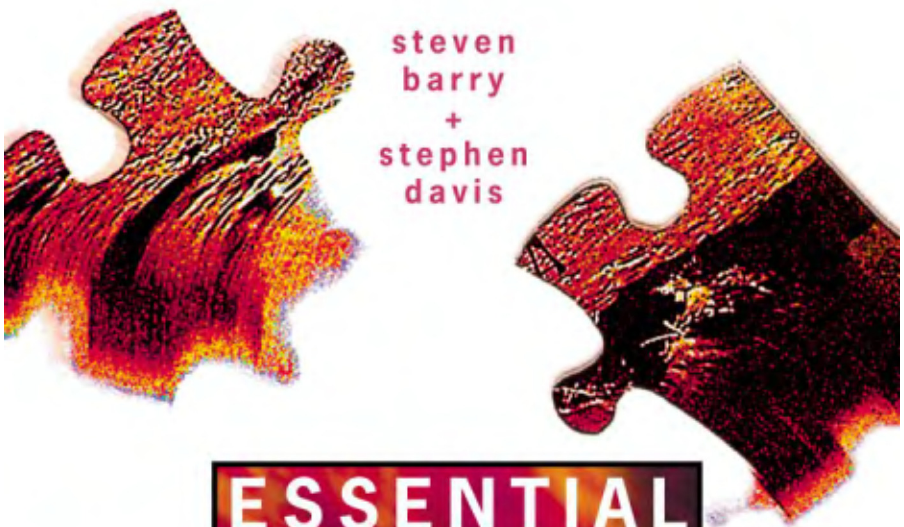

steven barry + stephen davis

ESSENTIAL MATHEMATICAL SKILLS

for engineering, science and applied mathematics

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+
Dr Stephen Alan Davis

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PREFACE

TO THE STUDENT

There are certain mathematical skills that are essential for any of your courses that use mathematics. Your lecturer will assume that you know them perfectly — not just a vague idea, but that you have completely mastered these skills. Without these necessary skills, you will find present and later subjects extremely difficult. You may also lose too many marks making 'silly' mistakes in exams.

So what skills do you need to have?

This book contains the mathematical skills we think are essential for you to not only know *but remember*. It is not a textbook and does not attempt to teach you, hence there are no long wordy explanations. This book should act as a reminder to you of material you have already learned. If you are having trouble with a section or chapter then we suggest you consult a more thorough textbook. We have left a number of blank pages at the back of the book for you to add in skills that you or your lecturers think are important to remember but we did not include.

This book covers the essential mathematics in the first one to two years of a science, engineering or applied mathematics degree. If you are in a first year undergraduate course you may not have covered some of the material included in this book.

As a guide, we expect our students at University College to have mastered (by the *start* of each semester) the following:

- **First Year — Semester One**: Chapters 1–3.
- **First Year — Semester Two**: Chapters 1–7.
- **Second Year**: Chapters 1–10.
- **Third Year**: Everything in the book!

There are practice tests in Chapter 13 based on these divisions.

Can you do the practice test at the end of these notes?

If you can't then perhaps there are some skills you need to do some revision on. If you can then you may need this book to help you revise those skills later on.

If you want more questions to practice on then see our extensive website:

http://www.ma.adfa.edu.au/∼sib/EMS.html

It contains extra questions, fully worked solutions, practice tests and also code for the Maple algebraic manipulation package giving solutions for every example and question.

TO THE LECTURER

What do you assume your students know? What material do you expect them to have a vague idea about (say the proof of Taylor's Theorem) and what material do you want students to know thoroughly (say the derivative of $\sin x$)? This book is an attempt to define what material students should have completely mastered at each year in an applied mathematics, engineering or science degree. Naturally we would like our students to know more than the bare essentials detailed in this book. However, most students do not get full marks in their previous courses and a few weeks after the exam will only remember a small fraction of a course. They are also doing many other courses not involving mathematics and are not constantly using their mathematical skills. This book can then act as guide to what material should realistically be remembered from previous courses. Naturally both the material and the year in which the students see this material will vary from university to university. This book represents what we feel is appropriate to our students during their degrees.

We invite you to look at our extensive web site:

http://www.ma.adfa.edu.au/∼sib/EMS.html

It contains more questions, solutions, practice tests and Maple code. There is a database of questions in LaTeX and pdf, which you can use to format your own tests and assignments. We are not concerned that students may access this database; if they can do the questions in the database then they have, in effect, learned the necessary skills.

