u(z + \zeta e^{i\theta})\, d\theta$$

is a subharmonic function of ζ and that $\mathfrak{N}(\zeta, z; u) = \mathfrak{N}(|\zeta|, z; u)$. By the Maximum Principle (the property 5) the function $\mathfrak{N}(r, z; u)$ is monotonic in r. To prove β), notice that

$$u(z) \le \mathfrak{N}(r, z; u) \le \max_{|\zeta - z| \le r} u(\zeta)$$

and, by upper semicontinuity,

$$\lim_{r\to 0} \max_{|\zeta - z|\le r} u(\zeta) = u(z) \ .$$

7. *Each subharmonic function can be represented as a pointwise limit of a decreasing sequence of infinitely differentiable functions.*

To prove this fact, we set

$$u_\varepsilon(z) = \iint u(\omega)\alpha_\varepsilon(z-\omega)\, d\sigma_\omega \ ,$$

where $d\sigma_\omega$ is the area element, $\alpha_\varepsilon(z) = \varepsilon^{-2}\alpha(\varepsilon^{-1}|z|)$, and $\alpha(t), t \ge 0$, is an infinitely differentiable function supported on $[0,1]$ and such that

$$2\pi \int_0^1 \alpha(s)s\, ds = 1 \ .$$

Then

$$\begin{aligned} u_\varepsilon(z) &= \int_0^\varepsilon s\alpha_\varepsilon(s) \int_0^{2\pi} u(z + se^{i\varphi})\, d\varphi\, ds \\ &= 2\pi \int_0^1 s\alpha(s)\mathfrak{N}(\varepsilon s, z; u)\, ds \ . \end{aligned}$$

Using properties α) and β) of the average $\mathfrak{N}(r)$, we complete the proof.

8. *A twice continuously differentiable function $u(z)$ is subharmonic in a domain D if and only if its Laplacian Δu is nonnegative in D.*

To prove this statement we need an analogue of the Jensen formula from Lecture 2. Let u, v be twice continuously differentiable functions, and let G be a plane domain with the smooth boundary. Then the Green formula is valid:

$$\iint_G (v\Delta u - u\Delta v)\, d\sigma = \int_{\partial G} \left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} \right) ds \ ,$$

where $\partial/\partial n$ is differentiation along the exterior normal. We apply this formula with $G = \{w : \varepsilon < |z - w| < R\}$, $v(w) = \log \dfrac{R}{|z-w|}$, and send ε to zero. We obtain

$$(1) \qquad u(z) + \frac{1}{2\pi} \iint_{|z-w|<R} \log \frac{R}{|z-w|} \Delta u(w)\, d\sigma_w = \mathfrak{N}(R, z; u) \ ,$$

proving our assertion.

PROBLEM 2. Prove the following statements:

1. Let us suppose that a function $u(z)$, $z = x + iy$, $a < x < b$ does not depend on y, i.e., $u(z) = \varphi(x)$. For $u(z)$ to be subharmonic it is necessary and sufficient that the function φ be convex.

2. Let $u(z)$ be a subharmonic function in an annulus $R_1 < |z| < R_2$. Then for $R_1 < r < R_2$ the functions $B(r) = \max_{|z|=r} u(z)$ and $\mathfrak{N}(r) = \mathfrak{N}(r, 0; u)$ are convex

functions of $\log r$. The first part of the statement with $u = \log|f|$, f being analytic, is called Hadamard's three-circle theorem.

3. Let a function $u(z)$, $z = re^{i\theta}$, $R_1 < r < R_2$ do not depend on θ, i.e., $u(z) = \psi(r)$. For $u(z)$ to be subharmonic it is necessary and sufficient that ψ be a convex function of $\log r$, i.e.,

$$\psi(r) \le \frac{\log r - \log r_1}{\log r_2 - \log r_1}\psi(r_2) + \frac{\log r_2 - \log r}{\log r_2 - \log r_1}\psi(r_1)$$

for $R_1 < r < R_2$.

PROBLEM 3. Prove that if a function $u(z)$ is subharmonic in a domain D, and if a function $z(w)$ is analytic in a domain G and with values in D, then the function $v(w) = u(z(w))$ is subharmonic in G.

7.2. The F. Riesz theorem and the Jensen formula

We state here without proof a theorem which is a fundamental fact of the theory of subharmonic functions.

THEOREM 1. *Let $u(z)$ be a subharmonic function in a domain D. Then there exists a unique nonnegative Borel measure μ in D such that $\mu(G) < \infty$ for every subdomain G compactly embedded into D, and $u(z)$ admits the representation*

$$u(z) = \iint_G \log|z-\zeta|\,d\mu_\zeta + h(z) \tag{2}$$

with a function $h(z)$ harmonic in G.

The measure μ is called *the Riesz measure* of the function $u(z)$, and the integral on the right-hand side of (2) is called the *logarithmic potential of μ*. Formula (2) is a generalization of the simple formula

$$\log|f(z)| = \sum_{z_k \in G} \log|z - z_k| + \log\left|\frac{f(z)}{P(z)}\right| ,$$

where $f(z)$ is an analytic function in D, $\{z_k\}$ is the set of its zeros in G, $P(z) = (z-z_1)\cdots(z-z_n)$ and the last term on the right-hand side is harmonic in G. In this case the measure μ is a linear combination of Dirac measures supported by the set $\{z_k\}$. If the function $u(z)$ is twice continuously differentiable, then (2) follows directly from (1), and the measure μ has the form

$$d\mu_\zeta = \frac{1}{2\pi}\Delta u(\zeta)\,d\sigma_\zeta .$$

The proof in the general case can be carried out using a careful limit process. Now we shall derive the Jensen formula for subharmonic functions, the Jensen formula of Lecture 2 being a particular case.

THEOREM 2. *Let $u(z)$ be a bounded subharmonic function in a disk $\mathbb{D}_R = \{z : |z| < R\}$, $u(0) \ne -\infty$, and let μ be the Riesz measure of $u(z)$. Then*

$$u(0) + \int_0^R \frac{\mu(t)}{t}\,dt = \mathfrak{N}(R, 0; u) ,$$

where $\mu(t) = \mu(\{z : |z| \le t\})$.

PROOF. The representation (2) can be written in a disk $\mathbb{D}_r$, $r < R$, in the form

$$u(z) = \iint_{\mathbb{D}_r} \log \left| \frac{r(\zeta - z)}{r^2 - \bar{\zeta} z} \right| d\mu_\zeta + h(z) . \tag{3}$$

Then

$$u(0) = \iint_{\mathbb{D}_r} \log \frac{|\zeta|}{r} \, d\mu_\zeta + h(0) .$$

Since the integrand in (3) vanishes for $|z| = r$, and $h(z)$ is a harmonic function, we obtain

$$\begin{aligned} u(0) &= \int_0^r \log \frac{t}{r} \, d\mu(t) + \mathfrak{N}(r, 0, h) \\ &= - \int_0^r \frac{\mu(t)}{t} \, dt + \mathfrak{N}(r, 0, u) , \end{aligned}$$

proving Theorem 2.

Proofs of the F. Riesz theorem as well as further results on subharmonic functions can be found in the monographs Ronkin [**116**] (short and elementary exposition), Landkof [**78**], Hayman and Kennedy [**54**].

7.3. Phragmén-Lindelöf theorems for subharmonic functions

The Phragmén-Lindelöf theorems proved above for analytic functions are valid for subharmonic functions. The following theorem is similar to Theorem 1 of the previous lecture.

THEOREM 3. *Let D be an angle of opening π/λ, and let $u(z)$ be a function subharmonic in this angle, satisfying an asymptotic estimate*

$$u(z) \overset{\text{as}}{<} |z|^\rho , \quad \rho < \lambda ,$$

and bounded by a constant M on the boundary of the angle. Then $u(z) \le M$ inside the full angle D.

PROOF. Without loss of generality we assume that $D = \{z = re^{i\theta} : |\theta| < \pi/2\lambda\}$ and consider the subharmonic function

$$\omega_\delta(z) = u(z) - \delta |z|^{\rho_1} \cos \rho_1 \theta , \quad \rho < \rho_1 < \lambda,$$

inside the sector $\{|z| < R, |\arg z| < \pi/2\lambda\}$. With R tending to infinity, we obtain by the Maximum Principle that $w_\delta(t) \le M$ for an arbitrary fixed z. Passing in this inequality to the limit as $\delta \to 0$, we complete the proof.

Other theorems of Phragmén-Lindelöf type can be derived in a similar way for subharmonic functions.

PROBLEM 4. Prove the following statement of Phragmén-Lindelöf type.

Let $u(z)$ be a subharmonic function in a domain D, and let $u(\zeta) \le M$, $\zeta \in \partial D \backslash E$, $E \subset \partial D$. Assume that there exists a negative harmonic function $h(z)$ in D such that, for every $\delta > 0$,

$$\limsup_{z \to \zeta,\, z \in D} (u(z) + \delta h(z)) \le M$$

at each point $\zeta \in E$. Then $u(z) \le M$ everywhere in D.

For example, if D is a bounded domain, $E = \{\zeta_1, \zeta_2, \dots\}$ is at most countable subset of the boundary ∂D, and $\sup_{z\in D} u(z) < \infty$, then the function

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \log|z - \zeta_n| - C$$

will be negative in D for an appropriate constant C. Hence, we conclude that $u(z) \le M$, $z \in D$.

7.4. Logarithmically subharmonic functions

A notion of logarithmically subharmonic function is rather useful.

DEFINITION. A nonnegative function $u(z)$ is called logarithmically subharmonic if the function $v(z) = \log u(z)$ is subharmonic.

For example, if $f(z)$ is analytic, then $|f(z)|^p$, $p > 0$, is a logarithmically subharmonic function.

Let $u_1, \dots, u_n$ be logarithmically subharmonic functions. Evidently, their product $u_1 \cdot \ldots \cdot u_n$ is also a logarithmically subharmonic function. To verify that the finite sum of logarithmically subharmonic functions has the same property, we start with the identity

$$u^2 \Delta \log u = u\Delta u - |\nabla u|^2,$$

which can be easily checked. Here, as usual, ∇u is the gradient of the function u. Using this identity, we obtain

$$\begin{aligned}(u+v)^2\Delta\log(u+v) &= \Big(1+\frac{v}{u}\Big)u\Delta u + \Big(1+\frac{u}{v}\Big)v\Delta v - |\nabla u + \nabla v|^2 \\ &= \Big(1+\frac{v}{u}\Big)(u^2\Delta\log u + |\nabla u|^2) + \Big(1+\frac{u}{v}\Big)(v^2\Delta\log v + |\nabla v|^2) - |\nabla u + \nabla v|^2 \\ &= \Big(1+\frac{v}{u}\Big)u^2\Delta\log u + \Big(1+\frac{u}{v}\Big)v^2\Delta\log v + \Big(\frac{v}{u}|\nabla u|^2 - 2(\nabla u, \nabla v) + \frac{u}{v}|\nabla v|^2\Big).\end{aligned}$$

This implies that the sum of two, and hence of an arbitrary finite number, of logarithmically subharmonic functions is logarithmically subharmonic.

A passage to the limit proves that integration with respect to a parameter preserves logarithmical subharmonicity.

PROBLEM 5 (Hardy). Let $f(z)$ be an analytic function in the disk $\{z : |z| < R\}$, and let

$$I(r, f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\, d\theta\,.$$

Then $I(r, f)$ is an increasing function of r, and $\log I(r, f)$ is a convex function of $\log r$.

THEOREM 4 (Plancherel and Pólya). *Let $f(z)$ be an analytic function in the upper half-plane $\{y > 0\}$, continuous up to the real axis, and let*

$$|f(z)| \overset{\text{as}}{<} e^{(\sigma+\varepsilon)|z|} \tag{3}$$

for an arbitrary $\varepsilon > 0$. If

$$\int_{-\infty}^{\infty} |f(x)|^p\, dx = M < \infty\,, \quad p > 0\,,$$

then

$$\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \leq M e^{p\sigma y}$$

for an arbitrary $y > 0$.

PROOF. Given $N > 0$, the function

$$w_N(z) = \int_{-N}^{N} |f(z+t)|^p \, dt$$

is logarithmically subharmonic in $\mathbb{C}_+$ and bounded on $\mathbb{R}$:

$$w_N(x) \leq \int_{-\infty}^{\infty} |f(x+t)|^p \, dt = M \, .$$

Further, by (3),

$$w_N(z) \overset{\text{as}}{<} e^{p(\sigma+\varepsilon)|z|} \, , \quad \varepsilon > 0 \, .$$

The Phragmén-Lindelöf theorem applied to the subharmonic function $\log w_N(z)$ implies that

$$w_N(x+iy) \leq M e^{p\sigma y} \, ,$$

or

$$\int_{-N}^{N} |f(x+iy+t)|^p \, dt \leq M e^{p\sigma y} \, .$$

Sending N to infinity, we obtain the desired estimate.

REMARK. If $f(z)$ is an entire function of exponential type σ_f such that

$$\int_{-\infty}^{\infty} |f(x)|^p \, dx \leq M$$

for some $p > 0$, then the function $f(z)$ is bounded on the real axis.

Indeed, the function $|f(z)|^p$ is subharmonic, and

$$\begin{aligned} |f(x)| &\leq \left\{ \frac{1}{\pi} \iint_{|\zeta|<1} |f(x+\zeta)|^p \, d\sigma_\zeta \right\}^{1/p} \\ &\leq \left\{ \frac{1}{\pi} \int_{-1}^{1} d\eta \int_{-\infty}^{\infty} |f(x+\tau+i\eta)|^p \, d\tau \right\}^{1/p} \\ &\leq \left\{ \frac{2}{\pi p \sigma} M (e^{p\sigma_f} - 1) \right\}^{1/p} . \end{aligned}$$

In connection with the Plancherel-Pólya theorem we would like to mention the papers Dzhrbashyan and Avetisyan [**28**] and Luxemburg [**85**].

In the second part of this book we will return to the entire functions belonging to the space $L^p(-\infty, \infty)$ on the real axis.

LECTURE 8

The Indicator Function

8.1. The definition and ρ-trigonometric convexity of the indicator

Let us consider a function $f(z)$ which is analytic inside an angle $D = \{z = re^{i\theta} : \alpha < \theta < \beta\}$ and satisfies the estimate

$$M_f(r) \overset{\text{as}}{<} e^{Ar^\rho} \tag{1}$$

with $M_f(r) = \sup_{\alpha<\theta<\beta} |f(z)|$.

DEFINITION. The function

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\rho}$$

is called the *indicator function* of $f(z)$ with respect to the order ρ.

The indicator function describes the growth of the function $f(z)$ along a ray $\{z : \arg z = \theta\}$.

It follows directly from the definition that the indicator of the product of two functions does not exceed the sum of the indicators of the factors, i.e.,

$$h_{fg}(\theta) \le h_f(\theta) + h_g(\theta) ,$$

and that the indicator of the sum of two functions does not exceed the larger of the two indicators:

$$h_{f+g}(\theta) \le \max\big(h_f(\theta),\ h_g(\theta)\big).$$

For the function

$$f(z) = e^{(A-iB)z^\rho}$$

holomorphic in an angle $\{z = re^{i\theta} : \alpha < \theta < \beta\}$, $\beta - \alpha \le 2\pi$, we have

$$|f(re^{i\theta})| = e^{(A\cos\rho\theta + B\sin\rho\theta)r^\rho} ,$$

and its indicator is equal to

$$H(\theta) = A\cos\rho\theta + B\sin\rho\theta .$$

Such functions are called sinusoidal or ρ-trigonometric. If $0 < \theta_2 - \theta_1 < \pi/\rho$, then the sinusoidal function $H(\theta)$ assuming values h_1 and h_2 at the points θ_1 and θ_2 is unique and can be expressed by the formula

$$H(\theta) = \frac{h_1\sin\rho(\theta_2-\theta) + h_2\sin\rho(\theta-\theta_1)}{\sin\rho(\theta_2-\theta_1)} , \quad \theta_1 \le \theta \le \theta_2 . \tag{2}$$

DEFINITION. A function $K(\theta)$ is called *ρ-trigonometrically convex* on the closed segment $[\alpha, \beta]$ if for $\alpha \leq \theta_1 < \theta_2 \leq \beta$, $0 < \theta_2 - \theta_1 < \pi/\rho$ the equations

$$K(\theta_1) = h_1 \ , \quad K(\theta_2) = h_2$$

imply the inequality

$$K(\theta) \leq H(\theta) \ , \quad \theta_1 \leq \theta \leq \theta_2 \ ,$$

where $H(\theta)$ is a ρ-trigonometric function assuming the values h_1 and h_2 at the points θ_1 and θ_2. A function $K(\theta)$ is called ρ-trigonometrically convex in an open interval if it is ρ-trigonometrically convex on each closed subinterval.