If you have any questions or queries please do not hesitate to email us.

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CHAPTER 1

ALGEBRA AND GEOMETRY

1.1 ELEMENTARY NOTATION

1. $\{\}$: A set of objects.
2. $\in$: A member of a set. For example $3 \in \{1, 2, 3\}$.
3. R: The set of real numbers. For example $-1, 3, 3.2, \sqrt{2} \in R$.
4. Z: The set of integers. For example $-2, 0, 3 \in Z$.
5. $<, >$: Less than, greater than. For example $5 < 6, 7 > 5$.
6. $\leq, \geq$: Less than or equal to, greater than or equal to.
7. $\Longrightarrow$: Becomes. For example $x - 2 = 3 \quad \Longrightarrow \quad x = 5$.
8. $[a, b]$: Bounds of a variable. For example $x \in [1, 3]$ means $1 \leq x \leq 3$.
9. (a, b): Bounds of a variable. For example $x \in (1, 3)$ means $1 < x < 3$.
10. $\rightarrow$: Tends to. For example $1/x \rightarrow 0$ as $x \rightarrow \infty$.
11. $\approx$: Approximately equal to. For example $3.02 \approx 3$.

EXAMPLES

1. $W = \{f(x) = a + bx : a, b \in R\}$ means W is the set of all functions $f(x) = a + bx$ where a, b are real numbers (constants). Hence $1 + 2x \in W$ and $3 - 1.2x \in W$.
2. $S = \{x : x \geq 5, x \in R\}$ means that S is the set of all numbers bigger than or equal to 5. This is also written as $x \in [5, \infty)$.

1.2 FRACTIONS

A fraction is of the form $\frac{a}{b}$ where a is called the *numerator* and b is called the *denominator*.

Rules for operating on fractions

1. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c} \qquad (c \neq 0)$
2. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad (b, d \neq 0)$
3. $\frac{a}{c} \times \frac{b}{d} = \frac{ab}{cd} \qquad (c, d \neq 0)$
4. $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc} \qquad (b, c, d \neq 0)$

EXAMPLES

1. $\frac{2}{9} \times \frac{3}{8} = \frac{6}{72} = \frac{1}{12}$
2. $\frac{1}{3} + \frac{1}{6} = \frac{2}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$
3. $\frac{x+2}{x-2} - \frac{x-2}{x+2} = \frac{(x+2)^2 - (x-2)^2}{(x-2)(x+2)} = \frac{(x^2+4x+4) - (x^2-4x+4)}{(x^2-4)} = \frac{8x}{x^2-4}$
4. To rearrange the equation $\frac{1}{x} + \frac{1}{y} = \frac{1}{10}$ to find y write

$$\begin{aligned} \frac{1}{y} &= \frac{1}{10} - \frac{1}{x} \\ \Longrightarrow \quad \frac{1}{y} &= \frac{x-10}{10x} \qquad \textbf{NOT} \qquad y = 10 - x \\ \Longrightarrow \quad y &= \frac{10x}{x-10}. \end{aligned}$$

1.3 MODULUS

The absolute value or modulus of x, written $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

The absolute value is the magnitude of a number and ignores whether it is positive or negative.

EXAMPLES

1. $|+5| = 5$
2. $|-3| = 3$
3. $|-x|\,|y| = |x|\,|y| = |xy|$

1.4 INEQUALITIES

1. If $x > y$ then $x + a > y + a$ for any a.
2. If $x > y$ then $ax > ay$ if a is positive, but $ax < ay$ if a is negative.
3. If $x > y$ and $u > v$, then $x + u > y + v$.

EXAMPLES

1. To find x such that $-5x - 2 \leq 3$ write

$$\begin{aligned} & -5x - 2 \leq 3 \\ \Longrightarrow \quad & -5x \leq 5 \\ \Longrightarrow \quad & x \geq -1. \end{aligned}$$

2. To find values of x such that $x + 1 > 2x - 5$ we write

$$\begin{aligned} & x + 1 > 2x - 5 \\ \Longrightarrow \quad & x < 6. \end{aligned}$$

Inequalities with modulus

1. The inequality $|x-b| < a$ can be written as $-a < x-b < a$.
2. The inequality $|x-b| > a$ can be written as $x-b > a$ **or** $(x-b) < -a$.

EXAMPLES

1. To find x such that $|2x-1| \leq 3$ write

$$\begin{aligned} |2x-1| \leq 3 &\implies -3 \leq 2x-1 \leq 3 \\ &\implies -2 \leq 2x \leq 4 \\ &\implies -1 \leq x \leq 2. \end{aligned}$$

2. To find x such that $\left|\frac{3x-1}{4}\right| \geq 2$ write

$$\begin{aligned} \left|\frac{3x-1}{4}\right| \geq 2 &\implies \frac{3x-1}{4} \geq 2 \quad \text{or} \quad \frac{3x-1}{4} \leq -2 \\ &\implies 3x \geq 9 \quad \text{or} \quad 3x \leq -7 \\ &\implies x \geq 3 \quad \text{or} \quad x \leq -\frac{7}{3}. \end{aligned}$$

1.5 EXPANSION AND FACTORISATION

$$\begin{aligned} (a+b)(c+d) &= a(c+d)+b(c+d) = ac+ad+bc+bd \\ (a-b)(a+b) &= a^2-b^2 \\ (a \pm b)^2 &= a^2 \pm 2ab + b^2 \end{aligned}$$

EXAMPLES

1. $(x^2-3)^2 = x^4 + 2(-3)x^2 + 9 = x^4 - 6x^2 + 9$

2.
$$\begin{aligned}(x-3)(x+5)^2(x+3) &= (x-3)(x+3)(x+5)^2\\ &= (x^2-9)(x^2+10x+25)\\ &= x^4+10x^3+16x^2-90x-225\end{aligned}$$

3.
$$\begin{aligned}\frac{s^2-4}{2+s} &= \frac{(s-2)(s+2)}{2+s}\\ &= s-2\end{aligned}$$

4.
$$\begin{aligned}(a+1)^3 &= (a+1)(a^2+2a+1)\\ &= a^3+3a^2+3a+1\end{aligned}$$

1.5.1 BINOMIAL EXPANSION

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n$$

(See also Section 1.13). To remember the coefficients of each term use Pascal's triangle where each number is the sum of the two numbers above it.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

Each term in a row represents the coefficients of the corresponding term in the expansion.

EXAMPLES

1. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

2. $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

3. The coefficient of x^3 in $(2+x)^5$ is $10 \times 2^2 = 40$.

1.5.2 FACTORISING POLYNOMIALS

Factorising a polynomial is the opposite of the expansion described above, that is, splitting the polynomial into its factors:

$$p(x) = (x - a_1)(x - a_2)\cdots(x - a_n).$$

EXAMPLES

1. $x^2 - 1 = (x-1)(x+1)$
2. $x^2 - 3x + 2 = (x-2)(x-1)$
3. $3x^2 - 7x + 2 = (3x-1)(x-2)$
4. $x^3 - 4x^2 + 4x = x(x-2)^2$
5. $a^3 + 3a^2 + 3a + 1 = (a+1)^3$

1.6 PARTIAL FRACTIONS

It is sometimes convenient to write

$$\frac{cx+d}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b}$$

where A and B are constants found by equating the numerators of both sides once the right hand side is written as one fraction:

$$cx + d = A(x+b) + B(x+a).$$

Some similar partial fraction expansions are

$$\begin{aligned} \frac{1}{(x+a)^2(x+b)} &= \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{x+b} \\ \frac{1}{(x^2+bx+c)(x+a)} &= \frac{Ax+B}{x^2+bx+c} + \frac{C}{x+a}. \end{aligned}$$

EXAMPLES

1. Writing $\dfrac{1}{(x+1)(x-1)}$ in the form $\dfrac{A}{x+1}+\dfrac{B}{x-1}$ implies

$$A(x-1)+B(x+1)=1.$$

The constants A and B can be found two simple ways. First, setting

$$x=1 \implies B=\frac{1}{2}$$
$$x=-1 \implies A=-\frac{1}{2}.$$

Alternatively the equation could be expanded as

$$Ax+Bx-A+B=1$$

and the coefficients of x^1 and x^0 equated giving

$$A+B=0$$
$$-A+B=1.$$

Solving these equations simultaneously gives $A=-1/2$ and $B=1/2$. Thus

$$\frac{1}{(x+1)(x-1)}=\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right).$$