For $\rho = 1$, the corresponding functions are called *trigonometric* and *trigonometrically convex*, respectively.

THEOREM 1. *Let $f(z)$ be a holomorphic function inside an angle, and satisfy inequality* (1). *Then its indicator function h_f with respect to the order ρ is a ρ-trigonometrically convex function.*

PROOF. Let values θ_1 and θ_2 in $[a, b]$ be such that $0 < \theta_2 - \theta_1 < \pi/\rho$, and let $H_\varepsilon(\theta) = A_\varepsilon \cos \rho\theta + B_\varepsilon \sin \rho\theta$ be the ρ-trigonometric function which assumes values $h_f(\theta_j) + \varepsilon$ at θ_j, $j = 1, 2$, $\varepsilon > 0$. Consider the holomorphic function

$$\varphi_\varepsilon(z) = f(z) e^{-(A_\varepsilon - iB_\varepsilon) z^\rho} \ .$$

We have

$$|\varphi_\varepsilon(re^{i\theta_j})| = |f(re^{i\theta_j})| e^{-H_\varepsilon(\theta_j) r^\rho} \overset{\text{as}}{<} e^{-\frac{\varepsilon}{2} r^\rho} \ .$$

Hence the function φ_ε is bounded on the rays $\{z : \arg z = \theta_j\}$, $j = 1, 2$, and by the Phragmén-Lindelöf theorem we have

$$|\varphi_\varepsilon(re^{i\theta})| \leq M_\varepsilon \ , \quad \theta \in [\theta_1, \theta_2] \ , \quad r > 0 \ .$$

The latter inequality yields

$$|f(re^{i\theta})| \leq M_\varepsilon e^{r^\rho H_\varepsilon(\theta)}$$

and, according to the definition of the indicator function,

$$h_f(\theta) \leq H_\varepsilon(\theta) \ .$$

Passing to the limit as $\varepsilon \to 0$, we obtain

$$h_f(\theta) \leq H(\theta) \ , \quad \theta \in [\theta_1, \theta_2] \ , \tag{3}$$

where $H(\theta)$ is the ρ-trigonometric function assuming the values $h(\theta_j)$ at θ_j, $j = 1, 2$. The theorem is proved.

Relations (2) and (3) imply the fundamental relation for the indicator function:

$$h(\theta_1) \sin \rho(\theta - \theta_2) + h(\theta) \sin \rho(\theta_2 - \theta_1) + h(\theta_2) \sin \rho(\theta_1 - \theta) \leq 0 \tag{4}$$

for $\theta_1 < \theta < \theta_2$, $0 < \theta_2 - \theta_1 < \pi/\rho$, which is equivalent to its ρ-trigonometric convexity.

REMARK. If $f(z)$ is an entire function of order ρ, then its indicator h_f is a 2π-periodic ρ-trigonometrically convex function. It is known that for every 2π-periodic ρ-trigonometrically convex function $h(\theta)$ there exists an entire function of order ρ whose indicator coincides with $h(\theta)$. This statement is due to V. Bernstein. The proof of a somewhat more general theorem is given in Levin [**82**].

PROBLEM 1 (Lindelöf). Let a function $f(z)$ analytic in a vertical strip satisfy the estimate

$$|f(x+iy)| = O(|y|^k) , \quad |y| \to \infty ,$$

with some $K < \infty$. Then the function

$$h_f(x) = \limsup_{|y|\to\infty} \frac{\log|f(x+iy)|}{\log|y|}$$

is convex.

8.2. Properties of trigonometrically convex functions

1. *The maximum of two ρ-trigonometrically convex functions is ρ-trigonometrically convex. Similarly, the upper envelope of a uniformly bounded family of ρ-trigonometrically convex functions is ρ-trigonometrically convex.*

2. *Let $h(\theta)$ be a ρ-trigonometrically convex function in an interval (α,β). If $h(\theta_1) = -\infty$ for some $\theta_1 \in (\alpha,\beta)$, then $h(\theta) \equiv -\infty$ for each $\theta \in (\alpha,\beta)$.*

PROOF. Assertion 1 follows from the definition. Now, in 2, let us prove that $h(\theta) = -\infty$ for each $\theta \in (\alpha,\beta)$ satisfying the condition $\theta_1 < \theta < \theta_1 + \pi/\rho$. We choose a point $\theta_2 \in (\alpha,\beta)$ such that $\theta_1 < \theta < \theta_2 < \theta_1 + \pi/\rho$ and introduce the ρ-trigonometric function $H_\varepsilon(\theta)$ which assumes values $H_\varepsilon(\theta_1) = -1/\varepsilon$, $H_\varepsilon(\theta_2) = \max\{-1/\varepsilon,\, h(\theta_2)\}$. The ρ-trigonometric convexity of the function h implies that $h(\theta) \le H_\varepsilon(\theta)$, $\theta_1 \le \theta \le \theta_2$. Sending ε to zero, we obtain $h(\theta) = -\infty$. Thus $h(\theta) = -\infty$ for $\theta_1 < \theta < \min(\beta,\, \theta_1 + \pi/\rho)$. The required assertion now follows for each $\theta \in (\theta_1,\beta)$ and, similarly, for each $\theta \in (\alpha,\theta_1)$.

3. *If a ρ-trigonometrically convex function $h(\theta)$ is bounded, i.e., $|h(\theta)| < K$ for $\theta \in (\alpha,\beta)$, then it is a continuous function of $\theta \in (\alpha,\beta)$, and in each closed subinterval it satisfies a Lipschitz condition.*

PROOF. The proof is based on the fundamental relation (4). We write it in the form

$$\begin{aligned}
&[h(\theta) - h(\theta_1)] \sin\rho(\theta_2 - \theta_1) \\
&\quad \le h(\theta_2)\sin\rho(\theta - \theta_1) + h(\theta_1)[\sin\rho(\theta_2 - \theta) - \sin\rho(\theta_2 - \theta_1)] \\
&\quad = h(\theta_2)\sin\rho(\theta - \theta_1) + 2h(\theta_1)\sin\rho\frac{\theta_1 - \theta}{2}\cos\rho\Big(\theta_2 - \frac{\theta_1 + \theta}{2}\Big) ,
\end{aligned}$$

which implies

$$h(\theta) - h(\theta_1) \le K_1(\theta - \theta_1) , \quad \theta > \theta_1 ; \quad K_1 = 2K\rho/\sin\rho(\theta_2 - \theta_1) .$$

On the other hand,

$$\begin{aligned}
&[h(\theta_2) - h(\theta)] \sin\rho(\theta - \theta_1) \\
&\quad \ge h(\theta_1)\sin\rho(\theta - \theta_2) + h(\theta)[\sin\rho(\theta_2 - \theta_1) - \sin\rho(\theta - \theta_1)] \\
&\quad = h(\theta_1)\sin\rho(\theta - \theta_2) + 2h(\theta)\sin\rho\frac{\theta_2 - \theta}{2}\cos\rho\Big(\theta_1 - \frac{\theta + \theta_2}{2}\Big) .
\end{aligned}$$

Fixing θ_1, and sending θ to θ_2, we obtain

$$h(\theta_2) - h(\theta) > -K_1(\theta_2 - \theta) , \quad \theta_2 > \theta ,$$

or after changing the notation,

$$|h(\theta'') - h(\theta')| < K_1|\theta' - \theta''| .$$

REMARK. A ρ-trigonometrically convex function does not have to be continuous on a closed segment. Its limit values at the endpoints can be smaller than the values of the function.

PROBLEM 2. Prove that a function $h(\theta)$ is ρ-trigonometrically convex for $\theta \in (\alpha, \beta)$ if and only if the function $u(re^{i\theta}) = r^\rho h(\theta)$ is subharmonic within the angle $D = \{z = re^{i\theta} : \alpha < \theta < \beta, r > 0\}$.

PROBLEM 3. Construct a function f analytic within the angle D, continuous up to the bounding rays and satisfying the estimate (1) with the indicator h_f discontinuous at the endpoints of the segment $[\alpha, \beta]$.

4. *Let $h(\theta)$ be a ρ-trigonometrically convex function on the segment $[\alpha, \beta]$. Then*

$$h(\varphi) + h(\varphi + \pi/\rho) \geq 0 , \quad \alpha \leq \varphi < \varphi + \pi/\rho \leq \beta . \tag{5}$$

PROOF. Let us substitute the values $\theta_1 = \varphi + \tau$, $\theta = \varphi + \pi/2\rho$, $\theta_2 = \varphi + \pi/\rho$ into the fundamental relation (4) and pass to the limit as $\tau \to 0$. Using the continuity of the indicator at the interior points of the segment $[\alpha, \beta]$ and the inequality $h(\varphi) \geq h(\varphi + 0)$ for $\varphi = \alpha$, we obtain (5).

5. *If the equality is attained in* (5), *then $h(\theta)$ is a ρ-trigonometric function in the segment $[\varphi, \varphi + \pi/\rho]$.*

PROOF. Let $h(-\pi/2\rho) = h(\pi/2\rho) = 0$. Then the fundamental relation (4) yields $h(\theta) \leq h(0)\cos\rho\theta$. If for some $\theta_0 \in (0, \pi/2\rho)$ the inequality holds, then, applying again the fundamental relation with $\theta_1 = -\theta_0$, $\theta_2 = \theta_0$, $\theta = 0$, we obtain $h(0) < h(0)$. Thus $h(\theta) = h(0)\cos\rho\theta$ everywhere on $[-\pi/\rho, \pi/\rho]$. The general case can be reduced easily to the examined one.

THEOREM 2. *Let $f(z)$ be an analytic function in the angle $D = \{z = re^{i\theta} : \alpha \leq \theta \leq \beta\}$, which satisfies the asymptotic estimate* (1), *and let its indicator with respect to the order ρ be a continuous function on $[\alpha, \beta]$. Then*

$$|f(re^{i\theta})| \leq e^{r^\rho(h(\theta)+\varepsilon)} , \quad r > r_\varepsilon , \quad \alpha \leq \theta \leq \beta . \tag{6}$$

PROOF. We divide the segment $[\alpha, \beta]$ into subintervals with endpoints $\alpha = \theta_0 < \theta_1 < \ldots < \theta_n = \beta$, $\theta_{j+1} - \theta_j < \pi/\rho$. For each subinterval $[\theta_j, \theta_{j+1}]$ we construct the sinusoidal functions $H_j(\theta) = A_j \cos\rho\theta + B_j \sin\rho\theta$ assuming values $h(\theta_j) + \varepsilon/3$ and $h(\theta_{j+1}) + \varepsilon/3$ at the points θ_j and θ_{j+1}, respectively. The segments $[\theta_j, \theta_{j+1}]$ can be chosen small enough that the oscillation of the functions $h(\theta)$ and $H_j(\theta)$ on each of these segments be less than $\varepsilon/3$.

The function

$$\varphi_j(z) = f(z)e^{-(A_j - iB_j)z^\rho}$$

is bounded on the sides of the angle $\theta_j \le \arg z \le \theta_{j+1}$. By the Phragmén-Lindelöf theorem it is also bounded inside the angle. Hence

$$|f(re^{i\theta})| \le C_j e^{H_j(\theta) r^\rho} , \quad \theta_j \le \theta \le \theta_{j+1} ,$$

and for sufficiently large $r_j(\varepsilon)$ and $r > r_j(\varepsilon)$

$$\log |f(re^{i\theta})| < \Big[H_j(\theta) + \frac{\varepsilon}{3}\Big] r^\rho , \quad \theta_j \le \theta \le \theta_{j+1} .$$

Thus,

$$\log |f(re^{i\theta})| \le [h(\theta) + \varepsilon] r^\rho , \quad \alpha \le \theta \le \beta ,$$

for $r > r(\varepsilon) = \max r_j(\varepsilon)$.

REMARK 1. Similar arguments show that if the indicator of the function $f(z)$ equals $-\infty$ identically, then

$$\frac{\log |f(re^{i\theta})|}{r^\rho} \rightrightarrows -\infty , \quad r \to -\infty ,$$

uniformly on each closed subangle.

REMARK 2. The previous remark implies that if $f(z)$ is an entire function of order ρ and if its indicator with respect to the order ρ equals $-\infty$ at one point, then $f \equiv 0$. This remark is a particular case of the following principle: "No nontrivial entire function which grows not too fast in the complex plane can approach zero too fast as z tends to infinity along any ray."

REMARK 3. There are several different definitions of the order and type of a function $f(z)$ analytic inside an angle $D = \{z = re^{i\theta} : \alpha < \theta < \beta\}$. We shall use the following definitions:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^+ \log^+ M_f(r)}{\log r} ; \quad \sigma_f = \limsup_{r\to\infty} \frac{\log^+ M_f(r)}{r^\rho} .$$

PROBLEM 4. Using the Phragmén-Lindelöf theorem, prove that, for $\rho_f > \pi/(\beta - \alpha)$, the order of growth is simultaneously the order of decrease, i.e.,

$$\limsup_{r\to\infty} \frac{\log |f(re^{i\theta})|}{r^{\rho+\varepsilon}} \equiv 0 , \quad \alpha < \theta < \beta,$$

for each $\varepsilon > 0$.

Other definitions of the order of functions analytic inside an angle can be found in Govorov [**46**], Goldberg and Ostrovskii [**43**], Hayman [**53**], Grishin [**47, 48**].

PROBLEM 5. Let $f(z)$ be a function analytic inside an angle $D = \{z : \alpha < \arg z < \beta\}$ and satisfying estimate (1), and let $h_f(\theta)$ be its indicator with respect to the order ρ. Prove that the indicators of $f(z)$ and $f'(z)$ satisfy the relation $h_{f'}(\theta) \le h_f(\theta)$, $\alpha < \theta < \beta$, where the inequality can hold at the point θ_0 only if $h_f(\theta) \equiv 0$ in some neighborhood of θ_0.

8.3. Applications of properties of the indicator function

THEOREM 3 (Carlson). *Let $f(z)$ be a function analytic and of exponential type in the right half-plane $\{z : \operatorname{Re} z > 0\}$, and let*

$$h_f\Big(\frac{\pi}{2}\Big) + h_f\Big(-\frac{\pi}{2}\Big) < 2\pi .$$

If $f(n) = 0$, $n = 0, 1, 2, \ldots$, then $f \equiv 0$.

PROOF. Outside the disks $\{z : |z - n| < \delta\}$ we have

$$|\sin \pi z| > m_\delta e^{\pi |y|} .$$

Thus the function $\varphi(z) = f(z)/\sin \pi z$ is analytic in $\{z : \operatorname{Re} z \geq 0\}$ and

$$|\varphi(z)| \overset{\text{as}}{<} \frac{C}{m_\delta} e^{A|z|}$$

outside the same disks. Assuming $\delta < 1/2$, these disks are pairwise disjoint, and by the Maximum Principle the latter inequality holds inside these disks as well. Also, the lower estimate of $\sin \pi z$ implies that

$$h_\varphi(\pm\pi/2) = h_f(\pm\pi/2) - \pi$$

and therefore $h_\varphi(-\pi/2) + h_\varphi(\pi/2) < 0$. By the property 4 of ρ-trigonometrically convex functions it follows that $h_\varphi(\theta) \equiv -\infty$, $-\pi/2 < \theta < \pi/2$. Let us consider a function

$$\varphi_{\alpha,\beta}(z) = \varphi(z) e^{i\alpha z + \beta z} , \quad \alpha \text{ real}, \quad \beta > 0 .$$

Its indicator equals:

$$h_{\varphi_{\alpha,\beta}}(\theta) = \begin{cases} -\infty, & -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ h_\varphi\Big(\pm\frac{\pi}{2}\Big) \mp \alpha, & \theta = \pm\frac{\pi}{2} . \end{cases}$$

Choosing an appropriate α we can assume that $h_{\varphi_{\alpha,\beta}}(\pm\pi/2) \leq -\eta < 0$. Hence the function $\varphi_{\alpha,\beta}(z)$ is bounded on the imaginary axis by a constant M which does not depend on β. Since $\varphi_{\alpha,\beta}(z)$ is bounded in the right half-plane (this follows from the obvious modification of Theorem 2), we can apply the Phragmén-Lindelöf theorem to arrive at the inequalities

$$|\varphi_{\alpha,\beta}(x)| \leq M$$

and

$$|\varphi(x)| \leq M e^{-\beta x} .$$

With β tending to infinity, we obtain $\varphi(x) \equiv 0$ and therefore $f(x) \equiv 0$.