2. To expand $\dfrac{3x+1}{(x+7)(x-3)}$ using partial fractions write

$$\frac{3x+1}{(x+7)(x-3)}=\frac{A}{x+7}+\frac{B}{x-3}$$

giving

$$A(x-3)+B(x+7)=3x+1.$$

Setting $x=3$ implies $B=1$ and setting $x=-7$ implies $A=2$. Alternatively, equating the coefficients of

$$Ax+Bx-3A+7B=3x+1$$

gives

$$A+B=3$$
$$-3A+7B=1.$$

These simultaneous equations are solved for A and B to give $A=2$ and $B=1$. Hence

$$\frac{3x+1}{(x+7)(x-3)}=\frac{2}{x+7}+\frac{1}{x-3}.$$

3. The partial fraction for $\dfrac{1}{(x+1)^2(x+2)}$ is

$$\frac{1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}$$

giving

$$1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$$

so that

$$\begin{array}{llllll} x=-1 & \Longrightarrow & 1=B & & \\ x=-2 & \Longrightarrow & 1=C & & \\ \text{order } x^2 & \Longrightarrow & 0=A+C & \Longrightarrow & A=-1. \end{array}$$

Thus

$$\frac{1}{(x+1)^2(x+2)} = \frac{1}{(x+1)^2} - \frac{1}{x+1} + \frac{1}{x+2}.$$

4. The partial fraction for $\dfrac{3}{(x^2+x+1)(x+2)}$ is

$$\frac{3}{(x^2+x+1)(x+2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x+2}$$

giving

$$3 = (Ax+B)(x+2) + C(x^2+x+1).$$

Hence

$$\begin{array}{llllll} x=-2 & \Longrightarrow & C{=}1 & & \\ x=0 & \Longrightarrow & 3{=}2B+C & \Longrightarrow & B=1 \\ \text{order } x^2 & \Longrightarrow & 0{=}A+C & \Longrightarrow & A=-1. \end{array}$$

Thus

$$\frac{3}{(x^2+x+1)(x+2)} = \frac{1}{x+2} - \frac{x-1}{x^2+x+1}.$$

1.7 POLYNOMIAL DIVISION

> Polynomial division is a type of long division for polynomials best illustrated by the following examples.

EXAMPLES

1. When dividing $x^2 + 3x + 4$ by $x + 1$ consider only the leading order terms to begin with. Thus x goes into x^2, x times. Thus $x(x + 1) = x^2 + x$, which is subtracted from $x^2 + 3x + 4$. The first step is therefore

$$\begin{array}{r} x \\ x+1 \overline{\smash{)}\, x^2 + 3x + 4} \\ \underline{x^2 + x } \\ 2x + 4 \end{array}$$

The division is completed by considering that x (the leading order of $x + 1$) goes into $2x + 4$ two times. Subtracting $2(x + 1)$ from $2x + 4$ gives

$$\begin{array}{r} x + 2 \\ x+1 \overline{\smash{)}\, x^2 + 3x + 4} \\ \underline{x^2 + x } \\ 2x + 4 \\ \underline{2x + 2} \\ 2 \end{array}$$

Thus $$\frac{x^2 + 3x + 4}{x + 1} = (x + 2) + \frac{2}{x + 1}.$$

2. Dividing $3x^3 + 2x^2 + x + 1$ by $x - 1$ gives

$$\frac{3x^3 + 2x^2 + x + 1}{x - 1} = 3x^2 + 5x + 6 + \frac{7}{x - 1}.$$

3. $$\frac{4x^3 + 6x^2 + 4x + 1}{2x + 1} = 2x^2 + 2x + 1$$

1.8 SURDS

A *surd* is of the form $\sqrt[n]{a}$ $(= a^{1/n})$:

1. $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
2. $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
3. $b\sqrt{a} \pm c\sqrt{a} = (b \pm c)\sqrt{a}$

EXAMPLES

1. $\sqrt{5} \times \sqrt{2} = \sqrt{10}$
2. $\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}$
3. $3\sqrt{10} - 2\sqrt{10} = \sqrt{10}$
4. $\dfrac{\sqrt{14}}{\sqrt{2}} = \sqrt{\dfrac{14}{2}} = \sqrt{7}$