THEOREM 4 (Shilov). *Let $f(x)$ be an infinitely differentiable function on the real axis and let*

$$\sup_{-\infty < x < \infty} |x^p f^{(q)}(x)| \leq C A^p B^q p^{\alpha p} q^{\beta q} , \quad p, q = 0, 1, \ldots$$

with some positive A, B, C, α, β, $\alpha + \beta < 1$. Then $f(x) \equiv 0$.

PROOF. The assumptions of the theorem imply that the remainder term in the Taylor formula for the function f tends to zero for every $x \in R$. Hence f can be continued into the whole complex plane as an entire function. Using its Taylor expansion at a point x, we obtain

$$|x^p f(x+iy)| \leq CA^p p^{\alpha p} \sum_q \frac{(B|y|)^q q^{\beta q}}{q!} .$$

By the Stirling formula and Lemma 2 from Section 1.3 we obtain

$$|x^p f(x+iy)| \leq CA^p p^{\alpha p} e^{b|y|^{(1-\beta)^{-1}}} .$$

Hence

$$|f(x+iy)| \leq \frac{Ce^{b|y|^{1/(1-\beta)}}}{\sup_p \left\{ \left(\frac{|x|}{A}\right)^p p^{-\alpha p} \right\}} ,$$

and

$$|f(x+iy)| \leq Ce^{-c|x|^{1/\alpha}+b|y|^{1/(1-\beta)}}$$

with some $c > 0$. Thus f is an entire function of order $\rho \leq 1/(1-\beta)$. Since $\alpha + \beta < 1$, we conclude that

$$\lim_{|x|\to\infty} \frac{\log|f(x)|}{|x|^\rho} = -\infty$$

and hence $f \equiv 0$.

THEOREM 5 (Morgan). *Let $f(t)$, $-\infty < t < \infty$, be a function such that $f(t)e^{A|t|^p}$ is bounded as $|t| \to \infty$ for some $A > 0$ and $p > 1$, and let its Fourier transform $g(x)$ decrease in such a way that for some $B > 0$, $l > 1$ the function $g(x)e^{B|x|^l}$ is bounded as $|x| \to \infty$. If $1/p + 1/l < 1$, then $f(t) \equiv 0$.*

PROOF. For $z \in \mathbb{C}$ we set

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{-izt}\, dt ,$$

and observe that $g(z)$ is an entire function which coincides with the Fourier transform of the function f on the real axis. Further,

$$|g(x+iy)| < C\int_{-\infty}^{\infty} e^{-A|t|^p+|yt|}\, dt .$$

Without loss of generality we may assume that $A > 1/p$ (otherwise, we introduce a function $f(\lambda t)$ instead of $f(t)$). Using the inequality

$$|yt| < \frac{1}{p}|t|^p + \frac{1}{q}|y|^q , \quad \frac{1}{p} + \frac{1}{q} = 1 ,$$

we obtain

$$|g(x+iy)| < Ce^{\frac{1}{q}|y|^q},$$

which means that the order ρ of the function $g(z)$ does not exceed $q < l$. On the other hand, the assumptions of the theorem imply that

$$h_g(0) = \limsup_{x\to+\infty} \frac{\log|g(x)|}{x^\rho} = -\infty .$$

Thus $g(x) \equiv 0$ and hence $f(t) \equiv 0$.

Let us remark that Morgan [**100**] found a precise condition on positive values A and B which guarantees that the assertion of Theorem 5 holds in the case $1/p+1/l = 1$.

Theorem 5 shows that in a study of connection between the decrease of a function and that of its Fourier transform the functions decreasing as fast as e^{-cx^2} play an essential role. The following theorem deals with such functions.

THEOREM 6 (Hardy). *Suppose that, for some nonnegative integer n and for all real x, the following estimates hold:*

$$|f(x)| \le C(1+|x|^n)e^{-x^2/2}\,, \quad |g(x)| \le C(1+|x|^n)e^{-x^2/2}\,,$$

where

$$g(x) = \int_{-\infty}^{\infty} f(t)e^{itx}\,dt$$

is the Fourier transform of f. Then

$$f(x) = e^{-x^2/2}P_n(x)\,, \quad g(x) = e^{-x^2/2}Q_n(x)\,,$$

where P_n and Q_n are polynomials of degrees not exceeding n.

PROOF. The estimate for $f(x)$ implies that

$$g(z) = \int_{-\infty}^{\infty} f(t)e^{itz}\,dt$$

is an entire function, and

$$\begin{aligned} |g(x+iy)| &\le C\int_{-\infty}^{\infty}(1+|t|^n)e^{-t^2/2+yt}\,dt \\ &= C\Big[\int_{|t|\le 4|y|} + \int_{|t|\ge 4|y|}\Big](1+|t|^n)e^{-t^2/2+yt}\,dt\,. \end{aligned} \tag{7}$$

We will denote by C positive constants which depend on n but do not depend on x and y. For the first integral on the right-hand side of (7) we have

$$\begin{aligned} \int_{|t|\le 4|y|}(1+|t|^n)e^{-t^2/2+yt}\,dt &\le C(1+|y|^n)e^{y^2/2}\int_{-4|y|}^{4|y|} e^{-t^2/2+yt-y^2/2}\,dt \\ &= C(1+|y|^n)e^{y^2/2}\int_{-4|y|}^{4|y|} e^{-\frac{1}{2}(t+y)^2}\,dt \\ &\le C(1+|z|^n)e^{y^2/2}\,. \end{aligned} \tag{8}$$

The second integral is bounded since

$$\begin{aligned} \int_{|t|\ge 4|y|}(1+|t|^n)e^{-t^2/2+yt}\,dt &\le 2\int_{4|y|}^{\infty}(1+|t|^n)e^{-t^2/4-(t^2/4-|y|t)}\,dt \\ &\le 2\int_{-\infty}^{\infty}(1+|t|^n)e^{-t^2/4}\,dt \le C\,. \end{aligned} \tag{9}$$

Inserting the estimates (8) and (9) into (7) we obtain

$$|g(x+iy)| \le C(1+|z|^n)e^{y^2/2}\,. \tag{10}$$

Now we observe that $\varphi(z) = e^{z^2/2}g(z)$ is an entire function of order not exceeding two. This function does not grow faster than a polynomial along the coordinate axes. Moreover, the estimate (10) implies that the indicator of $\varphi(z)$ with respect to the order $\rho = 2$ satisfies the estimate

$$h_\varphi(\theta) \leq \frac{1}{2}\cos^2\theta\,, \quad 0 \leq \theta \leq 2\pi. \tag{11}$$

Let us show that $\varphi(z)$ is a polynomial, which will prove the Hardy theorem.

We have $h_\varphi(0) = h_\varphi(\pi/2) \leq 0$. By Property 5 of the indicator it follows that

$$h_\varphi(\theta) = k\sin 2\theta\,, \quad 0 \leq \theta \leq \pi/2\,. \tag{12}$$

Comparing (11) and (12), we conclude that $k\sin 2\theta \leq (\cos^2\theta)/2$, or $2k\sin\theta \leq (\cos\theta)/2$, $0 \leq \theta \leq \pi/2$, whence $k \leq 0$. Since $\varphi(z)$ has a polynomial bound on the coordinate axes, we obtain by the Phragmén-Lindelöf theorem that $|\varphi(z)| \leq C(1+|z|^n)$ in the first quadrant. Similar estimates are valid in the remaining quadrants. Therefore, $\varphi(z)$ is a polynomial whose degree does not exceed n.

REMARK. If the function $f(x)$ is as stated in Theorem 6, and if $g(x)e^{x^2/2} \to 0$ as $|x| \to \infty$, then we conclude that $Q(x) \to 0$ as $|x| \to \infty$, i.e., $g(x) = f(x) \equiv 0$. This gives a uniqueness theorem.

In Lecture 25, Part III we will present uniqueness theorems generalizing theorems of this section. The recent developments are exposed in Nazarov [**101**].

LECTURE 9

The Pólya Theorem

9.1. Supporting functions of convex sets

We start with the introduction of a notion of *supporting function* $k(\theta)$ of a set $K \subset \mathbb{C}$:

$$k(\theta) = \sup_{z \in K}\{x\cos\theta + y\sin\theta\} = \sup_{z \in K}\{\mathrm{Re}(ze^{-i\theta})\}\ , \quad \theta \in [0, 2\pi]\ .$$

It is not difficult to prove that the supporting function of a set coincides with the supporting function of its closed convex hull. In what follows we assume that K is a convex compact set.

For each $\theta \in [0, 2\pi]$ the line $l_\theta = \{z : \mathrm{Re}(ze^{-i\theta}) = k(\theta)\}$ is called a supporting line of K. Evidently, it is orthogonal to the ray $\{z : \arg z = \theta\}$, has nonvoid intersection with K, and the set K itself is contained completely in a closed half-plane with the boundary l_θ. The value $|k(\theta)|$ is equal to the length of the segment of the ray $\{z : \arg z = \theta\}$ if $k(\theta) > 0$ or of the ray $\{z : \arg z = \theta + \pi\}$ if $k(\theta) < 0$, cut off by the line l_θ. If $k(\theta)$ is the supporting function of K, then $k(-\theta)$ is the supporting function of $\overline{K}$, where the bar means, as usual, the complex conjugation.

EXAMPLES. The supporting function of the disk $\{z : |z| \leq R\}$ is $k(\theta) = R$. The supporting function of a single point $\{z_0 = r_0 e^{i\theta_0}\}$ is $k(\theta) = r_0\cos(\theta - \theta_0)$. The latter function is sinusoidal. The converse statement is also true: every sinusoidal function is the supporting function of a set consisting of a single point. The supporting function of the segment $[-id, id]$ is $k(\theta) = d|\sin\theta|$.

THEOREM 1. *The supporting function of a convex compact set is trigonometrically convex. Conversely, every 2π-periodic trigonometrically convex function $k(\theta)$ is the supporting function of some convex compact set K.*

PROOF. Let $k(\theta)$ be the supporting function of a convex compact set K. Then $k(\theta)$ is the upper envelope of a uniformly bounded family of 2π-periodic sinusoidal functions $h(\theta) = \mathrm{Re}(ze^{-i\theta})$, $z \in K$. Hence $k(\theta)$ is trigonometrically convex, which proves the first statement of Theorem 1.

To prove the converse statement, let us consider the set

$$K = \bigcap_{0 \leq \theta \leq \pi} \Pi_\theta\ ,$$

where $\Pi_\theta = \{z : \mathrm{Re}(ze^{-i\theta}) \leq k(\theta)\}$ is a half-plane, and prove that this set is not empty. Moreover, we shall see that every line $l_\theta = \{z : \mathrm{Re}(ze^{-i\theta}) = k(\theta)\}$ contains points of K. It will show that K is a convex compact set and that l_θ are supporting lines of this compact set.

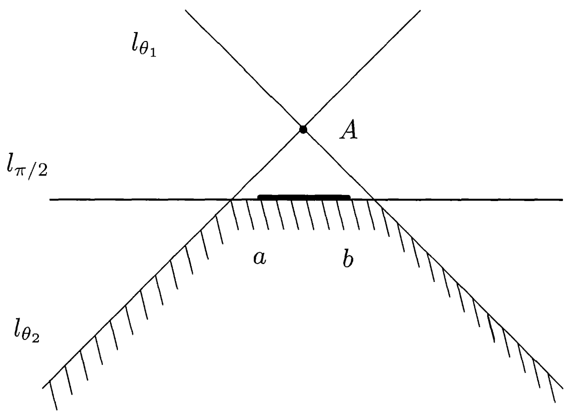

FIGURE 1

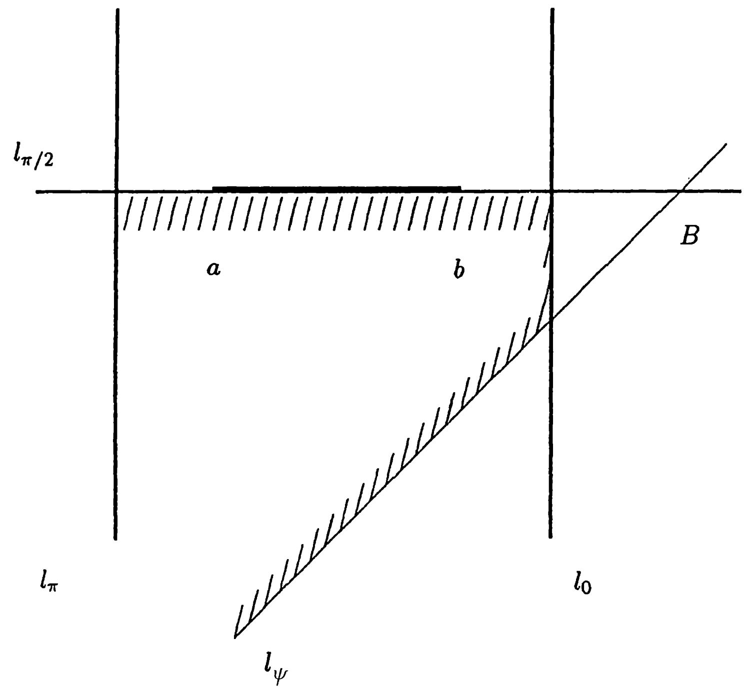

FIGURE 2

Without loss of generality, we fix $\theta = \pi/2$ and prove that $l_{\pi/2}$ contains at least one point which belongs to all half-planes Π_θ. Let us assume that $0 < \theta_1 < \pi/2 < \theta_2 < \pi$, and denote by A the intersection point of the lines l_{θ_1} and l_{θ_2}. Since the function $k(\theta)$ is trigonometrically convex, the point A cannot lie below the line $l_{\pi/2}$ (see Figure 1). Thus, the half-planes Π_{θ_1}, Π_{θ_2} and the line $l_{\pi/2}$ have a common segment which we denote by $[a_{\theta_2}, b_{\theta_1}]$. It may happen that $a_{\theta_2} = b_{\theta_1}$. Setting $a = \sup\{a_{\theta_2} : \pi/2 \leq \theta_2 \leq \pi\}$, $b = \inf\{b_{\theta_1} : 0 \leq \theta \leq \pi/2\}$, we find that $a \leq b$, and hence the segment $[a, b] \subset l_{\pi/2}$ belongs to all half-planes Π_θ, $0 \leq \theta \leq \pi$, and, in particular, to the strip $\{z = x + iy : k(\pi) \le x \le k(0)\}$. Let us show that each half-plane Π_ψ, $-\pi/2 < \psi < 0$ also intersects this segment. Using the definition of trigonometrically convex functions with $\psi = \theta_1 < 0 < \theta_2 = \pi/2$, we find that the intersection point of $l_{\pi/2}$ and l_ψ cannot lie to the left of the intersection point B of the lines $l_{\pi/2}$ and l_0 (see Figure 2). Hence the segment $[a, b]$ is in the half-plane Π_ψ. The case $-\pi < \psi \le -\pi/2$ can be examined using the same arguments, which proves the theorem.

Let F be an entire function of exponential type (EFET). By Theorem 1 from Section 8.1, its indicator h_F is trigonometrically convex, and hence h_F is the supporting function of a convex compact set $I_F \subset \mathbb{C}$. This compact set is called the *indicator diagram* of the function F. It gives a geometrical representation of the growth of F in various directions.

This definition allows us to give, in particular, a simple geometrical interpretation of certain properties of indicators of EFET. For example, let $F(z)$, $G(z)$ be such functions; then the inequality

$$h_{F+G}(\theta) \le \max\{h_F(\theta), h_G(\theta)\}$$

means that the indicator diagram I_{F+G} is contained in the convex hull of the diagrams I_F and I_G.

PROBLEM 1. Prove that if one of the indicator diagrams I_F and I_G can be obtained from the other by a parallel translation, then the indicator diagram of the sum I_{F+G} coincides with the convex hull of the indicator diagrams I_F and I_G.

The sum of sets, $K = K_1 + K_2$, is the set of points $\{z = z_1 + z_2 : z_1 \in K_1, z_2 \in K_2\}$. It is evident that the sum of convex compact sets is a convex compact set. It follows from the definition that the supporting function of the sum of convex compact sets equals $k(\theta) = k_1(\theta) + k_2(\theta)$. Therefore, if $h_{FG}(\theta) = h_F(\theta) + h_G(\theta)$, then $I_{FG} = I_F + I_G$, and conversely, the latter equality implies the former.

9.2. The Borel transform and the Pólya theorem

Let

$$F(z) = \sum_{n=0}^{\infty} \frac{c_n}{n!} z^n$$

be an EFET. It is easy to deduce from the formula for the type of an entire function that

$$\sigma = \sigma_F = \limsup_{n\to\infty} \sqrt[n]{|c_n|}\,.$$

The function

$$f(z) = \sum_{n=0}^{\infty} \frac{c_n}{z^{n+1}}$$

is called the *Borel transform* of the function $F(z)$. By the Hadamard theorem the series converges outside the disk $\{|z| \le \sigma_F\}$ and diverges inside this disk. It is possible that the function $f(z)$ can be analytically continued into the disk $\{z : |z| < \sigma_F\}$. The smallest convex compact set containing all singularities of $f(z)$ is called the *conjugate indicator diagram* of $F(z)$. We denote by $k_F(\theta)$ the supporting function of this compact set.

The following theorem establishes the remarkable connection between the conjugate diagram and the indicator diagram of EFET.

THEOREM 2 (Pólya). *For every EFET $F(z)$ the relation*

$$h_F(\theta) = k_F(-\theta)$$

holds, and hence the conjugate diagram is the reflection in the real axis of the indicator diagram I_F.

PROOF. Let us denote by $\overline{I}$ the conjugate diagram. The proof is based on two integral formulas linking the function $F(z)$ and its Borel transform. The first of them has the form

$$F(z) = \frac{1}{2i\pi} \int_{\partial(\overline{I}+K_\varepsilon)} f(\zeta) e^{\zeta z}\, d\zeta, \tag{2}$$

where K_ε is the disk $\{z : |z| \le \varepsilon\}$.

Indeed, the integration over the curve $\partial(\overline{I} + K_\varepsilon)$ on the right-hand side may be replaced by the integration over the circle $\partial K_{\sigma+\varepsilon}$ (here σ is the type of the function F). Thus the formula (2) is obtained by the integration of the series

$$e^{\zeta z} f(\zeta) = \sum_{n=0}^{\infty} \frac{c_n}{\zeta^{n+1}} e^{\zeta z} .$$

It follows from (2) that

$$\begin{aligned} |F(re^{i\theta})| &\le C_\varepsilon \exp\{r \max_{\zeta\in\overline{I}+K_\varepsilon} \mathrm{Re}(\zeta z)\} \\ &= C_\varepsilon \exp\{r\big(k(-\theta)+\varepsilon\big)\} . \end{aligned}$$

Hence

$$h(\theta) \le k(-\theta) . \tag{3}$$

The transform inverse to (2) has the form

$$f(\zeta) = \int_0^\infty F(te^{-i\theta}) e^{-\zeta t e^{-i\theta}}\, d(te^{-i\theta}) ; \quad \mathrm{Re}(\zeta e^{-i\theta}) > h(-\theta) . \tag{4}$$

To prove this formula we observe that the inequality

$$|F(te^{-i\theta})| \overset{\text{as}}{<} e^{(h(-\theta)+\varepsilon)t}$$

implies that the integral in (4) converges uniformly in the half-plane $\{\zeta = \xi + i\eta : \xi\cos\theta + \eta\sin\theta \ge h(-\theta) + 2\varepsilon\}$, and hence this integral represents a holomorphic function in the domain $\{\zeta : \mathrm{Re}(\zeta e^{-i\theta}) > h(-\theta)\}$. Let us check that this function coincides with $f(\zeta)$ for $\zeta = re^{i\theta}$, $r > 3\sigma$. Indeed, in this case the integral (4) can be written as

$$\int_0^\infty F(te^{-i\theta}) e^{-rt} e^{-i\theta}\, dt .$$

For the series

$$F(te^{-i\theta}) = \sum_{n=0}^{\infty} \frac{c_n}{n!} t^n e^{-in\theta} ,$$

we have the estimate of the general term

$$\left|\frac{c_n}{n!}\right| \le \frac{M_F(2t)}{(2t)^n} \overset{\text{as}}{<} \frac{e^{2(\sigma+\varepsilon)t}}{(2t)^n} , \quad n = 0, 1, \dots ,$$

and of the remainder

$$|R_n(t)| \le \sum_{k=n+1}^{\infty} \left|\frac{c_k}{k!}\right| t^k \le \sum_{k=n+1}^{\infty} \frac{1}{2^k} e^{2(\sigma+\varepsilon)t} = \frac{1}{2^n} e^{2(\sigma+\varepsilon)t} .$$

If $r > 3\sigma$ we obtain the estimate

$$\Big| \int_0^\infty F(te^{-i\theta}) e^{-tr} e^{-i\theta}\, dt - \int_0^\infty e^{-tr} e^{-i\theta} \sum_{k=0}^{n} \frac{c_k}{k!} t^k e^{-ik\theta}\, dt \Big| \le \frac{1}{2^n} \int_0^\infty e^{-(\sigma-2\varepsilon)t}\, dt .$$

Integrating termwise, we obtain the equality (4). Thus, $f(\zeta)$ is analytic in the domain $\{\zeta = \xi + i\theta : \xi\cos\theta + \eta\sin\theta > h(-\theta)\}$. Hence, $k(\theta) \le h(-\theta)$. Combining this estimate with (3), we obtain $h(\theta) = k(-\theta)$. The theorem is proved.

REMARK 1. It follows from formula (4) with $\theta = 0$ that the function $f(\zeta)$ coincides with the Laplace transform of the function $F(z)$.

REMARK 2. Let $F(z)$ be a holomorphic function of exponential type inside an angle $\{z : \alpha \le \arg z \le \beta\}$. Then the function $f(\zeta)$ defined by (4) is analytic in the union of half-planes $\{\zeta = \xi + i\eta : \xi\cos\theta + \eta\sin\theta > h(-\theta),\ \alpha \le \theta \le \beta\}$. The complement to this domain is the intersection of closed half-planes $\{\zeta = \xi + i\eta : \xi\cos\theta + \eta\sin\theta \le h(-\theta)\}$, and its boundary contains two rays orthogonal to the rays $\arg\zeta = -\alpha$ and $\arg\zeta = -\beta$. This closed convex set is minimal among all convex sets of such a form containing all singularities of $f(\zeta)$, which is not difficult to prove using the well-known inversion formula for the Laplace transform

$$F(z) = \frac{1}{2i\pi} \lim_{a\to\infty} \int_{a-i\infty}^{a+i\infty} f(\zeta) e^{\zeta z}\, d\zeta .$$

The notions of indicator, indicator diagram, and conjugate diagram can be extended to entire functions with values in a Banach space. To this end, the modulus $|f(re^{i\theta})|$ in the corresponding definitions should be replaced by the norm $\|f(re^{i\theta})\|$.

PROBLEM 2. Prove that the convex hull of the spectrum of an arbitrary element x of a Banach algebra coincides with the indicator diagram of the entire function $e^{\lambda x}$.

EXAMPLE 1. Let $F(z) = \sum_{k=1}^{n} P_k(z) e^{\lambda_k z}$, where λ_k, $k = 1, 2, \dots$, are complex numbers, and $P_k(z)$ are polynomials. The Borel transform of a monomial $z^p e^{\lambda_k z}$ equals $p!(\zeta - \lambda_k)^{-p-1}$. Therefore, the poles at the points $\lambda_k, k = 1, 2, \dots$, are the only singularities of the Borel transform of $F(z)$. The conjugate diagram of $F(z)$ coincides with the convex hull of points $\{\lambda_1, \dots, \lambda_n\}$, and by the Pólya theorem the indicator diagram of $F(z)$ coincides with the convex hull of points $\{\overline{\lambda}_1, \dots, \overline{\lambda}_n\}$.

EXAMPLE 2. Let K be an arbitrary convex compact set. Choose a countable set of points $\{\lambda_k\}$ dense on ∂K, and consider a function

$$f(\zeta) = \sum_{k=0}^{\infty} \frac{c_k}{\zeta - \lambda_k} , \quad \sum |c_k| < \infty .$$

Evidently, the function $f(\zeta)$ is analytic outside K, equals zero at infinity, and cannot be analytically continued through any part of ∂K. Hence K is the indicator diagram of the entire function

$$F(z) = \sum_{k=0}^{\infty} c_k e^{\lambda_k z} .$$

Thus for every closed compact set $K \subset \mathbb{C}$ there exists an entire function of exponential type whose indicator diagram coincides with K.

LECTURE 10

Applications of the Pólya Theorem

10.1. The Paley-Wiener theorem

The following theorem gives a description of the class of entire functions of exponential type that are square integrable on the real axis.

THEOREM 1 (Paley-Wiener). *For a function g to be representable in the form*

$$g(x) = \frac{1}{2\pi} \int_a^b \psi(t) e^{itx} \, dt \,, \quad \psi \in L^2(a,b) \,, \tag{1}$$

it is necessary and sufficient that

a) *it be possible to extend $g(x)$ to the whole complex plane as an EFET;*

b) $g \in L^2(-\infty, \infty)$.

If the interval (a,b) cannot be replaced by a smaller interval, then the segment $[ia, ib]$ of the imaginary axis coincides with the conjugate diagram of $g(z)$.

NECESSITY. By the Fourier-Plancherel theorem, we have $g \in L^2(-\infty, \infty)$ and $\sqrt{2\pi} \|g\|_{L^2(-\infty,\infty)} = \|\psi\|_{L^2(a,b)}$. Further, the function

$$g(z) = \frac{1}{2\pi} \int_a^b e^{itz} \psi(t) \, dt$$

is entire since the integrand is an entire function of $z \in \mathbb{C}$. For $y \geq 0$, we have

$$|g(x+iy)| \leq \frac{1}{2\pi} \int_a^b |\psi(t)| e^{\mathrm{Re}(itz)} \, dt \leq e^{-ay} \frac{1}{2\pi} \int_a^b |\psi(t)| \, dt \,.$$

Similarly, for $y \leq 0$,

$$|g(x+iy)| \leq e^{-by} \frac{1}{2\pi} \int_a^b |\psi(t)| \, dt \,.$$

It follows that $g(z)$ is an EFET and its indicator diagram is contained in the segment $[-ib, -ia]$ of the imaginary axis.

SUFFICIENCY. The Borel transform $\varphi(w)$ of the function g is holomorphic outside the conjugate diagram of g, and, in particular, outside the disk $\{w : |w| \leq \sigma_g\}$. The function φ can be represented using the Laplace transform

$$\varphi(u+iv) = \int_0^\infty g(x) e^{-x(u+iv)} \, dx, \quad u > \sigma_g \,, \tag{2}$$

and, similarly,

$$\varphi(u+iv) = -\int_{-\infty}^{0} g(x)e^{-x(u+iv)}\,dx\,, \quad u < -\sigma_g\,. \tag{3}$$

Since $g \in L^2(-\infty,\infty)$, equations (2) and (3) imply that φ is holomorphic in the half-planes $\{w: \pm\operatorname{Re} w > 0\}$. Therefore, the conjugate diagram of the function g coincides with a segment $[i\alpha, i\beta]$ of the imaginary axis.

The function $\varphi(u+iv)$ is square integrable on every vertical line which is not the imaginary axis. Let us show that there exist the mean square limits $\underset{u\to\pm 0}{\text{l.i.m.}}\,\varphi(u+iv)$ equal to

$$\varphi(+0+iv) = \int_0^\infty g(x)e^{-ixv}\,dx\,,$$
$$\varphi(-0+iv) = -\int_{-\infty}^0 g(x)e^{-ixv}\,dx\,,$$

where the integrals are mean square convergent as well. Indeed, by the Plancherel theorem we have

$$\int_{-\infty}^{\infty} |\varphi(+0+iv)-\varphi(u+iv)|^2\,dv = \frac{1}{2\pi}\int_0^\infty |g(x)|^2|1-e^{-ux}|^2\,dx \to 0\,, \quad u \searrow 0\,.$$

Similarly, $\varphi(-0+iv) = \underset{u\to -0}{\text{l.i.m.}}\,\varphi(u+iv)$.

The functions $\varphi(\pm 0+iv)$ coincide with $\varphi(iv)$ and, consequently, with each other, at imaginary points lying outside the indicator diagram. Thus, if $\psi(v)$ is the Fourier-Plancherel transform of the function $g(x)$, then for such values v we obtain

$$\begin{aligned}\psi(v) &= \int_{-\infty}^0 g(x)e^{-ixv}\,dx + \int_0^\infty g(x)e^{-ixv}\,dx \\ &= \varphi(+0+iv) - \varphi(-0+iv) = 0\,.\end{aligned}$$

The inversion formula implies equation (1) which shows that the indicator diagram of the function $g(z)$ contains the segment $[-ib,-ia]$. The theorem is proved.

10.2. Analytic continuation of a power series

The next application of the Pólya theorem is related to analytic continuation of a power series

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{4}$$

converging in a neighborhood of the origin. First, observe that there exists an EFET $F(w)$ such that $a_n = F(n)$, $n = 0, 1, \ldots$. Indeed, if the series (4) converges for $|z| < r$, then

$$a_n = \frac{1}{2\pi i}\int_{|z|=r} \frac{\varphi(z)}{z^{n+1}}\,dz\,. \tag{5}$$

Choosing a continuous branch of the logarithm $\log z$ for $|\arg z| < \pi$ and setting $\zeta = -\log z$, we obtain $a_n = F(n)$, $n = 0, 1, \ldots$, where

$$F(w) = -\frac{1}{2\pi i} \int_{-\log r - i\pi}^{-\log r + i\pi} \varphi(e^{-\zeta}) e^{w\zeta} \, d\zeta$$

is an EFET. Hence the converging series (4) can be represented in the form

$$\varphi(z) = \sum_{n=0}^{\infty} F(n) z^n, \tag{6}$$

where the width of the indicator diagram of the function F along the imaginary axis does not exceed 2π, i.e., $h_F(\pi/2) + h_F(-\pi/2) \leq 2\pi$. If this width is smaller than 2π, then the following theorem asserts that the series (6) may be analytically extended.

THEOREM 2 (Carlson). *If the width of the indicator diagram I_F of the function F along the imaginary axis is less than 2π, then the function φ defined in neighborhood of the origin by the series (6) may be analytically continued into a domain $\mathbb{C}\backslash G$, where $G = \{w = e^{-z},\, z \in \overline{I}_F\}$, and $\varphi(\infty) = 0$.*

Conversely, let a function φ be represented in a neighborhood of the origin by series (4) and also can be continued analytically into the exterior of a set

$$G = \{w : \, w = e^{-z} \, z \in K\}\ , \tag{7}$$

where K is a convex compact set whose width along the imaginary axis is less than 2π, and let $\varphi(\infty) = 0$. Then there exists an EFET F such that $\overline{I}_F \subset K$ and $a_n = F(n)$, $n = 0, 1, 2, \ldots$.

PROOF. To prove the first assertion of the theorem, we use the inversion formula (2) for the Borel transform defined in Section 9.2. Using this formula, we write equation (6) in the form

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial(\overline{I}_F + K_\varepsilon)} f(\zeta) \sum_{n=0}^{\infty} e^{n\zeta} z^n \, d\zeta,$$

where $\varepsilon > 0$ is chosen so small that the width along the imaginary axis of the compact set $\overline{I}_F + K_\varepsilon$ is less than 2π. For small enough values of $|z|$ the series in the integrand converges and hence

$$\varphi(z) = \frac{1}{2\pi i} \int_{\partial(\overline{I}_F + K_\varepsilon)} \frac{f(\zeta)}{1 - z e^{\zeta}} \, d\zeta\ . \tag{8}$$

The function defined by the integral is analytic outside $G = \{e^{-z},\, z \in \overline{I}_F\}$ and equals zero at infinity. Since the width of $\overline{I}_F$ along the imaginary axis is less than 2π, the set $\overline{\mathbb{C}}\backslash G$ is connected. The first part of the theorem is proved.