1.8.1 RATIONALISING SURD DENOMINATORS

For an expression of the form

$$\frac{a}{b + \sqrt{c}}$$

it may be preferable to have a rational denominator. A surd denominator is *rationalised* by multiplying the expression by $\dfrac{b - \sqrt{c}}{b - \sqrt{c}}$ $(= 1)$:

$$\begin{aligned} \frac{a}{b + \sqrt{c}} &= \frac{a}{b + \sqrt{c}} \times \frac{b - \sqrt{c}}{b - \sqrt{c}} \\ &= \frac{a(b - \sqrt{c})}{b^2 - c}. \end{aligned}$$

EXAMPLES

1.
$$\frac{5}{1+\sqrt{5}} = \frac{5}{1+\sqrt{5}} \times \frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{5-5\sqrt{5}}{(1)^2-(\sqrt{5})^2} = \frac{5-5\sqrt{5}}{(-4)} = \frac{5\sqrt{5}-5}{4}$$

2.
$$\frac{6x}{1+2\sqrt{x}} = \frac{6x}{1+2\sqrt{x}} \times \frac{1-2\sqrt{x}}{1-2\sqrt{x}} = \frac{6x-12x\sqrt{x}}{1-4x}$$

1.9 QUADRATIC EQUATION

A quadratic equation is of the form

$$y = ax^2 + bx + c$$

where a, b, c are constants. The roots of a quadratic equation (when $y = 0$) are

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

A quadratic is factorised if it is written in the form

$$y = a(x - x_1)(x - x_2).$$

EXAMPLES

1. The solutions to $x^2 + 3x + 1 = 0$ are

$$x = \frac{-3+\sqrt{5}}{2} \quad \text{or} \quad \frac{-3-\sqrt{5}}{2}.$$

2. The quadratic $y = x^2 + x - 6$ is factorised into $y = (x+3)(x-2)$.
3. The quadratic $y = x^2 + 2x + 1$ is factorised into $y = (x+1)^2$.

4. The solutions to $3x^2 + 5x + 1 = 0$ are

$$x = \frac{-5 \pm \sqrt{25 - 12}}{6}$$

so that

$$x = \frac{-5 + \sqrt{13}}{6} \quad \text{or} \quad \frac{-5 - \sqrt{13}}{6}.$$

1.10 SUMMATION

The summation sign $\sum$ is defined as

$$\sum_{i=1}^{n} f(i) = f(1) + f(2) + f(3) + \cdots + f(n-1) + f(n).$$

EXAMPLE $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30$

1.11 FACTORIAL NOTATION

The factorial notation is defined as follows:

$$n! = n.(n-1).(n-2)\ldots 3.2.1 \qquad \text{where } n \text{ is an integer.}$$

EXAMPLES

1. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
2. $0! = 1$ by definition.
3. $n! = n(n-1)!$
4. $2.4.6.8\ldots 2n = (2.2.2\ldots 2)(1.2.3\ldots n) = 2^n n!$

1.12 PERMUTATIONS

A **permutation** is a particular ordering of a set of unique objects. The **number of permutations** of r unique objects, chosen from a group of n, is given by

$$P_r^n = \frac{n!}{(n-r)!}.$$

EXAMPLE

The number of ways a batting lineup of 3 can be chosen from a squad of 8 cricket players is given by

$$P_3^8 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336.$$

1.13 COMBINATIONS

If order is not important when choosing r things from a group of n then the number of possible **combinations** is given by

$$C_r^n = \frac{n!}{r!(n-r)!}.$$

EXAMPLES

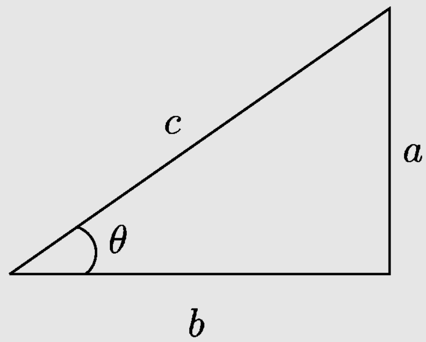
1. The number of possible groups of 4 delegates chosen from a group of 11 is given by

$$C_4^{11} = \frac{11!}{4!(11-4)!} = \frac{11!}{4!7!} = \frac{11 \times 10 \times 9 \times 8}{4 \times 3 \times 2 \times 1} = 330.$$