Conversely, let all singularities of the function φ represented by series (4) lie in a set G of the form (7), where K is a convex compact set whose width along the imaginary axis is less than 2π, and let $\varphi(\infty) = 0$. Under these assumptions the set $\overline{\mathbb{C}}\backslash G$ is a domain containing the points 0 and ∞. We have

$$a_n = \frac{1}{2\pi i} \int_C \frac{\varphi(z)}{z^{n+1}} \, dz\ , \quad n = 0, 1, \ldots\ ,$$

where C is a circumference of small radius centered at the origin. Since $\varphi(\infty) = 0$, each of these integrals can be replaced by the integral over an arbitrary contour L which surrounds G and does not surround the origin. Setting

$$F(w) = \frac{1}{2\pi i}\int_L \frac{\varphi(\zeta)}{\zeta^{w+1}}\, d\zeta\ ,$$

we obtain $F(n) = a_n$, $n = 0, 1, \dots$, and

$$F(w) = \frac{1}{2\pi i}\int_\Gamma \varphi(e^{-\zeta})e^{\zeta w}\, d\zeta\ ,$$

where the contour Γ surrounds the compact set K. It follows that F is an EFET and its indicator diagram is contained in K. The theorem is proved.

COROLLARY (Leau, Wigert). *In order that the function φ represented by the series (4) have a singularity at $z = 1$ only, and have a zero at infinity, it is necessary and sufficient that the coefficients of series (4) have the form $a_n = F(n)$, where $F(w)$ is an entire function whose growth does not exceed order one and minimal type.*

PROBLEM 1. Let $\varphi(z) = G(1/(1-z))$, where G is an entire function and $G(0) = 0$. Then the order ρ_G of the function G and the order ρ_F of the entire function $F(w)$ interpolating the coefficients of $\varphi(z)$ are related by the equation $\rho_G = \rho_F/(1-\rho_F)$.

PROBLEM 2. In order that the function

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

holomorphic in a neighborhood of the origin have only a finite number of isolated singularities (in whose vicinities the function φ is single-valued) in the disk $|z| < R$, it is necessary and sufficient that

$$a_n = \sum_{k=1}^{N} F_k(n) z_k^{-n} + O\Big(\frac{1}{R^n}\Big)\ , \quad n \to \infty\ ,$$

where $F_k(w)$ are entire functions of minimal exponential type. Moreover, $z_1, \dots,$ z_N are singularities of $\varphi(z)$ in the disk $\{|z| < R\}$. In order that these singularities be poles, it is necessary and sufficient that the functions $F_k(w)$ be polynomials.

REMARK 1. Equation (8) yields that if the assumptions of the Carlson theorem hold, then the expansion of $\varphi(z)$ in a neighborhood of infinity has the form

$$\varphi(z) = -\sum_{n=1}^{\infty} \frac{F(-n)}{z^n}\ .$$

REMARK 2. If the assumptions of the Carlson theorem hold, then the function $\varphi(z)$ can be continued to infinity along some ray emanating from the origin. If the indicator diagram I_F of the interpolating EFET $F(w)$ is such that the compact set e^{-I_F} does not separate 0 and ∞, then, as before, $\varphi(z)$ can be analytically continued at infinity. Evidently, e^{-I_F} does not separate 0 and ∞ if and only if the sets $I_F + 2m\pi i$, $m \in \mathbb{Z}$, are pairwise disjoint.

PROBLEM 3. Let $F(w)$ be an EFET whose indicator diagram has the width along the imaginary axis less than 2π. Prove that

$$h_F(0) = \limsup_{n\to\infty} \frac{\log|F(n)|}{n} .$$

Other applications of the Pólya theorem to problems of analytic continuation may be found in Bieberbach [**16**].

10.3. Analytic functionals

As was stated in Section 3.4, every linear functional F on the space $A(D)$ of analytic functions in a simply connected domain $D \in \mathbb{C}$ is defined by the equation

$$F(f) = \frac{1}{2\pi i}\int_l f(\zeta)\varphi(\zeta)\,d\zeta\ ,$$

where the function φ determined by F is analytic on the complement of D, $\varphi(\infty) = 0$, and the simple closed curve l surrounds all singularities of φ and lies in D. Thus, the space $A^*(D)$ of linear functionals on $A(D)$ (they are called analytic functionals) is isomorphic to the space $A_0(\mathbb{C}\backslash D)$ of functions analytic on $\mathbb{C}\backslash D$ which are zero at infinity.

The Pólya theorem gives another representation of the space $A^*(D)$ if D is a convex domain. In this case we may assume that compact sets $G_1 \Subset G_2 \Subset \cdots \Subset G_m \Subset \cdots$ exhausting the domain D from the inside are convex and their supporting functions satisfy the condition

$$h_1(\theta) < h_2(\theta) < \cdots < h_m(\theta) < \cdots .$$

Then the function $H(\theta) = \lim_{m\to\infty} h_m(\theta)$ is supporting for D. Given a linear functional F, let us introduce the function

$$\Phi(\lambda) = F(e^{\lambda z}) \tag{9}$$

usually called the Fourier-Borel transform of F. According to formulas (8) and (10) from Lecture 3, there exist an integer $m \geq 1$ and a constant C such that

$$|\Phi(\lambda)| \leq C \max_{z\in D_m} |e^{\lambda z}| = C\exp\big(h_m(-\arg\lambda)|\lambda|\big).$$

So $\Phi(\lambda)$ is an EFET and by the Pólya theorem its conjugate diagram is contained in the domain D. Denote by $\varphi(\zeta)$ the Borel transform of the function $\Phi(\lambda)$ and check that

$$F(f) = \frac{1}{2\pi i}\int_l f(\zeta)\varphi(\zeta)\,d\zeta \tag{10}$$

for each function $f \in A(D)$, where l is a simple closed contour in D surrounding all singularities of the function φ.

Indeed,

$$F(e^{\lambda z}) = \Phi(\lambda) = \frac{1}{2\pi i}\int_l e^{\lambda\zeta}\varphi(\zeta)\,d\zeta\ .$$

Differentiating this equation with respect to λ and setting $\lambda = 0$ afterwards, we obtain

$$F(z^k) = \frac{1}{2\pi i}\int_l \zeta^k\varphi(\zeta)\,d\zeta\ .$$

Hence (10) holds for every polynomial. Since polynomials are dense in $A(D)$, equation (10) holds for every function $f \in A(D)$.

Let $\Phi(\lambda)$ be an entire function satisfying an estimate

$$|\Phi(\lambda)| \leq C \exp\left[h_m(-\arg\lambda)|\lambda|\right]$$

for some C and m, and let $\varphi(\zeta)$ be its Borel transform. Then equation (10) defines a linear functional F on $A(D)$ such that $\Phi(\lambda) = F(e^{\lambda z})$. Thus, we have proved

THEOREM 3. *Equation* (9) *defines an isomorphism between the space* $A^*(D)$ *of linear functionals on the space of analytic functions in a convex domain* D *with supporting function* $H(\theta)$, *and the space of entire functions of exponential type whose indicators satisfy the condition*

$$h(\theta) < H(\theta)\ .$$

If $D = \mathbb{C}$, then $H(\theta) \equiv +\infty$, and $A(D)$ is the space of all entire functions endowed with the topology of uniform convergence on each compact set in $\mathbb{C}$. According to Theorem 3, the space $A^*(D)$ can be identified with the space of all EFET.

LECTURE 11

Lower Bounds for Analytic and Subharmonic Functions

11.1. The Carathéodory inequality

For a function $f(z) = u(z) + iv(z)$ analytic in the disk $\{z : |z| \le R\}$ we set $A_f(r) = \max\{u(z) : |z| \le r\}$. It follows from the Maximum Principle for harmonic functions that $A_f(r)$ is a monotonically increasing function of r, and that $|A_f(r)| \le M_f(r)$. It appears that for $R > r$ the value $M_f(r)$ can be estimated from above by means of $A_f(R)$.

THEOREM 1 (Carathéodory). *Let $f(z)$ be an analytic function in the disk $\{z : |z| \le R\}$, and let $f(0) = 0$. Then*

$$M_f(r) \le \frac{2r}{R-r} A_f(R) .$$

PROOF. The Schwarz formula, Section 2.1, states that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\psi}) \frac{Re^{i\psi} + z}{Re^{i\psi} - z} \, d\psi , \quad |z| < R .$$

Further, by the condition $f(0) = 0$, we have

$$0 = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\psi}) \, d\psi .$$

Hence

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(Re^{i\psi}) \frac{2z}{Re^{i\psi} - z} \, d\psi .$$

According to the Cauchy theorem

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{Re^{i\psi} - z} = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{d\zeta}{\zeta(\zeta - z)} = 0 .$$

Thus

$$-f(z) = \frac{1}{2\pi} \int_0^{2\pi} [A_f(R) - u(Re^{i\psi})] \frac{2z}{Re^{i\psi} - z} \, d\psi ,$$

and then

$$|f(z)| \le \frac{1}{2\pi} \int_0^{2\pi} [A_f(R) - u(Re^{i\psi})] \frac{2r}{R-r} \, d\psi = \frac{2r}{R-r} A_f(R) ,$$

proving Theorem 1.

The proven inequality yields *a lower bound for the harmonic function* $u(z)$ *in the disk* $\{z : |z| \le R\}$ provided that $u(0) = 0$:

$$u(z) \ge -\frac{2r}{R-r} \max\{u(Re^{i\psi}) : 0 \le \psi \le 2\pi\} , \quad r = |z| .$$

It immediately implies

THEOREM 2. *If an analytic function* $f(z)$ *has no zeros in a disk* $\{z : |z| \le R\}$ *and if* $|f(0)| = 1$, *then*

$$\log |f(z)| \ge -\frac{2r}{R-r} \log M_f(R)$$

as $|z| = r < R$. *In particular,* $\log |f(z)| > -2 \log M_f(2r)$.

PROBLEM 1. Prove that

$$M_f(r) \le [A_f(R) - \operatorname{Re} f(0)] \frac{2r}{R-r} + |f(0)| , \quad r < R$$

for a function analytic in the disk $\{z : |z| \le R\}$.

PROBLEM 2. Let $f(z)$ be an analytic function in the upper half-plane $\{z : \operatorname{Im} z > 0\}$ such that $\operatorname{Im} f(z) > 0$. Prove the estimates

$$\frac{1}{5}|f(i)|\frac{\sin\theta}{r} \le |f(z)| \le 5|f(i)|\frac{r}{\sin\theta} , \quad z = re^{i\theta} , \quad 0 < \theta < \pi , \quad r \ge 1 .$$

This is the so-called Carathéodory inequality for a half-plane.

HINT. Consider the function

$$F(u) = if\Big(-i\frac{u+1}{u-1}\Big), \ |u| < 1;$$

it satisfies $\operatorname{Re} F(0) \le 0$. Then apply the inequality from the previous problem.

11.2. The Cartan estimate

If the function $f(z)$ has zeros in the disk $\{z : |z| \le R\}$, then

$$\log |f(z)| = \log |P(z)| + \log \left|\frac{f(z)}{P(z)}\right| ,$$

where $P(z)$ is a polynomial and the second term is a harmonic function. Therefore, the problem of estimating the function $f(z)$ from below reduces to that for the first term. Evidently, such an estimate is possible only outside some neighborhood of the zeros of the function $f(z)$.

We shall consider a more general problem of estimating from below the logarithmic potential of a finite measure.

THEOREM 3. *Let*

$$u(z) = \iint_{\mathbb{C}} \log |z - \zeta| \, d\mu(\zeta)$$

where μ *is Borel measure,* $\mu(\mathbb{C}) = n < \infty$. *Given* H, $0 < H < 1$, *there exists a system of disks in the complex plane such that*

$$\sum r_j \le 5H ,$$

where r_j are radii of these disks, and

$$u(z) \geq n \log \frac{H}{e}$$

everywhere outside these disks.

PROOF. Fix $p > 0$. A point $z \in \mathbb{C}$ is said to be p-normal, if $n(t;z) < pt$, $t > 0$, where $n(t;z) = \mu(\{\zeta : |z-\zeta| \leq t\})$. If z is a p-abnormal point, then there exists a number t such that $n(t;z) \geq pt$. Let ρ_z be the l.u.b. of the set of such values t. Since $\mu(\mathbb{C}) < \infty$, the value ρ_z is finite and attained for some t. Indeed, let $t_m \nearrow \rho_z$ and $n(t_m;z) \geq pt_m$. Then

$$n(\rho_z;z) \geq pt_m \to p\rho_z\ , \quad m \to \infty\ .$$

Thus, for every p-abnormal point z there exists a radius ρ_z and an exceptional disk $C_z = \{\zeta : |z-\zeta| < \rho_z\}$. For normal points we set $\rho_z = 0$.

Let $r_1 = \sup\{\rho_z : z \in \mathbb{C}\}$. We will prove that this l.u.b. is attained at some point. Note that for a given $\varepsilon > 0$ one can choose a value R_ε such that the measure μ of the domain $\{z : |z| > R_\varepsilon\}$ is less than ε, and hence $\rho_z \to 0$ as $z \to \infty$. Let $\{z_m\}$ be a sequence such that $\rho_{z_m} \nearrow r_1$. Since the sequence of points z_m lies in some disk $|z| \leq R$, we may assume, without loss of generality, that $z_m \to \zeta$. We have $n(r_1+\varepsilon;\zeta) \geq pr_1$, and then $n(r_1;\zeta) \geq pr_1$.

We delete from the plane the open exceptional disk C_1 with the center at $\zeta_1 = \zeta$. Similarly, in the remaining part of the plane, the l.u.b. of radii of abnormality is attained at some point ζ_2. We select the corresponding disk C_2. Continuing this construction, we obtain a sequence of exceptional disks $C_1, C_2, \dots$ with centers at $\zeta_1, \zeta_2, \dots$, and radii $r_1 \geq r_2 \geq \cdots$.

Let us show that no point of the plane will be covered by more than five disks C_j. Indeed, let a point z' be covered by disks $C'_1, \dots, C'_k$ with radii $r'_1 \geq r'_2 \geq \cdots \geq r'_k$. Draw vectors from z' to the centers $\zeta'_1, \dots, \zeta'_k$ of these disks. Since the center of each disk lies outside other disks, the angle between each pair of these vectors is larger than $\pi/3$. Thus there are no more than five such vectors.

The disks C_j are exceptional, i.e., $n(r_j;\zeta_j) \geq pr_j$. Therefore,

$$p\sum_j r_j \leq \sum_j n(r_j;\zeta_j) \leq 5\mu(\mathbb{C}) = 5n\ .$$

Choosing $p = n/H$, we obtain

$$\sum_j r_j \leq 5H. \tag{1}$$

Evidently, if there is an infinite number of disks, then $r_j \to 0$ as $j \to \infty$, and since for each p-abnormal point z there is a radius of abnormality $\rho_z > 0$, every p-abnormal point will be covered by some disk C_j.

It remains to estimate the potential $u(z)$ at an arbitrary normal point z. It is evident that

$$u(z) \geq \iint_{|z-\zeta|\leq 1} \log|z-\zeta|\, d\mu(\zeta) = \int_0^1 \log t\, dn(t;z)$$
$$= n(t;z)\log t\Big|_0^1 - \int_0^1 \frac{n(t;z)}{t}\, dt\ ,$$

and since $n(t;z) < pt$, we have

$$u(z) \geq -\int_0^1 \frac{n(t;z)}{t}\, dt \, .$$

In addition, $n(t;z) \leq n = pH$. Therefore,

$$u(z) \geq -\int_0^H \frac{pt}{t}\, dt - \int_H^1 \frac{n}{t}\, dt = -n - n\log\frac{1}{H} = n\log\frac{H}{e} \, ,$$

and the theorem is proved.

In particular, if

$$P(z) = \prod_{k=1}^{n}(z - z_k) \, ,$$

then the inequality

$$|P(z)| \geq \left(\frac{H}{e}\right)^n \tag{2}$$

holds outside exceptional disks (C_j) with the sum of radii not exceeding $5H$. Due to the Maximum Principle, one can assume that each exceptional disk contains at least one zero z_k. Notice that the estimate (2) is not precise. In the paper Cartan [**24**] (see also Levin [**82,** Chapter 1]) a more precise estimate is proven: for a polynomial $P(z)$ the inequality (2) holds outside disks (C_j) with the sum of radii not exceeding $2H$. In the paper Grishin [**47**] it is proven that in the statement of Theorem 3 one can replace $5H$ by $2H$ as well.[7] The method of proving Theorem 3 presented here is essentially due to L. Ahlfors. This method is frequently used in potential theory (see the monographs Nevanlinna [**102**], Landkof [**78**]) for estimating integral operators with kernels depending on the difference of arguments:

$$\int_X g(|x - \zeta|)\, d\mu(\zeta) = \int_0^\infty g(t)\, d\mu(t;x) \, .$$

In the paper Gorin, Koldobskii [**45**] infinite-dimensional analogs of the Cartan estimate are found.

Ahlfors' method has applications in approximation of a subharmonic function by the logarithm of modulus of entire function. The first general result of such a type was proved by V. S. Azarin. In the paper [**124**] by Yulmukhametov the following theorem is proved.

Let $u(z)$ be an arbitrary subharmonic function of finite order. Then there exists an entire function $f(z)$ such that

$$\big|u(z) - \log|f(z)|\big| = O(\log|z|) \, , \quad |z| \to \infty \, ,$$

outside a set of disks (C_j) with finite sum of radii.

[7] It seems that the best constant for the sum of radii of exceptional disks is still unknown either for logarithmic potentials or for the logarithm of modulus of monic polynomials, see Hayman [**52,** Problem 4.7].

11.3. Lower bounds for the modulus of an analytic function in a disk

THEOREM 4. *Let $f(z)$ be a function analytic in the disk $\{z : |z| \leq 2eR\}$, $|f(0)| = 1$, and let η be an arbitrary small positive number. Then the estimate*

$$\log |f(z)| > -H(\eta) \log M_f(2eR) , \quad H(\eta) = \log \frac{15e^3}{\eta} ,$$

is valid everywhere in the disk $\{z : |z| \leq R\}$ except a set of disks (C_j) with sum of radii

$$\sum r_j \leq \eta R .$$

PROOF. First, we construct the function

$$\varphi(z) = \frac{(2R)^n}{a_1 \cdots a_n} \prod_{k=1}^{n} \frac{2R(z - a_k)}{(2R)^2 - \bar{a}_k z} ,$$

$a_1, \ldots , a_n$ being the zeros of the function $f(z)$ in the disk $\{z : |z| \leq 2R\}$ with account taken, as usual, of their multiplicities. We have $|\varphi(0)| = 1$ and

$$|\varphi(2Re^{i\theta})| = \frac{(2R)^n}{|a_1 \cdots a_n|} .$$

The function $\psi(z) = f(z)/\varphi(z)$ has no zeros in the disk $|z| \leq 2R$, and by Theorem 2 we conclude

$$\begin{aligned} \log |\psi(z)| &\geq -2 \log M_\psi(2R) = -2 \log M_f(2R) + 2 \log \frac{(2R)^n}{|a_1 \cdots a_n|} \\ &\geq -2 \log M_f(2R) > -2 \log M_f(2eR) \end{aligned}$$

for $|z| \leq R$. Thus

$$\log |f(z)| \geq -2 \log M_f(2eR) + \log |\varphi(z)| \tag{3}$$

for $|z| \leq R$. Let us now estimate the second term on the right-hand side of this inequality.

For $|z| \leq R$ we have

$$\prod_{k=1}^{n} |(2R)^2 - \bar{a}_k z| \leq (6R^2)^n . \tag{4}$$

Applying Theorem 3 with $H = \eta R/5$, we obtain the inequality

$$\log \prod_{k=1}^{n} |z - a_k| > n \log \frac{\eta R}{5e} \tag{5}$$

everywhere outside the disks (C_j) with the sum of radii not exceeding ηR. Taking into account (4) and (5), we obtain

$$\begin{aligned} \log |\varphi(z)| &= \log \frac{(2R)^{2n}}{|a_1 \cdots a_n|} + \log \prod_{k=1}^{n} |z - a_k| - \log \prod_{k=1}^{n} |(2R)^2 - \bar{a}_k z| \\ &\geq n \log \frac{\eta R}{5e} - n \log 6R^2 + n \log 2R = n \log \frac{\eta}{15e} \end{aligned}$$

for $|z| \le R$, but outside the exceptional disks (C_j). Using the corollary to the Jensen formula, Section 2.3,

$$n = n(R, f) \le \log M_f(2eR) ,$$

we have

$$\log|\varphi(z)| > -\log M_f(2eR) \log \frac{15e}{\eta} .$$

Inserting this inequality into (3), we obtain

$$\log|f(z)| > -\log M_f(2eR) \log \frac{15e^3}{\eta}$$

for $|z| \le R$, but outside the disks (C_j). The theorem is proved.

Using the Nevanlinna characteristic, we have proved in Section 2.4 the theorem on division of entire functions. Another way of deriving division theorems is based on lower estimates of analytic functions.

THEOREM 5. *Let $f_1(z)$ be an analytic function inside the angle $D = \{z : \alpha < \arg z < \beta\}$, and let $f_2(z)$ be an entire function. Assume that both functions have order ρ and mean type. If the quotient $\varphi(z) = f_1(z)/f_2(z)$ is analytic inside the same angle, and if*

$$|\varphi(Re^{i\alpha})| \overset{\text{as}}{<} e^{AR^\rho} , \quad |\varphi(Re^{i\beta})| \overset{\text{as}}{<} e^{AR^\rho} \tag{6}$$

on the boundary of the angle, then $\varphi(z)$ also has the order ρ and mean type in D.

PROOF. By Theorem 4, we have

$$\log|f_2(z)| \overset{\text{as}}{>} -H(\eta)(\sigma_{f_2} + \varepsilon)(2e)^\rho R^\rho$$

for $|z| \le R$ and outside the exceptional disks (C_j). This implies that the estimate

$$\log|\varphi(z)| \le [H(\eta)(\sigma_{f_2} + \varepsilon)(2e)^\rho + \sigma_{f_1} + \varepsilon]R^\rho = BR^\rho \tag{7}$$

holds for $z \in D$, $|z| \le R$, but outside (C_j).

The exceptional disks (C_j) satisfy the condition

$$\sum_{|z_j|<R} r_j < \eta R ,$$

where z_j are the centers of the disks, and r_j are their radii. Hence, there is a number R_1 lying between R and $(1 - 2\eta)^{-1}R$ such that the circumference $|z| = R_1$ does not intersect the disks (C_j). Using (6) and the Maximum Principle we conclude that the estimate (7) is fulfilled for all R (possibly, with another constant B). The theorem is proved.

LECTURE 12

Entire Functions with Zeros on a Ray

12.1. Asymptotic behavior of canonical products

Let $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$, let $n(r)$ be the counting function of the sequence $\{\lambda_n\}$, and let the limit

$$\Delta = \lim_{r\to\infty} \frac{n(r)}{r^\rho} \tag{1}$$

exist for a noninteger ρ. We consider the canonical product

$$\Pi(z) = \prod_{n=1}^{\infty} G\Big(\frac{z}{\lambda_n}, p\Big) ,$$

where $p = [\rho]$, and

$$G(u,p) = (1-u)e^{u+u^2/2+\cdots+u^p/p} ,$$

and assume that $-\pi < \arg(1-u) < \pi$. For such u the function $\log G(u,p)$ is single-valued in the complex plane cut along the ray $[1,\infty)$.

THEOREM 1. *If a sequence $\{\lambda_n\}$ satisfies* (1), *then, uniformly with respect to θ, $0 \le \theta \le 2\pi$, the asymptotic relation*

$$\Big|\log \Pi(re^{i\theta}) - \frac{\pi\Delta}{\sin \pi\rho} e^{i\rho(\theta-\pi)} r^\rho\Big| \sin\frac{\theta}{2} = o(r^\rho) , \qquad r\to\infty \tag{2}$$

holds in the complex plane cut along the positive ray of the real axis.

PROOF. We have

$$\begin{aligned}\log \Pi(re^{i\theta}) &= \sum_{n=1}^{\infty} \log G\Big(\frac{z}{\lambda_n}, p\Big) = \int_0^\infty \log G\Big(\frac{z}{t}, p\Big)\, dn(t) \\ &= -z^{p+1}\int_0^\infty \frac{n(t)}{t^{p+1}(t-z)}\, dt .\end{aligned}$$

Let us estimate the modulus of the expression

$$\begin{aligned} S &= \log \Pi(re^{i\theta}) + r^{p+1}e^{i(p+1)\theta}\int_0^\infty \frac{\Delta t^\rho dt}{t^{p+1}(t-re^{i\theta})} \\ &= -z^{p+1}\int_0^\infty \frac{n(t)-\Delta t^\rho}{t^{p+1}(t-z)}\, dt .\end{aligned} \tag{3}$$

Because of (1) we obtain

$$
\begin{aligned}
|S| &\le r^{p+1}\int_0^\infty \frac{|n(t)-\Delta t^\rho|}{t^{p+1}|t-re^{i\theta}|}\,dt \\
&< r^{p+1}\int_0^N \frac{|n(t)-\Delta t^\rho|}{t^{p+1}|t-re^{i\theta}|}\,dt+\varepsilon r^{p+1}\int_0^\infty \frac{t^{\rho-p-1}}{|t-re^{i\theta}|}\,dt \\
&= J_1(r,\theta)+J_2(r,\theta)
\end{aligned} \tag{4}
$$

as $N > N(\varepsilon)$. Then

$$
J_1(r,\theta)=r^{p+1}\int_0^N \frac{|n(t)-\Delta t^\rho|}{t^{p+1}|t-re^{i\theta}|}\,dt=O(r^p)=o(r^\rho)\ ,\quad r\to\infty\ , \tag{5}
$$

uniformly with respect to θ, $0 \le \theta \le 2\pi$.

To estimate the term J_2 we set $t = r\tau$. Then

$$
\begin{aligned}
J_2(r,\theta) &= \varepsilon r^\rho \int_0^\infty \frac{\tau^{\rho-p-1}}{|\tau-e^{i\theta}|}\,d\tau=\varepsilon r^\rho\Big(\int_0^2+\int_2^\infty\Big)\frac{\tau^{\rho-p-1}}{|\tau-e^{i\theta}|}\,d\tau \\
&< \varepsilon r^\rho\Big(\int_0^2 \frac{\tau^{\rho-p-1}}{\sin(\theta/2)}\,d\tau+\int_2^\infty \frac{\tau^{\rho-p-2}}{|1-e^{i\theta}/\tau|}\,d\tau\Big)<\frac{\varepsilon c_\rho r^\rho}{\sin(\theta/2)}\ .
\end{aligned} \tag{6}
$$

Moreover,

$$
\Delta r^{p+1}e^{i(p+1)\theta}\int_0^\infty \frac{t^\rho dt}{t^{p+1}(t-re^{i\theta})}=\Delta r^\rho e^{i(p+1)\theta}\int_0^\infty \frac{t^{\rho-p-1}dt}{t-e^{i\theta}}\ .
$$

The latter integral is easily calculated using residues. It is equal to

$$
-\frac{\pi}{\sin\pi\rho}e^{i(\rho-1-p)\theta-i\pi\rho}\ .
$$

Thus the relations (3)–(6) imply the assertion of the theorem.

PROBLEM 1. Prove asymptotic relation (2) under the assumptions that λ_n are complex numbers such that each angle around the positive ray contains all but finitely many λ_n, $n/\lambda_n^\rho \to \Delta$, $\Delta > 0$, ρ is noninteger, and $\delta \le \theta \le 2\pi-\delta$, $\delta > 0$.

REMARK. Taking the real part of both sides of equation (2), we obtain

$$
\log|\Pi(re^{i\theta})|=\frac{\pi\Delta r^\rho\cos\rho(\theta-\pi)}{\sin\pi\rho}+\frac{o(r^\rho)}{\sin(\theta/2)}\ . \tag{2$'$}
$$

Evidently, this yields

$$
h_\Pi(\theta)=\frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi)\ ,\qquad 0<\theta<2\pi\ ,
$$

and, by the continuity of the indicator, the latter equation holds for all θ, $0 \le \theta \le 2\pi$. By the property of the indicator stated in Theorem 2, Section 8.2, we obtain

$$
\log|\Pi(re^{i\theta})| \overset{\text{as}}{<} \Big[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi)+\varepsilon\Big]r^\rho\ .
$$

In what follows, the function $\cos\rho(\theta-\pi)$ is meant to be extended from the interval $\{\theta : 0 \le \theta \le 2\pi\}$ as a 2π-periodic function.

THEOREM 2. *If λ_n is a sequence of positive numbers such that $n/\lambda_n \to \Delta$, $n \to \infty$, and if*

$$\Pi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right),$$

then

$$h_\Pi(\theta) = \pi\Delta|\sin\theta|, \tag{7}$$

and for $\theta \neq 0, \pi$, the limit

$$h_\Pi(\theta) = \lim_{r\to\infty} \frac{\log|\Pi(re^{i\theta})|}{r}$$

exists.

PROOF. The sequence $\{\lambda_n^2\}$ has density Δ with respect to the order $\rho = 1/2$. According to the previous theorem,

$$\log|\Pi(\sqrt{z})| = \pi\Delta R^{1/2}\cos\frac{1}{2}(\varphi - \pi) + \frac{o(R^{1/2})}{\sin(\varphi/2)}, \qquad R \to \infty,$$

where $R^{1/2} = r = |z|$, $\varphi/2 = \theta = \arg z$; i.e.,

$$\log|\Pi(re^{i\theta})| = \pi\Delta|\sin\theta|r + \frac{o(r)}{|\sin\theta|}, \qquad r \to \infty, \qquad 0 < |\theta| < \pi,$$

which completes the proof.

12.2. Theorem on a segment on the boundary of the indicator diagram

THEOREM 3. *Let $f(z)$ be an entire function of exponential type. If f vanishes at a point set $\{\lambda_n\}$ having the density*

$$\Delta = \lim_{n\to\infty} \frac{n}{\lambda_n},$$

then the supporting line of the indicator diagram of $f(z)$, which is orthogonal to the direction $\arg z = 0$, *and the indicator diagram itself have a common segment of length at least $2\pi\Delta$.*

PROOF. Consider the function

$$\varphi(z) = \frac{f(z)}{\Pi(z)}$$

with

$$\Pi(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right).$$

By Theorem 5 from the previous lecture, $\varphi(z)$ is of exponential type in the upper half-plane.

Theorem 2 implies

$$h_f(\theta) = h_\varphi(\theta) + h_\Pi(\theta) = h_\varphi(\theta) + \pi\Delta|\sin\theta|;$$

i.e., for $|\theta| \leq \pi/2$ the supporting function of the indicator diagram I_f coincides with the supporting function of the sum of the indicator diagram I_φ and the segment $[-i\pi\Delta, i\pi\Delta]$. To complete the proof, we observe that the boundary of the convex compact set $I_\varphi + [-i\pi\Delta, i\pi\Delta]$ contains a segment of length at least $2\pi\Delta$ which is parallel to the imaginary axis.

DEFINITION. A domain G bounded by continuous curves $y = a(x)$ and $y = a(x) + d$, $x \in \mathbb{R}$, is called a curvilinear strip of width d.

THEOREM 4. *Assume that all singularities of the Borel transform of an entire function $F(z)$ of exponential type are in a curvilinear strip of width d. If $F(z)$ vanishes at points $\{\lambda_n\}$ and if*

$$\Delta = \lim_{n\to\infty} \frac{n}{\lambda_n} \geq \frac{d}{2\pi},$$

then $F(z) \equiv 0$.

PROOF. Consider the smallest convex compact set K containing all the singularities of the Borel transform of $F(z)$. We see that the vertical line supporting K contains no segment of length $2\pi\Delta$ lying on the boundary of K. Then by the previous theorem we obtain $F(z) \equiv 0$, proving Theorem 4.

Theorems 3 and 4 can be used for the study of the completeness of exponential systems $\{e^{\lambda_n z}\}$.

THEOREM 5. [8] *If $\lambda_n/n \to 1$, $n \to \infty$, then the system of functions $\{e^{\lambda_n z}\}$ is complete in $A(\Omega)$, where Ω is an arbitrary curvilinear strip of width 2π, and is not complete in any simply connected domain which contains a closed segment of length 2π parallel to the imaginary axis.*

PROOF. If a system $\{e^{\lambda_n z}\}$ is not complete, then there exists a function $f(z) \not\equiv 0$ in a neighborhood of the point at infinity satisfying $f(\infty) = 0$, with all its singularities lying in Ω and such that

$$\int_L e^{\lambda_n z} f(z)\, dz = 0,$$

where $L \subset \Omega$ is a simple closed curve surrounding all these singularities. The function

$$\Phi(\lambda) = \int_L e^{\lambda z} f(z)\, dz$$

is an entire function of exponential type, and all singularities of its Borel transform $f(z)$ are in the strip Ω. By Theorem 4, the equations $\Phi(\lambda_n) = 0$, $n = 1, 2, \dots$ yield $\Phi(\lambda) \equiv 0$, and $f(z) \equiv 0$ giving a contradiction.