2. The number ways of choosing a team of 5 people from 7 is $C_5^7 = 21$.

1.14 GEOMETRY

The trigonometric ratios can be expressed in terms of the sides of a right-angled triangle:

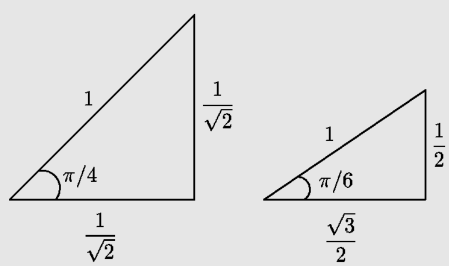


$$\sin\theta = \frac{a}{c}, \quad \cos\theta = \frac{b}{c}, \quad \tan\theta = \frac{a}{b} = \frac{\sin\theta}{\cos\theta}.$$

The longest length, opposite the right angle, is called the **hypotenuse**.
Pythagoras' Theorem states

$$a^2 + b^2 = c^2.$$

The sine, cosine and tangent of the common angles can be related to the following triangles:

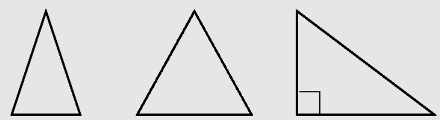


EXAMPLE

$$\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}},\quad \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2},\quad \sin\frac{\pi}{6}=\frac{1}{2},\quad \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}.$$

The three common triangles are the

1. **isosceles**: any two sides are of equal length.
2. **equilateral**: all three sides of of equal length.
3. **right angled**: one of the angles is $\frac{\pi}{2}$.



All triangles have three angles that sum to π.

EXAMPLES

1. A right angled triangle has one other angle $\frac{\pi}{6}$. Hence the third angle is $\frac{\pi}{3}$.
2. An equilateral triangle must have three identical angles of $\frac{\pi}{3}$.

1.14.1 CIRCLES

A circle of radius r has

1. **area** $= \pi r^2$
2. **circumference** $= 2\pi r$

EXAMPLES

1. The area of a circle with diameter $d = 6$ is $\pi 3^2 = 9\pi$.
2. The circumference of the circle with diameter $d = 7$ is 7π.

1.15 EXAMPLE QUESTIONS

(Answers are given in Chapter 14)

1. Simplify the following.

   (i) $\frac{1}{2} - \frac{5}{6} + \frac{1}{10}$

   (ii) $\frac{x}{x-3} - \frac{5}{x+2}$

   (iii) $\frac{x-1}{x+2} - \frac{2x}{x-2}$

   (iv) $\frac{2x+1}{x-4} - \frac{x-1}{x+2}$

   (v) $\frac{x^2-2}{x-1} + \frac{x+3}{x}$

   (vi) $\frac{x^3-x^2}{x(x^2-1)}$

2. Find the solution set for the following inequalities.

   (i) $2d + 2 \leq 4d - 3$

   (ii) $3d - 2 > 4d + 6$

   (iii) $|x - 10| < 5$

   (iv) $|z + 3| \geq 8$

   (v) $|a + 4| > 1$

   (vi) $\left|\frac{x}{2} - \frac{1}{2}\right| < 2$

3. Expand the following.

   (i) $(x - 3)(x + 3)$

   (ii) $(4 - 3x)^2$

   (iii) $(x + y)^2(x - y)$

   (iv) $(3 + x)(3x + 2)(x - 3)$

   (v) $(x - 4)^3$

4. Use Pascal's triangle (Binomial theorem) to find

   (i) the expansion of $(2 + x)^4$

   (ii) the expansion of $(1 + x)^8$

   (iii) the coefficient of x^5 in $(1 + x)^7$.