Now, let us prove the second assertion of the theorem. Let G be an arbitrary simply connected domain containing a closed segment $\overline{I} = [\overline{a} - i\pi, \overline{a} + i\pi]$. Using (7), we see that the indicator diagram of the function

$$F(\lambda) = e^{\overline{a}\lambda} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_n^2}\right)$$

coincides with the segment $I = [a - i\pi, a + i\pi]$, and the conjugate diagram coincides with the segment $\overline{I} = [\overline{a} - i\pi, \overline{a} + i\pi]$. By the Pólya theorem

$$F(\lambda) = \frac{1}{2\pi i} \int_C e^{\lambda z} f(z)\, dz,$$

[8]This theorem was proved independently by A. F. Leont'ev and the author.

where $C \subset G$ is a simple closed curve surrounding the segment I. A nontrivial functional from $A^*(G)$ corresponding to the function $f(z)$ annihilates all the functions $e^{\lambda_n z}$, and hence this system is not complete.

Final results on completeness of the system of functions $\{e^{\lambda_n z}\}$ in curvilinear strips are established in the papers Malliavin and Rubel [**90**], and Khabibullin [**66**].

Theorem 3 may be used to solve completeness problems for systems of functions in spaces with another topology, for example, in the spaces $L^p(K)$ or $C(K)$ where K is a compact set.

THEOREM 6. *Let K be a rectifiable curve which is the graph of a continuous function defined on a closed segment, and let $\Lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of complex numbers satisfying the condition*

$$\frac{n}{\lambda_n} \to \Delta > 0\,, \qquad n \to \infty\,.$$

Then the system $E(\Lambda) = \{e^{i\lambda_n z}\}_{n=1}^\infty$ is complete in each of the spaces $L^p(K)$, $1 \le p < \infty$, and $C(K)$.

PROOF. For the sake of definiteness, consider the space $C(K)$. If the system $E(\Lambda)$ is not complete in this space, then there exists a nontrivial measure $d\mu(z)$ supported by K and orthogonal to all functions of the system $E(\Lambda)$. Using this measure we construct an entire function of exponential type

$$\Phi(\lambda) = \int_K e^{\lambda z}\, d\mu(z)$$

vanishing at the points of Λ. The Borel transform of this function has the form

$$\varphi(\zeta) = \int_K \frac{d\mu(z)}{\zeta - z}\,,$$

and hence the function $\varphi(\zeta)$ is holomorphic in $\overline{\mathbb{C}} \setminus K$. If this function does not vanish identically, the conjugate diagram of the function $\Phi(\lambda)$ coincides with the convex hull of K or with the convex hull of some part of it. It is evident that such a convex hull has no vertical segment on its boundary, which contradicts Theorem 3. Thus, $\Phi(\lambda) \equiv 0$. This yields

$$\Phi^{(k)}(0) = \int_K z^k\, d\mu(z) = 0\,, \qquad k = 0, 1, 2, \dots\,,$$

and since polynomials are dense in $C(K)$,[9] we obtain that the measure $d\mu$ is orthogonal to the whole space $C(K)$. The theorem is proved.

REMARK 1. Let K be an arbitrary compact set in the complex plane and let $A(K)$ be the closure of polynomials in the uniform norm on this compact set. If Λ and $E(\Lambda)$ are as in Theorem 6, and, for each real a, the diameter of the intersection of K and the vertical line $\operatorname{Re} z = a$ is less than $2\pi\Delta$ (or the intersection is empty), then the system $E(\Lambda)$ is complete in $A(K)$.

It is worth mentioning that by Mergelyan's theorem (Mergelyan [**98**]) the space $A(K)$ coincides with the space of functions continuous on K and holomorphic in the interior of K.

[9]This statement is a rather particular case of a well-known theorem of Mergelyan, see [**98**].

REMARK 2. If there are two points z_1, z_2 in a compact set K such that $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $|z_1 - z_2| = d > 0$, then the system of functions $E(\Lambda)$, $\Lambda = \{2\pi ki/d\}$, $k \in \mathbb{Z}$ is not complete in $C(K)$.

Indeed, each function of this system assumes equal values at the points z_1 and z_2, and hence the system $E(\Lambda)$ could not be complete in $A(K)$. Evidently, a subset of this system could not be complete as well.

PROBLEM 2. If $a > 0, b > 0$, and an entire function

$$F(\lambda) = \int_{-a}^{b} \cos(\lambda\sqrt{t})\psi(t)\,dt\,, \qquad \psi \in L(-a,b)$$

vanishes on a set $\{\lambda_n\}$, where

$$\frac{n}{\lambda_n} \to \Delta > 0\,, \qquad n \to \infty\,,$$

then $\psi(t) = 0$ a.e. on $(-a, 0)$.

12.3. Lower bound for the canonical product with positive zeros having density

For the canonical product

$$\Pi(z) = \prod_{n=1}^{\infty} G\Big(\frac{z}{\lambda_n}, p\Big)\,, \qquad p < \rho < p+1\,, \qquad \lambda_n > 0\,,$$

with

$$\lim_{r\to\infty} \frac{n(r)}{r^\rho} = \Delta$$

we have established the asymptotic formula

$$\log|\Pi(re^{i\theta})| = \frac{\pi\Delta r^\rho}{\sin\pi\rho}\cos\rho(\theta-\pi) + \frac{o(r^\rho)}{\sin(\theta/2)}\,, \qquad 0 < \theta < 2\pi\,.$$

The latter relation is of no meaning for $\theta = 0$. However, it can be defined more exactly to be valid for $\theta = 0$ as well. To this end, it is necessary to exclude from the complex plane some exceptional disks containing zeros of $\Pi(z)$, i.e., the points λ_n.

DEFINITION. A set of disks (C_j) in the complex plane will be called a C^0-set if

$$\lim_{R\to\infty} \frac{1}{R} \sum_{|z_j|<R} r_j = 0\,,$$

where z_j are the centers of (C_j), and r_j are their radii.

THEOREM 7. *Under the condition*

$$\lim_{t\to\infty} \frac{n(t)}{t^\rho}\,dt = \Delta \tag{8}$$

there exists a C^0-set of disks (C_j) outside which the asymptotic relation

$$\log|\Pi(re^{i\theta})| = \frac{\pi\Delta}{\sin\pi\rho} r^\rho \cos\rho(\theta-\pi) + o(r^\rho)\,, \qquad r\to\infty\,, \tag{9}$$

holds uniformly with respect to θ, $0 \le \theta \le 2\pi$.

PROOF. In the following proof we denote by K various numbers depending on ρ and Δ only. Firstly, we shall prove that, for each small enough value $\varepsilon > 0$, there exists a set of disks $(C_j(\varepsilon))$ with centers z_j and radii r_j such that

$$\limsup_{R\to\infty} \frac{1}{R} \sum_{|z_j|<R} r_j \le \eta(\varepsilon) , \tag{10}$$

where $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, and that outside $(C_j(\varepsilon))$ the inequality

$$\left| \log|\Pi(re^{i\theta})| - \frac{\pi\Delta}{\sin\pi\rho} \cos\rho(\theta-\pi) r^\rho \right| \le K\sqrt{\varepsilon}\, r^\rho \tag{11}$$

holds. We have already proved that the inequality

$$\log|\Pi(re^{i\theta})| < \Big[\frac{\pi\Delta}{\sin\pi\rho} \cos\rho(\theta-\pi) + \varepsilon\Big] r^\rho \tag{12}$$

holds as $r > r_\varepsilon$, and the inequality

$$\log|\Pi(re^{i\theta})| > \Big[\frac{\pi\Delta}{\sin\pi\rho} \cos\rho(\theta-\pi) - \varepsilon\Big] r^\rho \tag{13}$$

holds as $r > r_\varepsilon$ and $\varepsilon/2 \le \theta \le 2\pi - \varepsilon/2$.

For each natural n we set $R_n = (1+\varepsilon)^n$, $z_0 = R_n e^{i\varepsilon/2}$, and consider the function

$$\Phi(w) = \frac{\Pi(z_0+w)}{\Pi(z_0)} = \frac{\Pi(z)}{\Pi(z_0)} , \qquad z = w + z_0 ,$$

in the disk $|w| \le 2\varepsilon R_n$. We note that the disk $\{|z-z_0| < 2\varepsilon R_n\}$ contains the sector

$$S_n = \Big\{ re^{i\theta} : R_{n-1} \le r \le R_n,\ |\theta| \le \frac{\varepsilon}{2} \Big\} .$$

Moreover,

$$\max\{|\arg z| : |z - z_0| \le 2\varepsilon R_n\} = \frac{\varepsilon}{2} + \arcsin 2\varepsilon < 3\varepsilon ,$$
$$\max\{|z| : |z - z_0| \le 2\varepsilon R_n\} = R_n(1+2\varepsilon) .$$

By the lower bound of the modulus of a holomorphic function (Theorem 4 of the previous lecture), given $\eta > 0$, the inequality

$$\log|\Phi(w)| > -H(\eta)\log M_\Phi(4\varepsilon e R_n) , \qquad H(\eta) = \log\frac{15e^3}{\eta} ,$$

holds everywhere outside disks $(C_j^{(n)})$ such that

$$\sum_j r_j^{(n)} < 2\varepsilon\eta R_n = 2\varepsilon\eta(1+\varepsilon)^n .$$

Inequalities (12) and (13) imply

$$\log M_\Phi(4\varepsilon e R_n) \le \max_{|\theta|\le 3\varepsilon} \Big[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi) + \varepsilon\Big](1+2\varepsilon)^\rho R_n^\rho$$
$$- \Big[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho\big(\frac{\varepsilon}{2} - \pi\big) - \varepsilon\Big] R_n^\rho \le K\varepsilon R_n^\rho .$$

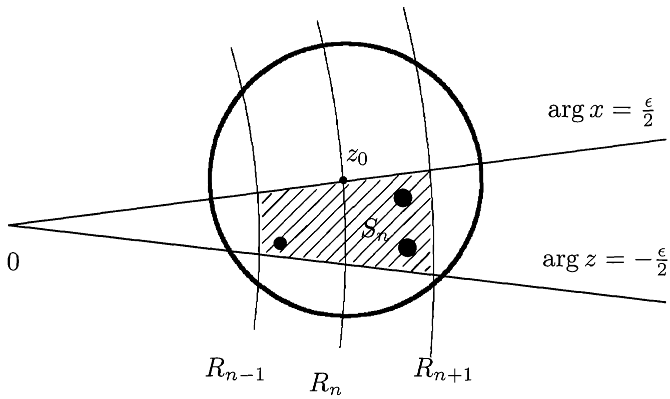

FIGURE 3

Thus, for $|w| \le 2\varepsilon R_n$, but outside the disks $(C_j^{(n)})$,

$$\log|\Phi(w)| > -H(\eta)K\varepsilon R_n^\rho .$$

Setting $w = z - z_0$, we obtain

$$\begin{aligned}\log|\Pi(z)| &= \log|\Pi(z_0)| + \log|\Phi(w)| \\ &> \Big[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho\Big(\frac{\varepsilon}{2}-\pi\Big) - \varepsilon - H(\eta)K\varepsilon\Big]R_n^\rho \\ &> \Big[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi) - \big(K_1 + KH(\eta)\big)\varepsilon\Big]r^\rho\end{aligned}$$

for $z = re^{i\theta}$, but outside the exceptional disks $(C_j^{(n)})$.

If $\eta = \eta(\varepsilon)$ is chosen such that

$$H(\eta) = \log\frac{15e^3}{\eta} = \frac{1}{\sqrt{\varepsilon}} ,$$

then

$$\log|\Pi(re^{i\theta})| > \Big[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi) - K\sqrt{\varepsilon}\Big]r^\rho \tag{14}$$

with $re^{i\theta} \in S_n$, but outside the set of exceptional disks $(C_j^{(n)})$. Let us now consider the system of disks

$$(C_j(\varepsilon)) = \bigcup_n (C_j^{(n)}) .$$

Let z_j be the centers of these disks and let r_j be their radii. Then for $R_{N-1} \le R < R_N$

$$\sum_{|z_j|<R} r_j \le \sum_{n=0}^{N}\sum_j r_j^{(n)} \le 2\varepsilon\eta\sum_{n=0}^{N}(1+\varepsilon)^n \le 2\eta(1+\varepsilon)^{N+1} \le 2\eta R .$$

Therefore, the system of disks $(C_j(\varepsilon))$ satisfies the condition (10). By estimates (12), (13), and (14), the inequality (11) holds.

In order to complete the proof of Theorem 7 we shall construct an exceptional C^0-set of disks. For this purpose let us choose a sequence $\varepsilon_p \searrow 0$; then $\eta_p = \eta(\varepsilon_p) \searrow 0$. For each p we have found, as described above, an exceptional set of disks $(C_j(\varepsilon_p))$ with centers $z_{j,p}$ and radii $r_{j,p}$. Starting with $R^{(0)} = 1$, we choose a value $R^{(p)} > pR^{(p-1)}$ so that

$$\sum_{|z_j|<R} r_{j,p} < 2\eta_p R$$

as $R \geq R^{(p)}$. We construct the system (C_j) including in it all disks from $(C_j(\varepsilon_p))$ whose centers are in the annulus $R^{(p)} \leq |t| \leq R^{(p+1)}$, $p = 0, 1, 2, \ldots$. Then, for $R^{(N)} \leq R < R^{(N+1)}$, we obtain

$$\begin{aligned}\sum_{|z_j|<R} r_j &= \sum_{p=1}^{N} \sum_{R^{(p-1)} \leq |z_{j,p}| \leq R^{(p)}} r_{j,p} + \sum_{R^{(N)} < |z_{j,N}| < R} r_{j,N} \\ &\leq \sum_{p=1}^{N} 2\eta_{p-1} R^{(p)} + 2\eta_n R \\ &\leq 2\eta_1 \Big(\frac{1}{2 \cdot 3 \cdot 4 \cdots N} + \frac{1}{3 \cdot 4 \cdots N} + \cdots + \frac{1}{N} \Big) R^{(N)} + 2\eta_{N-1} R^{(N)} + 2\eta_N R \\ &= o(1) R , \qquad N \to \infty .\end{aligned}$$

Evidently, the asymptotic relation (9) holds outside the disks (C_j). The theorem is proved.

REMARK 1. If

$$\Pi(z) = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{\lambda_n^2}\Big) , \qquad \frac{n}{\lambda_n} \to \Delta ,$$

then, making a change of variables $z^2 = w$, we obtain an entire function of non-integer order $\rho = 1/2$. Applying the latter theorem to this function, we find that outside a C^0-set

$$\log |\Pi(re^{i\theta})| = \pi \Delta |\sin \theta| r + o(r) , \qquad r \to \infty .$$

REMARK 2. If zeros $\{\lambda_n\}$ lie on the ray $\{z : \arg z = \psi\}$, then equation (9) is replaced by

$$\log |\Pi(re^{i\theta})| = \frac{\pi \Delta r^\rho}{\sin \pi\rho} \cos \rho(\theta - \psi - \pi) + o(r^\rho) , \qquad \psi \leq \theta \leq \psi + 2\pi ,$$

valid outside an exceptional C^0-set.

REMARK 3. As an analysis of the proof of Theorem 7 shows, one can replace the assumption (8) by its corollary

$$\log |f(re^{i\theta})| = \frac{\pi \Delta r^\rho}{\sin \pi\rho} \cos \rho(\theta - \pi) + o(r^\rho) , \qquad \delta \leq \theta \leq 2\pi - \delta .$$

PROBLEM 3. Let a sequence of positive integers $\{\lambda_n\}$ has density Δ with respect to the order $\rho = 1$ and let

$$\Pi(z) = \prod_{n=1}^{\infty} G\Big(\frac{z}{\lambda_n}, \rho\Big) .$$

Prove that, outside a C^0-set of disks, the asymptotic relation

$$\log |\Pi(re^{i\theta})| = \Delta \cos\theta r \log r + o(r \log r) , \qquad r \to \infty ,$$

is valid.

PROBLEM 4. Prove that if $f(z)$ is an entire function of minimal type with respect to an order ρ, then

$$\log |f(z)| = o(|z|^\rho) , \qquad |z| \to \infty ,$$

everywhere outside a C^0-set.

LECTURE 13

Entire Functions with Zeros on a Ray (Continuation)

13.1. The Valiron theorem

In this section we shall prove theorems which may be regarded as converse to the theorems proved in Lecture 12.

THEOREM 1. *Let $f(z)$ be an entire function of noninteger order ρ with positive zeros, and let, for each $\delta > 0$, the asymptotic relation*

$$\log|f(re^{i\theta})| = \frac{\pi\Delta r^{\rho}}{\sin\pi\rho}\cos\rho(\theta-\pi) + o(r^{\rho}) \tag{1}$$

hold uniformly with respect to θ, $\delta \le \theta \le 2\pi - \delta$. Then the limit

$$\lim_{t\to\infty}\frac{n(t)}{t^{\rho}} = \Delta \tag{2}$$

exists, where $n(t)$ is the zero-counting function of $f(z)$.

PROOF. The asymptotic relation (1) implies that

$$h_f(\theta) = \frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi)$$

for $0 < \theta < 2\pi$. Since the indicator is a continuous function, the latter equation holds for $\theta = 0$ as well. It yields

$$\log|f(re^{i\theta})| \overset{\text{as}}{<} \left[\frac{\pi\Delta}{\sin\pi\rho}\cos\rho(\theta-\pi)+\varepsilon\right]r^{\rho}, \qquad \varepsilon > 0\,. \tag{3}$$

According to Remark 3 to Theorem 7 of the previous lecture, relations (1) and (3) imply that the asymptotic relation (1) holds everywhere outside an exceptional C^0-set of disks (C_j) containing zeros of $f(z)$.

Let us now choose a number $R > 0$ such that the circle $\{z : |z| = R\}$ does not intersect the exceptional disks (C_j). Assume that $f(0) = 1$. Then, by the Jensen formula,

$$\begin{aligned} N(R) &= \int_0^R \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi}\log|f(Re^{i\theta})|\,d\theta \\ &= \frac{\pi\Delta R^{\rho}}{2\pi\sin\pi\rho}\int_0^{2\pi}\cos\rho(\theta-\pi)\,d\theta + o(R^{\rho}) = \frac{\Delta}{\rho}R^{\rho} + o(R^{\rho})\,. \end{aligned} \tag{4}$$

If R is a number large enough that the circle $\{z : |z| = R\}$ intersects the exceptional set, then for each $\delta > 0$ we can find $h = h(R)$, $0 < h < \delta$, such that the circles $|z| = R(1-h)$, $|z| = R(1+h)$, $R > R_0(\delta)$, do not intersect this set. We obtain

$$\frac{\Delta}{\rho}(1-h)^\rho R^\rho + o(R^\rho) \le \int_0^R \frac{n(t)}{t}\,dt \le \frac{\Delta}{\rho}(1+h)^\rho R^\rho + o(R^\rho),$$

which shows that the asymptotic relation (4) holds as $R \to \infty$.

Let us show that equation (4) yields relation (2); this is a standard Tauberian argument from real analysis. Choosing $k > 1$, we easily derive from (4)

$$n(R)\log k \le \int_R^{kR} \frac{n(t)}{t}\,dt = \frac{\Delta}{\rho}(k^\rho - 1)R^\rho + o(R^\rho)\ ,$$

and

$$n(kR)\log k \ge \int_R^{kR} \frac{n(t)}{t}\,dt = \frac{\Delta}{\rho}(k^\rho - 1)R^\rho + o(R^\rho)\ .$$

In other words,

$$\frac{n(R)}{R^\rho} \le \frac{\Delta}{\rho}\frac{k^\rho - 1}{\log k} + o(1)\ ,$$
$$\frac{n(kR)}{(kR)^\rho} \ge \frac{\Delta}{\rho}\frac{k^\rho - 1}{k^\rho \log k} + o(1)\ .$$

Using these two inequalities, we obtain

$$\frac{\Delta}{\rho}\frac{k^\rho - 1}{k^\rho \log k} \le \liminf_{R\to\infty} \frac{n(R)}{R^\rho} \le \limsup_{R\to\infty} \frac{n(R)}{R^\rho} \le \frac{\Delta}{\rho}\frac{k^\rho - 1}{\log k}$$

for each $k > 1$. Passing to the limit as $k \searrow 1$, we get

$$\liminf_{R\to\infty} \frac{n(R)}{R^\rho} = \limsup_{R\to\infty} \frac{n(R)}{R^\rho} = \Delta\ .$$

The theorem is proved.

In fact, Valiron showed that (2) follows from a far weaker form of (1). To do this we remind the reader that for a function $\varphi(z)$ which is analytic and bounded above in the upper half-plane we have defined $|\varphi(x)|$, $x \in \mathbb{R}$, as

$$|\varphi(x)| = \limsup_{z\to x,\, \operatorname{Im} z > 0} |\varphi(z)|\ .$$

THEOREM 2 (on two constants). *Let $\varphi(z)$ be a bounded analytic function in the half-plane $\{x + iy : y > 0\}$, let $|\varphi(x)| \le M_1$, for $x < 0$, and let $|\varphi(x)| \le M_2$ for $x > 0$. Then*

$$|\varphi(re^{i\theta})| \le M_1^{\theta/\pi} M_2^{1-\theta/\pi} \tag{5}$$

for $0 \le \theta \le \pi$.

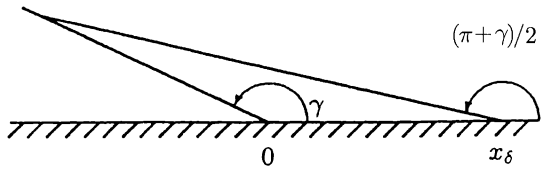

FIGURE 4

PROOF. Let

$$u(z) = \frac{\theta}{\pi} \log M_1 + \left(1 - \frac{\theta}{\pi}\right) \log M_2 \ , \qquad z = re^{i\theta} \ .$$

Evidently, $u(z)$ is a harmonic function in the half-plane $y > 0$. The function

$$\omega(z) = \log|\varphi(z)| - u(z)$$

is subharmonic. It is bounded above in the half-plane $\{x + iy : y > 0\}$ and its boundary values are nonpositive on the real axis. The Phragmén-Lindelöf theorem implies that $\omega(z) \le 0$ for $y > 0$, which is equivalent to (5).

THEOREM 3 (Lindelöf). *Let $\varphi(z)$ be a bounded analytic function in the upper half-plane, and let $|\varphi(x)| \to 0$ as $x \to +\infty$. Then $\varphi(z) \to 0$, $z \to \infty$, uniformly inside the angle $\{z : 0 \le \arg z \le \gamma\}$ for each $\gamma < \pi$.*

PROOF. Indeed, if $|\varphi(z)| \le M$ and if $|\varphi(x)| < \delta$ as $x > x_\delta$, then the inequality

$$|\varphi(z)| \le M^{(1+\gamma\pi^{-1})/2} \delta^{(1-\gamma\pi^{-1})/2}$$

holds inside the angle $\{z : 0 \le \arg(z - x_\delta) \le (\pi + \gamma)/2\}$. The conclusion of the theorem now follows by inspection of Figure 4.

COROLLARY. *Let $f(z)$ be analytic in the upper half-plane and continuous up to its boundary. Assume that $f(x)$ has distinct limits as $x \to \pm\infty$. Then the function $f(z)$ is unbounded.*

THEOREM 4 (Valiron). *Let the zeros of an entire function $f(z)$ of noninteger order ρ and mean type lie on the positive axis, and let*

$$\log|f(-r)| = \frac{\pi\Delta}{\sin \pi\rho} r^\rho + o(r^\rho) \ , \qquad r \to \infty \ .$$

Then

$$n(t) = \Delta t^\rho + o(t^\rho) \ , \qquad t \to \infty \ .$$

PROOF. According to the Hadamard theorem, Section 4.2, we have

$$\log|f(z)| = \log|\Pi(z)| + \operatorname{Re} P_q(z) \ ,$$

where $\Pi(z)$ is a canonical product, and P_q is a polynomial of degree $q < \rho$. Thus

$$\log|\Pi(-r)| = \frac{\pi\Delta}{\sin \pi\rho} r^\rho + o(r^\rho) \ , \qquad r \to \infty \ . \tag{6}$$

The function $\log \Pi(z)$ may be represented (see Section 12.1) in the form

$$\log \Pi(z) = -z^{p+1} \int_0^\infty \frac{n(t)dt}{t^{p+1}(t-z)} , \qquad 0 < \arg z < 2\pi , \qquad p = [\rho] .$$

Since $f(z)$ is of order ρ and mean type, $n(t) \overset{\text{as}}{<} ct^\rho$, and hence

(7) $$|\log \Pi(z)| < Cr^{p+1} \int_0^\infty \frac{t^{\rho-p-1}}{|t-z|} dt < C_\delta r^\rho$$

for $0 < \delta \le \theta \le 2\pi - \delta$. We choose the branch of the function $(-z)^\rho$ in the complex plane cut along the positive axis such that $(-z)^\rho > 0$ on the negative axis, and set

$$\varphi(z) = \frac{\log \Pi(z)}{(-z)^\rho} - \frac{\pi\Delta}{\sin \pi\rho} .$$

By (7), the function $\varphi(z)$ is bounded inside the angle $\{z : 0 < \delta \le \arg z \le 2\pi - \delta\}$. Further, the function $\log \Pi(z)$ is real on the negative axis. Together with (6) this implies that $\varphi(-r) \to 0$ as $r \to \infty$. By Theorem 3 we find that $\varphi(z) \to 0$ uniformly inside the angle $\delta \le \arg z \le 2\pi - \delta$ as $z \to \infty$. Hence

$$\log \Pi(re^{i\theta}) = \frac{\pi \Delta r^\rho}{\sin \pi\rho} e^{i\rho(\theta-\pi)} + o(r^\rho) , \qquad \delta \le \theta \le 2\pi - \delta .$$

According to Theorem 1, this asymptotic relation yields $n(t) = \Delta t^\rho + o(t^\rho)$, which proves Theorem 4.

13.2. Functions of completely regular growth

If zeros $\{\lambda_n\}$ are located on a finite number of rays $\arg z = \psi_k$ with densities Δ_k with respect to t^ρ, ρ being noninteger, then it follows from Theorem 7, Lecture 12, that the canonical product $\Pi(z)$ with zeros at $\{\lambda_n\}$ has the asymptotic behavior

$$\log |\Pi(z)| = \frac{\pi r^\rho}{\sin \pi\rho} \sum_k \Delta_k \cos \rho(\theta - \psi_k - \pi) + o(r^\rho) , \qquad \theta - 2\pi < \psi_k \le \theta ,$$

outside an exceptional C^0-set. The latter sum can be written as a Stieltjes integral:

$$\log |\Pi(z)| = \frac{\pi r^\rho}{\sin \pi\rho} \int_{[0,2\pi]} \cos \rho(\theta - \psi - \pi)\, d\Delta(\psi) + o(r^\rho) ,$$

where Δ is the measure supported by the points ψ_k, $\Delta(\{\psi_k\}) = \Delta_k$, and, as before, $\cos \rho(\theta - \pi)$ is the 2π-periodic continuation of the function $\cos \rho(\theta - \pi)$ from $[0, 2\pi]$ to the whole axis.

This result can be substantially generalized. To avoid technical difficulties, we restrict ourselves to a narrative exposition omitting the proofs.

For a noninteger ρ, every ρ-trigonometrically convex function $h(\theta)$ may be represented in the integral form

(8) $$h(\theta) = \frac{\pi}{\sin \pi\rho} \int_{[0,2\pi]} \cos \rho(\theta - \psi - \pi)\, d\Delta(\psi) ,$$

where Δ is a nonnegative Borel measure on $[0, 2\pi)$. Denote by $n_f(r; \psi_1, \psi_2)$ the number of zeros of an entire function $f(z)$ of order ρ in the sector $\{z : |z| \le$

$r,\ \psi_1 \le \arg z < \psi_2\}$. Assume that for each $\psi_1, \psi_2 \in [0, 2\pi)\setminus Q$, where Q is at most countable, there exists the limit

$$\Delta_f(\psi_1, \psi_2) = \lim_{r\to\infty} \frac{n_f(r, \psi_1, \psi_2)}{r^\rho} \tag{9}$$

which is called *the angular density of zeros* of the function $f(z)$. Then the asymptotic relation

$$\log|f(re^{i\theta})| = h_f(\theta) r^\rho + o(r^\rho)\ , \qquad r \to \infty\ , \tag{10}$$

holds everywhere outside an exceptional C^0-set. Here, the indicator $h_f(\theta)$ has the form (8) with the measure Δ coinciding with the angular density of zeros of $f(z)$. In other words, on each semi-interval $[\psi_1, \psi_2)$, $\psi_1, \psi_2 \notin Q$, the measure Δ is defined as $\Delta([\psi_1, \psi_2)) = \Delta_f(\psi_1, \psi_2)$.

For an integer $\rho > 0$, every 2π-periodic ρ-trigonometrically convex function $h(\theta)$ can be represented in the form

$$h_f(\theta) = \int_{[0,2\pi]} (\theta - \psi) \sin \rho(\theta - \psi)\, d\Delta(\psi) + \tau \cos \rho(\theta - \theta_0)\ . \tag{11}$$

In this case, in addition to the existence of the limit (9), the following condition must be imposed: there exists the limit

$$\delta_f = \lim_{R\to\infty} \delta_{f,R} = \lim_{R\to\infty} \Big(a_\rho + \frac{1}{\rho} \sum_{|\lambda_n| \le R} \lambda_n^{-\rho} \Big) \tag{12}$$

(see Lecture 5). The relations (9) and (12) imply that (10) holds outside a C^0-set with the indicator $h_f(\theta)$ defined by (11) with $\delta_f = \tau e^{i\theta_0}$.

The entire functions satisfying (10) are called *the functions of completely regular growth.* The following theorem is valid:

The zeros of an entire function of completely regular growth have an angular density (9) *with*

$$\begin{aligned} &\Delta_f(\psi_1 \pm 0, \psi_2 \pm 0) \\ &= \frac{1}{2\pi\rho} \Big\{ h'(\psi_2 \pm 0) - h'(\psi_1 \pm 0) + \rho^2 \int_{\psi_1}^{\psi_2} h(\psi)\, d\psi \Big\}\ , \\ &\qquad \psi_1 \le \psi_2\ . \end{aligned} \tag{13}$$

For an integer ρ, the relation (12) holds as well.

For $\rho = 1$ (i.e., for entire functions of exponential type) equation (13) has a simple geometrical interpretation. Namely, let us consider the indicator diagram of the function $f(z)$ and draw the supporting lines orthogonal to the directions $\arg z = \psi_1, \psi_2$ (see Figure 5). Let $S = S(\psi_1, \psi_2)$ be the length of the arc of the boundary of the indicator diagram between the supporting points z_1 and z_2 on these lines (for the sake of simplicity, we assume that the indicator diagram has no segment on its boundary). Then the angular density of zeros of the function $f(z)$ of completely regular growth is

$$\Delta_f(\psi_1, \psi_2) = \frac{1}{2\pi} S\ .$$

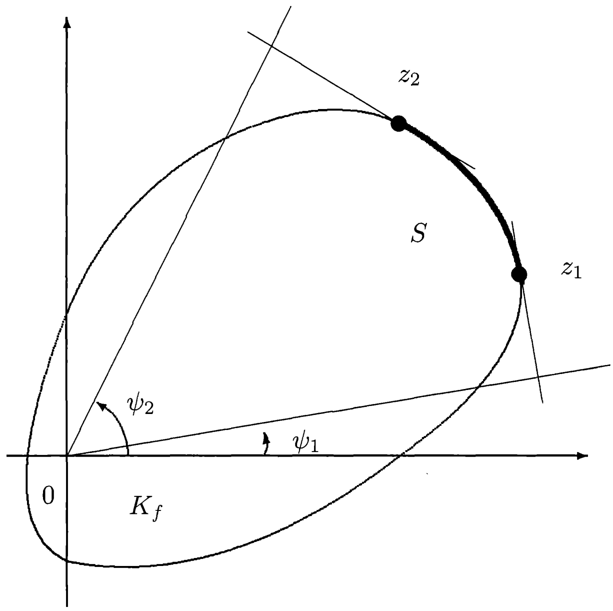

FIGURE 5

The proofs of these results can be found in Chapters II and III of the monograph Levin [**82**], which also contains numerous applications. Further results on functions of completely regular growth can be found in Appendix III to the same monograph. In the recent monograph Ronkin [**117**] the theory of functions of completely regular growth is constructed based on the concept of weak convergence.

V. S. Azarin [**7**] proposed a new approach to the investigation of the asymptotic behavior of entire functions of finite order. This approach is based on the theory of subharmonic functions and allows one to obtain many new results, as well as to simplify proofs of many well-known theorems.