5. Write the following expressions as partial fractions.

   (i) $\frac{3}{(x-2)(x-4)}$

   (ii) $\frac{4x-1}{(x-1)(x+2)}$

   (iii) $\frac{1}{x^2+5x+6}$

   (iv) $\frac{3x}{(x-2)(x+4)}$

   (v) $\frac{1}{(x+3)^2(x-2)}$

6. Simplify the following.

   (i) $\sqrt{27}\sqrt{3}$

   (ii) $\frac{\sqrt{5}}{\sqrt{45}}$

   (iii) $\frac{\sqrt{17}+5\sqrt{17}}{2\sqrt{17}}$

   (iv) $\frac{2}{3+\sqrt{3}}$

7. Factorise the following quadratic equations.

   (i) $y = x^2 + 6x + 5$

   (ii) $y = x^2 - 6x + 5$

   (iii) $y = x^2 + 4x - 5$

   (iv) $y = x^2 - 4x - 5$

   (v) $y = 2x^2 + x - 1$

8. Find the zeros of the following quadratics.

   (i) $y = x^2 + 4x + 4$

   (ii) $y = x^2 + 7x + 6$

   (iii) $y = x^2 + x - 12$

   (iv) $y = x^2 + x - 2$

   (v) $y = x^2 + 3x - 4$

   (vi) $y = x^2 + x - 3$

9. Use polynomial division to calculate the following.

   (i) $(x^2 + 3x + 4)/(x + 2)$

   (ii) $(x^2 + 3x + 2)/(x + 2)$

   (iii) $(x^3 + 5x^2 + 7x + 2)/(x + 2)$

10. Find the following.

   (i) $\frac{10!}{7!}$

   (ii) P_2^6

   (iii) C_2^6

   (iv) $\sum_{i=1}^{6}(i + 1)$

CHAPTER 2

FUNCTIONS AND GRAPHS

2.1 THE BASIC FUNCTIONS AND CURVES

The standard functions and shapes are

1. Straight Lines: $y = mx + c$
2. Quadratics (parabolas): $y = ax^2 + bx + c$
3. Polynomials: $y = a_n x^n + \cdots + a_1 x + a_0$
4. Hyperbola: $y = \dfrac{1}{x}$
5. Exponential: $y = e^x \equiv \exp x$
6. Logarithm: $y = \ln x$
7. Sine: $y = \sin x$
8. Cosine: $y = \cos x$
9. Tangent: $y = \tan x$
10. Circles: $y^2 + x^2 = r^2$
11. Ellipses: $\left(\dfrac{y}{a}\right)^2 + \left(\dfrac{x}{b}\right)^2 = 1.$

2.2 FUNCTION PROPERTIES

> A function is a rule for mapping one number to another. For example: $f(x) = x^2$ is a mapping from x to x^2 so that $f(3) = 3^2 = 9$.

EXAMPLES

1. If $f(x) = 3x + 1$ then $f(2) = 7$ and $f(a) = 3a + 1$.
2. If $f(z) = z^2 - 1$ then $f(1) = 0$.

> The **domain** of a function is the set of all possible input values for that function.

EXAMPLES

1. $y = x^2 + 4$ has domain of all real numbers.
2. $y = 1/(x - 1)$ has domain $x \neq 1$. That is, all real numbers *except* $x = 1$ can be used in this function. If $x = 1$ then the function is undefined because of division by zero.

Sometimes the domain is defined as part of the function such as $y = x^2$ for $0 < x < 3$ so that the domain is restricted to be in the interval zero to three.

> The **range** of a function is the set of all possible output values for that function.

EXAMPLES

1. $y = x^2$ has range $y \geq 0$ since any squared number is positive.
2. $y = \sin x$ has range $-1 \leq y \leq 1$ since the sine function is always between positive and negative one.
3. $y = x^2$, $0 < x < 3$ (so the domain is restricted to $x \in (0, 3)$) has range $0 < y < 9$.

The argument of a function could be the value of another function. For example if $f(x) = x^2$ and $g(x) = x + 1$ then

$$f(g(x)) = (g(x))^2 = (x+1)^2.$$

EXAMPLES

1. If $f(x) = 3x - 1$ then $f(x+1) = 3(x+1) - 1 = 3x + 2$.
2. If $f(x) = 2x + 1$ and $g(x) = \cos(x)$ then $f(g(x)) = 2\cos(x) + 1$ and $g(f(x)) = \cos(2x+1)$.

The **inverse** of a function is denoted $f^{-1}(x)$ and has the property that

$$f^{-1}(f(x)) = f(f^{-1}(x)) = x.$$

EXAMPLES

1. $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverses since $\sqrt{x^2} = (\sqrt{x})^2 = x$.
2. If $f(x) = 3x^2 + 1$ then the inverse is found by rearrangement:

$$\begin{aligned} f(x) &= 3x^2 + 1 \\ \Longrightarrow \quad \pm\sqrt{\frac{f(x)-1}{3}} &= x \\ \Longrightarrow \quad f^{-1}(x) &= \pm\sqrt{\frac{x-1}{3}}. \end{aligned}$$

The **zeros** of a function, $f(x)$, are the values of x when $f(x) = 0$.

EXAMPLES

1. $f(x) = 2x + 3$ has zero $x = -\frac{3}{2}$.
2. $f(x) = x^2 + 3x + 2$ has zeros $x = -1, -2$.

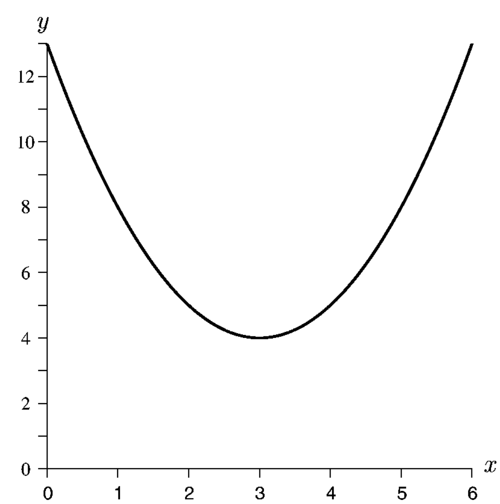
A graph $y = f(x)$ shifted from being centred on $(0, 0)$ to being centred on (a, b) is written in the form

$$y - b = f(x - a).$$

EXAMPLES

1. A circle with centre (1,2) has form $(x - 1)^2 + (y - 2)^2 = r^2$.

2. A parabola $y = x^2$ with turning point (0,0) if shifted to having turning point (3,4) has equation $(y - 4) = (x - 3)^2$.



A function is **even** if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$.

EXAMPLES

1. $y = f(x) = x^3$ is odd since $f(-x) = (-x)^3 = -x^3 = -f(x)$.

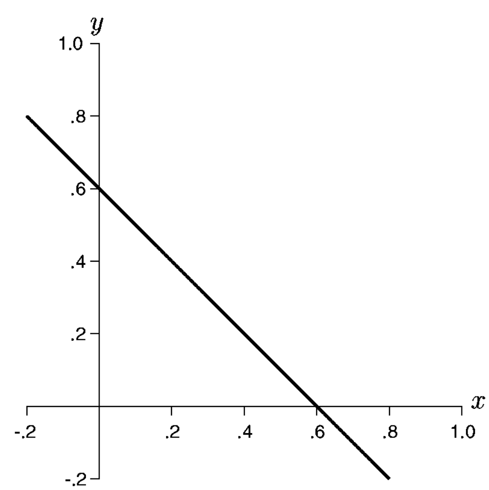
2. $y = f(x) = x^4$ is even since $f(-x) = (-x)^4 = x^4 = f(x)$.

2.3 STRAIGHT LINES

> A line has the general form
>
> $$y = mx + a$$
>
> where a and m are real numbers and m is the slope of the line.

EXAMPLES

1. Part of the straight line $y = 0.6 - x$ is drawn in the following diagram:



2. The line $y = 2x + 1$ cuts the x axis when $y = 0$ giving $x = -\frac{1}{2}$ as the zero.

3. The line $5y = x - 1$ has slope $m = \frac{1}{5}$ since it can be rewritten as $y = \frac{x}{5} - \frac{1}{5}$.

4. The equation of a line that passes through the points $(0, -1)$ and $(3, 0)$ is $y = \frac{x}{3} - 1$. The gradient is found from

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 + 1}{3 - 0} = \frac{1}{3}.$$

2.4 QUADRATICS

A quadratic (parabola) has the general form

$$y = ax^2 + bx + c$$

and can have either no real zeros, one real zero or two real zeros.

If the quadratic has two real zeros, c_1, c_2 then it can also be written as

$$y = a(x - c_1)(x - c_2).$$

EXAMPLE

Sections of the three quadratic functions

$$y = (x-1)^2 + 1, \quad y = (x-3)^2, \quad y = (x-5)(x-6)$$

are drawn in the following diagram: