CHAPTER 4

Exponential, Logarithmic, and Inverse Trigonometric Functions

EXERCISE SET 4.1

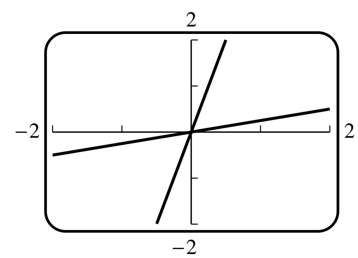
1. **(a)** $f(g(x)) = 4(x/4) = x$, $g(f(x)) = (4x)/4 = x$, f and g are inverse functions

(b) $f(g(x)) = 3(3x - 1) + 1 = 9x - 2 \neq x$ so f and g are not inverse functions

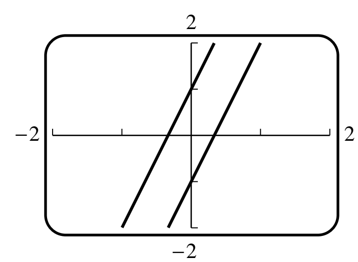
(c) $f(g(x)) = \sqrt[3]{(x^3 + 2) - 2} = x$, $g(f(x)) = (x - 2) + 2 = x$, f and g are inverse functions

(d) $f(g(x)) = (x^{1/4})^4 = x$, $g(f(x)) = (x^4)^{1/4} = |x| \neq x$, f and g are not inverse functions

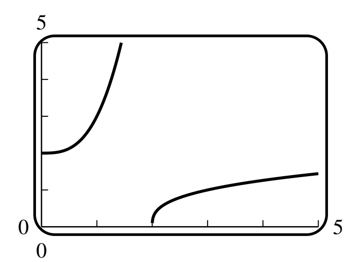
2. **(a)** They are inverse functions.



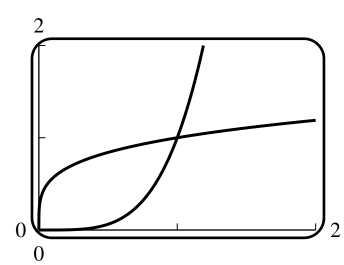
(b) The graphs are not reflections of each other about the line $y = x$.



(c) They are inverse functions provided the domain of g is restricted to $[0, +\infty)$



(d) They are inverse functions provided the domain of $f(x)$ is restricted to $[0, +\infty)$



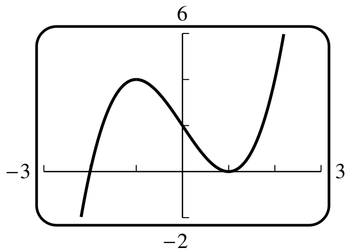
3. **(a)** yes; all outputs (the elements of row two) are distinct

(b) no; $f(1) = f(6)$

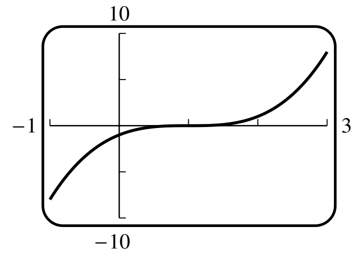
4. **(a)** no; it is easy to conceive of, say, 8 people in line at two different times
(b) no; perhaps your weight remains constant for more than a year
(c) yes, since the function is increasing, in the sense that the greater the volume, the greater the weight

5. **(a)** yes **(b)** yes **(c)** no **(d)** yes **(e)** no **(f)** no

6. **(a)** no, the horizontal line test fails



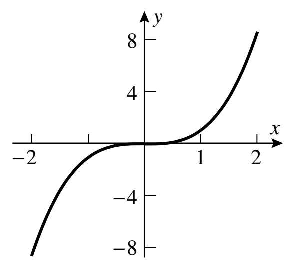
(b) yes, horizontal line test



7. **(a)** no, the horizontal line test fails
(b) no, the horizontal line test fails
(c) yes, horizontal line test

8. **(d)** no, the horizontal line test fails
(e) no, the horizontal line test fails
(f) yes, horizontal line test

9. **(a)** f has an inverse because the graph passes the horizontal line test. To compute $f^{-1}(2)$ start at 2 on the y-axis and go to the curve and then down, so $f^{-1}(2) = 8$; similarly, $f^{-1}(-1) = -1$ and $f^{-1}(0) = 0$.
(b) domain of f^{-1} is $[-2, 2]$, range is $[-8, 8]$
(c)



10. **(a)** the horizontal line test fails
(b) $-\infty < x \leq -1$; $-1 \leq x \leq 2$; and $2 \leq x < 4$.

11. **(a)** $f'(x) = 2x + 8$; $f' < 0$ on $(-\infty, -4)$ and $f' > 0$ on $(-4, +\infty)$; not one-to-one
(b) $f'(x) = 10x^4 + 3x^2 + 3 \geq 3 > 0$; $f'(x)$ is positive for all x, so f is one-to-one
(c) $f'(x) = 2 + \cos x \geq 1 > 0$ for all x, so f is one-to-one

12. **(a)** $f'(x) = 3x^2 + 6x = x(3x + 6)$ changes sign at $x = -2, 0$, so f is not one-to-one
(b) $f'(x) = 5x^4 + 24x^2 + 2 \geq 2 > 0$; f' is positive for all x, so f is one-to-one
(c) $f'(x) = \dfrac{1}{(x+1)^2}$; f is one-to-one because:
if $x_1 < x_2 < -1$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
if $-1 < x_1 < x_2$ then $f' > 0$ on $[x_1, x_2]$, so $f(x_1) \neq f(x_2)$
if $x_1 < -1 < x_2$ then $f(x_1) > 1 > f(x_2)$ since $f(x) > 1$ on $(-\infty, -1)$ and $f(x) < 1$ on $(-1, +\infty)$

13. $y = f^{-1}(x)$, $x = f(y) = y^5$, $y = x^{1/5} = f^{-1}(x)$

14. $y = f^{-1}(x)$, $x = f(y) = 6y$, $y = \frac{1}{6}x = f^{-1}(x)$

15. $y = f^{-1}(x)$, $x = f(y) = 7y - 6$, $y = \frac{1}{7}(x + 6) = f^{-1}(x)$

16. $y = f^{-1}(x)$, $x = f(y) = \frac{y+1}{y-1}$, $xy - x = y + 1$, $(x - 1)y = x + 1$, $y = \frac{x+1}{x-1} = f^{-1}(x)$

17. $y = f^{-1}(x)$, $x = f(y) = 3y^3 - 5$, $y = \sqrt[3]{(x+5)/3} = f^{-1}(x)$

18. $y = f^{-1}(x)$, $x = f(y) = \sqrt[5]{4y+2}$, $y = \frac{1}{4}(x^5 - 2) = f^{-1}(x)$

19. $y = f^{-1}(x)$, $x = f(y) = \sqrt[3]{2y-1}$, $y = (x^3 + 1)/2 = f^{-1}(x)$

20. $y = f^{-1}(x)$, $x = f(y) = \frac{5}{y^2+1}$, $y = \sqrt{\frac{5-x}{x}} = f^{-1}(x)$

21. $y = f^{-1}(x)$, $x = f(y) = 3/y^2$, $y = -\sqrt{3/x} = f^{-1}(x)$

22. $y = f^{-1}(x)$, $x = f(y) = \begin{cases} 2y, & y \le 0 \\ y^2, & y > 0 \end{cases}$, $\quad y = f^{-1}(x) = \begin{cases} x/2, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

23. $y = f^{-1}(x), x = f(y) = \begin{cases} 5/2 - y, & y < 2 \\ 1/y, & y \ge 2 \end{cases}$, $y = f^{-1}(x) = \begin{cases} 5/2 - x, & x > 1/2 \\ 1/x, & 0 < x \le 1/2 \end{cases}$

24. $y = p^{-1}(x)$, $x = p(y) = y^3 - 3y^2 + 3y - 1 = (y - 1)^3$, $y = x^{1/3} + 1 = p^{-1}(x)$

25. $y = f^{-1}(x)$, $x = f(y) = (y + 2)^4$ for $y \ge 0$, $y = f^{-1}(x) = x^{1/4} - 2$ for $x \ge 16$

26. $y = f^{-1}(x)$, $x = f(y) = \sqrt{y+3}$ for $y \ge -3$, $y = f^{-1}(x) = x^2 - 3$ for $x \ge 0$

27. $y = f^{-1}(x)$, $x = f(y) = -\sqrt{3-2y}$ for $y \le 3/2$, $y = f^{-1}(x) = (3 - x^2)/2$ for $x \le 0$

28. $y = f^{-1}(x)$, $x = f(y) = 3y^2 + 5y - 2$ for $y \ge 0$, $3y^2 + 5y - 2 - x = 0$ for $y \ge 0$,
$y = f^{-1}(x) = (-5 + \sqrt{12x + 49})/6$ for $x \ge -2$

29. $y = f^{-1}(x)$, $x = f(y) = y - 5y^2$ for $y \ge 1$, $5y^2 - y + x = 0$ for $y \ge 1$,
$y = f^{-1}(x) = (1 + \sqrt{1 - 20x})/10$ for $x \le -4$

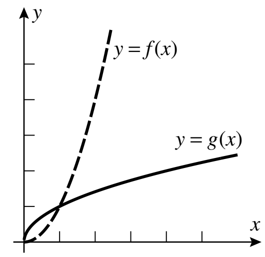
30. **(a)** $C = \frac{5}{9}(F - 32)$
(b) how many degrees Celsius given the Fahrenheit temperature
(c) $C = -273.15°$ C is equivalent to $F = -459.67°$ F, so the domain is $F \ge -459.67$, the range is $C \ge -273.15$

31. **(a)** $y = f(x) = (6.214 \times 10^{-4})x$ **(b)** $x = f^{-1}(y) = \frac{10^4}{6.214}y$
(c) how many meters in y miles

32. f and f^{-1} are continuous so $f(3) = \lim_{x\to 3} f(x) = 7$; then $f^{-1}(7) = 3$, and
$\lim_{x\to 7} f^{-1}(x) = f^{-1}\left(\lim_{x\to 7} x\right) = f^{-1}(7) = 3$

33. **(a)** $f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x, x > 1;$
$g(f(x)) = g(x^2) = \sqrt{x^2} = x, x > 1$

(b)



(c) no, because $f(g(x)) = x$ for every x in the domain of g is not satisfied (the domain of g is $x \geq 0$)

34. $y = f^{-1}(x)$, $x = f(y) = ay^2 + by + c$, $ay^2 + by + c - x = 0$, use the quadratic formula to get $y = \dfrac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$;

(a) $f^{-1}(x) = \dfrac{-b + \sqrt{b^2 - 4a(c-x)}}{2a}$ **(b)** $f^{-1}(x) = \dfrac{-b - \sqrt{b^2 - 4a(c-x)}}{2a}$

35. **(a)** $f(f(x)) = \dfrac{3 - \dfrac{3-x}{1-x}}{1 - \dfrac{3-x}{1-x}} = \dfrac{3 - 3x - 3 + x}{1 - x - 3 + x} = x$ so $f = f^{-1}$

(b) symmetric about the line $y = x$

36. $y = m(x - x_0)$ is an equation of the line. The graph of the inverse of $f(x) = m(x - x_0)$ will be the reflection of this line about $y = x$. Solve $y = m(x - x_0)$ for x to get $x = y/m + x_0 = f^{-1}(y)$ so $y = f^{-1}(x) = x/m + x_0$.

37. **(a)** $f(x) = x^3 - 3x^2 + 2x = x(x-1)(x-2)$ so $f(0) = f(1) = f(2) = 0$ thus f is not one-to-one.

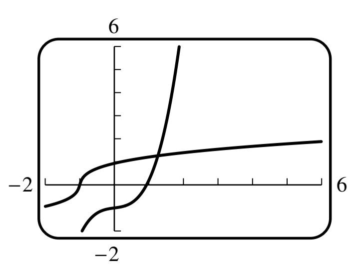
(b) $f'(x) = 3x^2 - 6x + 2$, $f'(x) = 0$ when $x = \dfrac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \sqrt{3}/3$. $f'(x) > 0$ (f is increasing) if $x < 1 - \sqrt{3}/3$, $f'(x) < 0$ (f is decreasing) if $1 - \sqrt{3}/3 < x < 1 + \sqrt{3}/3$, so $f(x)$ takes on values less than $f(1 - \sqrt{3}/3)$ on both sides of $1 - \sqrt{3}/3$ thus $1 - \sqrt{3}/3$ is the largest value of k.

38. **(a)** $f(x) = x^3(x-2)$ so $f(0) = f(2) = 0$ thus f is not one to one.
(b) $f'(x) = 4x^3 - 6x^2 = 4x^2(x - 3/2)$, $f'(x) = 0$ when $x = 0$ or $3/2$; f is decreasing on $(-\infty, 3/2]$ and increasing on $[3/2, +\infty)$ so $3/2$ is the smallest value of k.

39. if $f^{-1}(x) = 1$, then $x = f(1) = 2(1)^3 + 5(1) + 3 = 10$

40. if $f^{-1}(x) = 2$, then $x = f(2) = (2)^3/[(2)^2 + 1] = 8/5$

41.



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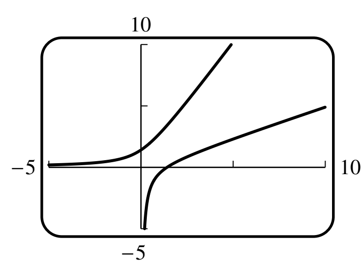
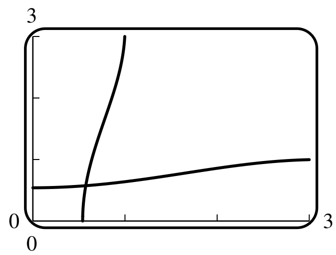
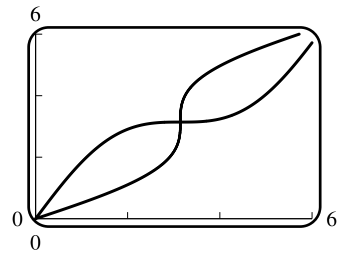


43. 

44. 

45. $y = f^{-1}(x)$, $x = f(y) = 5y^3 + y - 7$, $\dfrac{dx}{dy} = 15y^2 + 1$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2+1}$;

check: $1 = 15y^2\dfrac{dy}{dx} + \dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{15y^2+1}$

46. $y = f^{-1}(x)$, $x = f(y) = 1/y^2$, $\dfrac{dx}{dy} = -2y^{-3}$, $\dfrac{dy}{dx} = -y^3/2$;

check: $1 = -2y^{-3}\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = -y^3/2$

47. $y = f^{-1}(x)$, $x = f(y) = 2y^5 + y^3 + 1$, $\dfrac{dx}{dy} = 10y^4 + 3y^2$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4 + 3y^2}$;

check: $1 = 10y^4\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{10y^4+3y^2}$

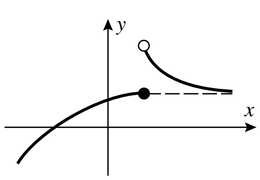
48. $y = f^{-1}(x)$, $x = f(y) = 5y - \sin 2y$, $\dfrac{dx}{dy} = 5 - 2\cos 2y$, $\dfrac{dy}{dx} = \dfrac{1}{5 - 2\cos 2y}$;

check: $1 = (5 - 2\cos 2y)\dfrac{dy}{dx}$, $\dfrac{dy}{dx} = \dfrac{1}{5 - 2\cos 2y}$

49. $f(f(x)) = x$ thus $f = f^{-1}$ so the graph is symmetric about $y = x$.

50. **(a)** Suppose $x_1 \neq x_2$ where x_1 and x_2 are in the domain of g and $g(x_1)$, $g(x_2)$ are in the domain of f then $g(x_1) \neq g(x_2)$ because g is one-to-one so $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one thus $f \circ g$ is one-to-one because $(f \circ g)(x_1) \neq (f \circ g)(x_2)$ if $x_1 \neq x_2$.

(b) f, g, and $f \circ g$ all have inverses because they are all one-to-one. Let $h = (f \circ g)^{-1}$ then $(f \circ g)(h(x)) = f[g(h(x))] = x$, apply f^{-1} to both sides to get $g(h(x)) = f^{-1}(x)$, then apply g^{-1} to get $h(x) = g^{-1}(f^{-1}(x)) = (g^{-1} \circ f^{-1})(x)$, so $h = g^{-1} \circ f^{-1}$

51. 

52. Suppose that g and h are both inverses of f then $f(g(x)) = x$, $h[f(g(x))] = h(x)$, but $h[f(g(x))] = g(x)$ because h is an inverse of f so $g(x) = h(x)$.

53. $F'(x) = 2f'(2g(x))g'(x)$ so $F'(3) = 2f'(2g(3))g'(3)$. By inspection $f(1) = 3$, so $g(3) = f^{-1}(3) = 1$ and $g'(3) = (f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(1) = 1/7$ because $f'(x) = 4x^3 + 3x^2$. Thus $F'(3) = 2f'(2)(1/7) = 2(44)(1/7) = 88/7$.
$F(3) = f(2g(3)), g(3) = f^{-1}(3)$; by inspection $f(1) = 3$, so $g(3) = f^{-1}(3) = 1, F(3) = f(2) = 25$.

EXERCISE SET 4.2

1. **(a)** -4 **(b)** 4 **(c)** $1/4$

2. **(a)** $1/16$ **(b)** 8 **(c)** $1/3$

3. **(a)** 2.9690 **(b)** 0.0341

4. **(a)** 1.8882 **(b)** 0.9381

5. **(a)** $\log_2 16 = \log_2(2^4) = 4$ **(b)** $\log_2\left(\dfrac{1}{32}\right) = \log_2(2^{-5}) = -5$
(c) $\log_4 4 = 1$ **(d)** $\log_9 3 = \log_9(9^{1/2}) = 1/2$

6. **(a)** $\log_{10}(0.001) = \log_{10}(10^{-3}) = -3$ **(b)** $\log_{10}(10^4) = 4$
(c) $\ln(e^3) = 3$ **(d)** $\ln(\sqrt{e}) = \ln(e^{1/2}) = 1/2$

7. **(a)** 1.3655 **(b)** -0.3011

8. **(a)** -0.5229 **(b)** 1.1447

9. **(a)** $2\ln a + \dfrac{1}{2}\ln b + \dfrac{1}{2}\ln c = 2r + s/2 + t/2$ **(b)** $\ln b - 3\ln a - \ln c = s - 3r - t$

10. **(a)** $\dfrac{1}{3}\ln c - \ln a - \ln b = t/3 - r - s$ **(b)** $\dfrac{1}{2}(\ln a + 3\ln b - 2\ln c) = r/2 + 3s/2 - t$

11. **(a)** $1 + \log x + \dfrac{1}{2}\log(x-3)$ **(b)** $2\ln|x| + 3\ln\sin x - \dfrac{1}{2}\ln(x^2+1)$

12. **(a)** $\dfrac{1}{3}\log(x+2) - \log\cos 5x$ **(b)** $\dfrac{1}{2}\ln(x^2+1) - \dfrac{1}{2}\ln(x^3+5)$

13. $\log\dfrac{2^4(16)}{3} = \log(256/3)$

14. $\log\sqrt{x} - \log(\sin^3 2x) + \log 100 = \log\dfrac{100\sqrt{x}}{\sin^3 2x}$

15. $\ln\dfrac{\sqrt[3]{x}(x+1)^2}{\cos x}$

16. $1 + x = 10^3 = 1000,\ x = 999$

17. $\sqrt{x} = 10^{-1} = 0.1,\ x = 0.01$

18. $x^2 = e^4,\ x = \pm e^2$

19. $1/x = e^{-2},\ x = e^2$

20. $x = 7$

21. $2x = 8,\ x = 4$

22. $\log_{10} x^3 = 30,\ x^3 = 10^{30},\ x = 10^{10}$

23. $\log_{10} x = 5,\ x = 10^5$

24. $\ln 4x - \ln x^6 = \ln 2,\ \ln\dfrac{4}{x^5} = \ln 2,\ \dfrac{4}{x^5} = 2,\ x^5 = 2,\ x = \sqrt[5]{2}$

25. $\ln 2x^2 = \ln 3,\ 2x^2 = 3,\ x^2 = 3/2,\ x = \sqrt{3/2}$ (we discard $-\sqrt{3/2}$ because it does not satisfy the original equation)

26. $\ln 3^x = \ln 2,\ x\ln 3 = \ln 2,\ x = \dfrac{\ln 2}{\ln 3}$

27. $\ln 5^{-2x} = \ln 3,\ -2x\ln 5 = \ln 3,\ x = -\dfrac{\ln 3}{2\ln 5}$

28. $e^{-2x} = 5/3,\ -2x = \ln(5/3),\ x = -\dfrac{1}{2}\ln(5/3)$

29. $e^{3x} = 7/2,\ 3x = \ln(7/2),\ x = \dfrac{1}{3}\ln(7/2)$

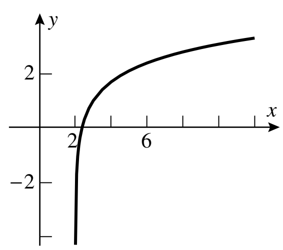
30. $e^x(1-2x) = 0$ so $e^x = 0$ (impossible) or $1-2x = 0,\ x = 1/2$

31. $e^{-x}(x+2) = 0$ so $e^{-x} = 0$ (impossible) or $x+2 = 0,\ x = -2$

32. $e^{2x} - e^x - 6 = (e^x - 3)(e^x + 2) = 0$ so $e^x = -2$ (impossible) or $e^x = 3$, $x = \ln 3$

33. $e^{-2x} - 3e^{-x} + 2 = (e^{-x} - 2)(e^{-x} - 1) = 0$ so $e^{-x} = 2$, $x = -\ln 2$ or $e^{-x} = 1$, $x = 0$

34. **(a)**



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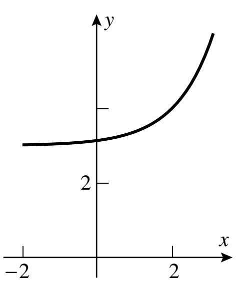
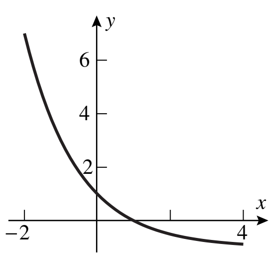


35. **(a)**



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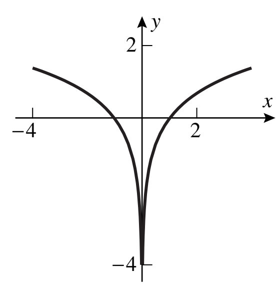
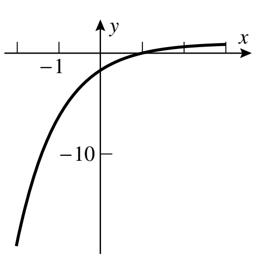


36. **(a)**



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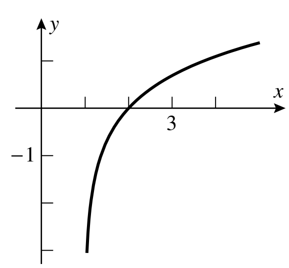
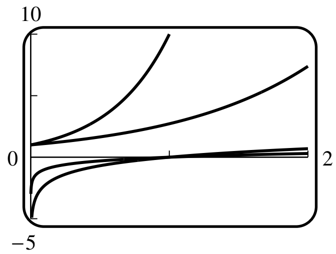


37. $\log_2 7.35 = (\log 7.35)/(\log 2) = (\ln 7.35)/(\ln 2) \approx 2.8777$;
$\log_5 0.6 = (\log 0.6)/(\log 5) = (\ln 0.6)/(\ln 5) \approx -0.3174$

38.



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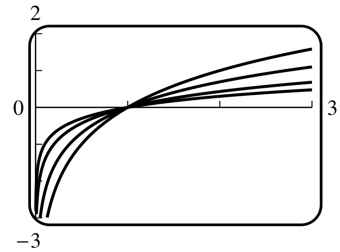
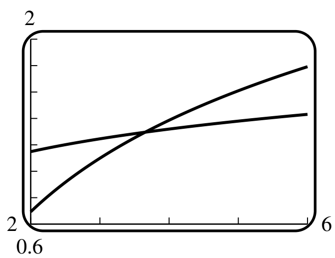


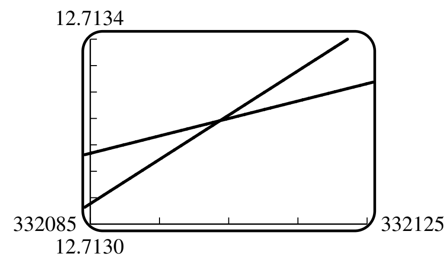
40. **(a)** Let $X = \log_b x$ and $Y = \log_a x$. Then $b^X = x$ and $a^Y = x$ so $a^Y = b^X$, or $a^{Y/X} = b$, which means $\log_a b = Y/X$. Substituting for Y and X yields $\dfrac{\log_a x}{\log_b x} = \log_a b, \log_b x = \dfrac{\log_a x}{\log_a b}$.

(b) Let $x = a$ to get $\log_b a = (\log_a a)/(\log_a b) = 1/(\log_a b)$ so $(\log_a b)(\log_b a) = 1$.
$(\log_2 81)(\log_3 32) = (\log_2[3^4])(\log_3[2^5]) = (4 \log_2 3)(5 \log_3 2) = 20(\log_2 3)(\log_3 2) = 20$

41. **(a)** $x = 3.6541, y = 1.2958$



(b) $x \approx 332105.11, y \approx 12.7132$



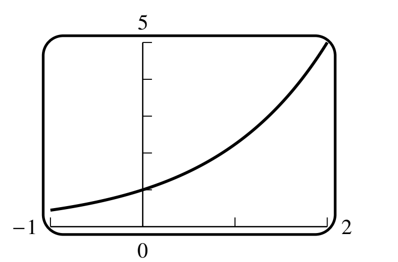
42. Since the units are billions, one trillion is 1,000 units. Solve $1000 = 0.051517(1.1306727)^x$ for x by taking common logarithms, resulting in $3 = \log 0.051517 + x \log 1.1306727$, which yields $x \approx 77.4$, so the debt first reached one trillion dollars around 1977.

43. **(a)** no, the curve passes through the origin

(b) $y = 2^{x/4}$

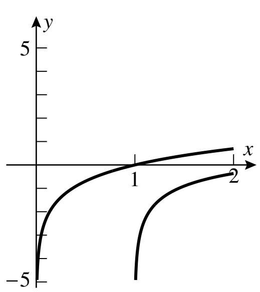
(c) $y = 2^{-x}$

(d) $y = (\sqrt{5})^x$



44. **(a)** As $x \to +\infty$ the function grows very slowly, but it is always increasing and tends to $+\infty$. As $x \to 1^+$ the function tends to $-\infty$.

(b)



45. $\log(1/2) < 0$ so $3\log(1/2) < 2\log(1/2)$

46. Let $x = \log_b a$ and $y = \log_b c$, so $a = b^x$ and $c = b^y$.
First, $ac = b^x b^y = b^{x+y}$ or equivalently, $\log_b(ac) = x + y = \log_b a + \log_b c$.
Secondly, $a/c = b^x/b^y = b^{x-y}$ or equivalently, $\log_b(a/c) = x - y = \log_b a - \log_b c$.
Next, $a^r = (b^x)^r = b^{rx}$ or equivalently, $\log_b a^r = rx = r\log_b a$.
Finally, $1/c = 1/b^y = b^{-y}$ or equivalently, $\log_b(1/c) = -y = -\log_b c$.

47. $75e^{-t/125} = 15, t = -125\ln(1/5) = 125\ln 5 \approx 201$ days.

48. **(a)** If $t = 0$, then $Q = 12$ grams **(b)** $Q = 12e^{-0.055(4)} = 12e^{-0.22} \approx 9.63$ grams
(c) $12e^{-0.055t} = 6, e^{-0.055t} = 0.5, t = -(\ln 0.5)/(0.055) \approx 12.6$ hours

49. **(a)** 7.4; basic **(b)** 4.2; acidic **(c)** 6.4; acidic **(d)** 5.9; acidic

50. **(a)** $\log[H^+] = -2.44, [H^+] = 10^{-2.44} \approx 3.6 \times 10^{-3}$ mol/L
(b) $\log[H^+] = -8.06, [H^+] = 10^{-8.06} \approx 8.7 \times 10^{-9}$ mol/L

51. **(a)** 140 dB; damage **(b)** 120 dB; damage
(c) 80 dB; no damage **(d)** 75 dB; no damage

52. Suppose that $I_1 = 3I_2$ and $\beta_1 = 10\log_{10} I_1/I_0$, $\beta_2 = 10\log_{10} I_2/I_0$. Then
$I_1/I_0 = 3I_2/I_0$, $\log_{10} I_1/I_0 = \log_{10} 3I_2/I_0 = \log_{10} 3 + \log_{10} I_2/I_0$, $\beta_1 = 10\log_{10} 3 + \beta_2$,
$\beta_1 - \beta_2 = 10\log_{10} 3 \approx 4.8$ decibels.

53. Let I_A and I_B be the intensities of the automobile and blender, respectively. Then
$\log_{10} I_A/I_0 = 7$ and $\log_{10} I_B/I_0 = 9.3$, $I_A = 10^7 I_0$ and $I_B = 10^{9.3} I_0$, so $I_B/I_A = 10^{2.3} \approx 200$.

54. The decibel level of the nth echo is $120(2/3)^n$;
$120(2/3)^n < 10$ if $(2/3)^n < 1/12, n < \dfrac{\log(1/12)}{\log(2/3)} = \dfrac{\log 12}{\log 1.5} \approx 6.13$ so 6 echoes can be heard.

55. **(a)** $\log E = 4.4 + 1.5(8.2) = 16.7, E = 10^{16.7} \approx 5 \times 10^{16}$ J
(b) Let M_1 and M_2 be the magnitudes of earthquakes with energies of E and $10E$, respectively. Then $1.5(M_2 - M_1) = \log(10E) - \log E = \log 10 = 1$,
$M_2 - M_1 = 1/1.5 = 2/3 \approx 0.67$.

56. Let E_1 and E_2 be the energies of earthquakes with magnitudes M and $M+1$, respectively. Then
$\log E_2 - \log E_1 = \log(E_2/E_1) = 1.5, E_2/E_1 = 10^{1.5} \approx 31.6$.

57. If $t = -2x$, then $x = -t/2$ and $\lim_{x\to 0}(1 - 2x)^{1/x} = \lim_{t\to 0}(1 + t)^{-2/t} = \lim_{t\to 0}[(1 + t)^{1/t}]^{-2} = e^{-2}$.

58. If $t = 3/x$, then $x = 3/t$ and $\lim_{x\to+\infty}(1 + 3/x)^x = \lim_{t\to 0^+}(1 + t)^{3/t} = \lim_{t\to 0^+}[(1 + t)^{1/t}]^3 = e^3$.

EXERCISE SET 4.3

1. $\dfrac{1}{2x}(2) = 1/x$

2. $\dfrac{1}{x^3}(3x^2) = 3/x$

3. $2(\ln x)\left(\dfrac{1}{x}\right) = \dfrac{2\ln x}{x}$

4. $\dfrac{1}{\sin x}(\cos x) = \cot x$

5. $\dfrac{1}{\tan x}(\sec^2 x) = \dfrac{\sec^2 x}{\tan x}$

6. $\dfrac{1}{2+\sqrt{x}}\left(\dfrac{1}{2\sqrt{x}}\right) = \dfrac{1}{2\sqrt{x}(2+\sqrt{x})}$

7. $\dfrac{1}{x/(1+x^2)}\left[\dfrac{(1+x^2)(1)-x(2x)}{(1+x^2)^2}\right] = \dfrac{1-x^2}{x(1+x^2)}$

8. $\dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) = \dfrac{1}{x\ln x}$

9. $\dfrac{3x^2-14x}{x^3-7x^2-3}$

10. $x^3\left(\dfrac{1}{x}\right) + (3x^2)\ln x = x^2(1+3\ln x)$

11. $\dfrac{1}{2}(\ln x)^{-1/2}\left(\dfrac{1}{x}\right) = \dfrac{1}{2x\sqrt{\ln x}}$

12. $\dfrac{\frac{1}{2}2(\ln x)(1/x)}{\sqrt{1+\ln^2 x}} = \dfrac{\ln x}{x\sqrt{1+\ln^2 x}}$

13. $-\dfrac{1}{x}\sin(\ln x)$

14. $2\sin(\ln x)\cos(\ln x)\dfrac{1}{x} = \dfrac{\sin(2\ln x)}{x} = \dfrac{\sin(\ln x^2)}{x}$

15. $3x^2\log_2(3-2x) + \dfrac{-2x^3}{(\ln 2)(3-2x)}$

16. $\left[\log_2(x^2-2x)\right]^3 + 3x\left[\log_2(x^2-2x)\right]^2 \dfrac{2x-2}{(x^2-2x)\ln 2}$

17. $\dfrac{2x(1+\log x) - x/(\ln 10)}{(1+\log x)^2}$

18. $1/[x(\ln 10)(1+\log x)^2]$

19. $7e^{7x}$

20. $-10xe^{-5x^2}$

21. $x^3e^x + 3x^2e^x = x^2e^x(x+3)$

22. $-\dfrac{1}{x^2}e^{1/x}$

23. $\dfrac{dy}{dx} = \dfrac{(e^x+e^{-x})(e^x+e^{-x}) - (e^x-e^{-x})(e^x-e^{-x})}{(e^x+e^{-x})^2}$

$= \dfrac{(e^{2x}+2+e^{-2x}) - (e^{2x}-2+e^{-2x})}{(e^x+e^{-x})^2} = 4/(e^x+e^{-x})^2$

24. $e^x\cos(e^x)$

25. $(x\sec^2 x + \tan x)e^{x\tan x}$

26. $\dfrac{dy}{dx} = \dfrac{(\ln x)e^x - e^x(1/x)}{(\ln x)^2} = \dfrac{e^x(x\ln x - 1)}{x(\ln x)^2}$

27. $(1-3e^{3x})e^{(x-e^{3x})}$

28. $\dfrac{15}{2}x^2(1+5x^3)^{-1/2}\exp(\sqrt{1+5x^3})$

29. $\dfrac{(x-1)e^{-x}}{1-xe^{-x}} = \dfrac{x-1}{e^x-x}$

30. $\dfrac{1}{\cos(e^x)}[-\sin(e^x)]e^x = -e^x\tan(e^x)$

31. $\dfrac{dy}{dx}+\dfrac{1}{xy}\left(x\dfrac{dy}{dx}+y\right)=0,\ \dfrac{dy}{dx}=-\dfrac{y}{x(y+1)}$

32. $\dfrac{dy}{dx}=\dfrac{1}{x\tan y}\left(x\sec^2 y\dfrac{dy}{dx}+\tan y\right),\ \dfrac{dy}{dx}=\dfrac{\tan y}{x(\tan y-\sec^2 y)}$

33. $\dfrac{d}{dx}\left[\ln\cos x-\dfrac{1}{2}\ln(4-3x^2)\right]=-\tan x+\dfrac{3x}{4-3x^2}$

34. $\dfrac{d}{dx}\left(\dfrac{1}{2}[\ln(x-1)-\ln(x+1)]\right)=\dfrac{1}{2}\left(\dfrac{1}{x-1}-\dfrac{1}{x+1}\right)$

35. $\ln|y|=\ln|x|+\dfrac{1}{3}\ln|1+x^2|,\ \dfrac{dy}{dx}=x\sqrt[3]{1+x^2}\left[\dfrac{1}{x}+\dfrac{2x}{3(1+x^2)}\right]$

36. $\ln|y|=\dfrac{1}{5}[\ln|x-1|-\ln|x+1|],\ \dfrac{dy}{dx}=\dfrac{1}{5}\sqrt[5]{\dfrac{x-1}{x+1}}\left[\dfrac{1}{x-1}-\dfrac{1}{x+1}\right]$

37. $\ln|y|=\dfrac{1}{3}\ln|x^2-8|+\dfrac{1}{2}\ln|x^3+1|-\ln|x^6-7x+5|$

$\dfrac{dy}{dx}=\dfrac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\left[\dfrac{2x}{3(x^2-8)}+\dfrac{3x^2}{2(x^3+1)}-\dfrac{6x^5-7}{x^6-7x+5}\right]$

38. $\ln|y|=\ln|\sin x|+\ln|\cos x|+3\ln|\tan x|-\dfrac{1}{2}\ln|x|$

$\dfrac{dy}{dx}=\dfrac{\sin x\cos x\tan^3 x}{\sqrt{x}}\left[\cot x-\tan x+\dfrac{3\sec^2 x}{\tan x}-\dfrac{1}{2x}\right]$

39. $f'(x)=2^x\ln 2;\ y=2^x,\ \ln y=x\ln 2,\ \dfrac{1}{y}y'=\ln 2,\ y'=y\ln 2=2^x\ln 2$

40. $f'(x)=-3^{-x}\ln 3;\ y=3^{-x},\ \ln y=-x\ln 3,\ \dfrac{1}{y}y'=-\ln 3,\ y'=-y\ln 3=-3^{-x}\ln 3$

41. $f'(x)=\pi^{\sin x}(\ln\pi)\cos x;$

$y=\pi^{\sin x},\ \ln y=(\sin x)\ln\pi,\ \dfrac{1}{y}y'=(\ln\pi)\cos x,\ y'=\pi^{\sin x}(\ln\pi)\cos x$

42. $f'(x)=\pi^{x\tan x}(\ln\pi)(x\sec^2 x+\tan x);$

$y=\pi^{x\tan x},\ \ln y=(x\tan x)\ln\pi,\ \dfrac{1}{y}y'=(\ln\pi)(x\sec^2 x+\tan x)$

$y'=\pi^{x\tan x}(\ln\pi)(x\sec^2 x+\tan x)$

43. $\ln y = (\ln x)\ln(x^3 - 2x),\ \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{3x^2-2}{x^3-2x}\ln x + \dfrac{1}{x}\ln(x^3-2x),$

$$\frac{dy}{dx} = (x^3-2x)^{\ln x}\left[\frac{3x^2-2}{x^3-2x}\ln x + \frac{1}{x}\ln(x^3-2x)\right]$$

44. $\ln y = (\sin x)\ln x,\ \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{\sin x}{x} + (\cos x)\ln x, \dfrac{dy}{dx} = x^{\sin x}\left[\dfrac{\sin x}{x} + (\cos x)\ln x\right]$

45. $\ln y = (\tan x)\ln(\ln x),\ \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x\ln x}\tan x + (\sec^2 x)\ln(\ln x),$

$$\frac{dy}{dx} = (\ln x)^{\tan x}\left[\frac{\tan x}{x\ln x} + (\sec^2 x)\ln(\ln x)\right]$$

46. $\ln y = (\ln x)\ln(x^2+3),\ \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{2x}{x^2+3}\ln x + \dfrac{1}{x}\ln(x^2+3),$

$$\frac{dy}{dx} = (x^2+3)^{\ln x}\left[\frac{2x}{x^2+3}\ln x + \frac{1}{x}\ln(x^2+3)\right]$$

47. $f'(x) = ex^{e-1}$

48. **(a)** because x^x is not of the form a^x where a is constant

(b) $y = x^x,\ \ln y = x\ln x,\ \dfrac{1}{y}y' = 1 + \ln x,\ y' = x^x(1+\ln x)$

49. **(a)** $\log_x e = \dfrac{\ln e}{\ln x} = \dfrac{1}{\ln x}, \dfrac{d}{dx}[\log_x e] = -\dfrac{1}{x(\ln x)^2}$

(b) $\log_x 2 = \dfrac{\ln 2}{\ln x}, \dfrac{d}{dx}[\log_x 2] = -\dfrac{\ln 2}{x(\ln x)^2}$

50. **(a)** From $\log_a b = \dfrac{\ln b}{\ln a}$ for $a, b > 0$ it follows that $\log_{(1/x)} e = \dfrac{\ln e}{\ln(1/x)} = -\dfrac{1}{\ln x}$, hence $\dfrac{d}{dx}\left[\log_{(1/x)} e\right] = \dfrac{1}{x(\ln x)^2}$

(b) $\log_{(\ln x)} e = \dfrac{\ln e}{\ln(\ln x)} = \dfrac{1}{\ln(\ln x)}$, so $\dfrac{d}{dx}\log_{(\ln x)} e = -\dfrac{1}{(\ln(\ln x))^2}\dfrac{1}{x\ln x} = -\dfrac{1}{x(\ln x)(\ln(\ln x))^2}$

51. **(a)** $f'(x) = ke^{kx},\ f''(x) = k^2e^{kx},\ f'''(x) = k^3e^{kx}, \ldots, f^{(n)}(x) = k^n e^{kx}$

(b) $f'(x) = -ke^{-kx},\ f''(x) = k^2e^{-kx},\ f'''(x) = -k^3e^{-kx}, \ldots, f^{(n)}(x) = (-1)^n k^n e^{-kx}$

52. $\dfrac{dy}{dt} = e^{-\lambda t}(\omega A\cos\omega t - \omega B\sin\omega t) + (-\lambda)e^{-\lambda t}(A\sin\omega t + B\cos\omega t)$

$$= e^{-\lambda t}[(\omega A - \lambda B)\cos\omega t - (\omega B + \lambda A)\sin\omega t]$$

53. $$f'(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\frac{d}{dx}\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$
$$= \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]\left[-\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\right]$$
$$= -\frac{1}{\sqrt{2\pi}\sigma^3}(x-\mu)\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

54. $y = Ae^{kt}$, $dy/dt = kAe^{kt} = k(Ae^{kt}) = ky$

55. $y = Ae^{2x} + Be^{-4x}$, $y' = 2Ae^{2x} - 4Be^{-4x}$, $y'' = 4Ae^{2x} + 16Be^{-4x}$ so
$y'' + 2y' - 8y = (4Ae^{2x} + 16Be^{-4x}) + 2(2Ae^{2x} - 4Be^{-4x}) - 8(Ae^{2x} + Be^{-4x}) = 0$

56. **(a)** $y' = -xe^{-x} + e^{-x} = e^{-x}(1-x)$, $xy' = xe^{-x}(1-x) = y(1-x)$

(b) $y' = -x^2e^{-x^2/2} + e^{-x^2/2} = e^{-x^2/2}(1-x^2)$, $xy' = xe^{-x^2/2}(1-x^2) = y(1-x^2)$

57. **(a)** $f(w) = \ln w$; $f'(1) = \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = \left.\frac{1}{w}\right|_{w=1} = 1$

(b) $f(w) = 10^w$; $f'(0) = \lim_{h\to 0}\frac{10^h - 1}{h} = \left.\frac{d}{dw}(10^w)\right|_{w=0} = \left.10^w\ln 10\right|_{w=0} = \ln 10$

58. **(a)** $f(x) = \ln x$; $f'(e^2) = \lim_{\Delta x\to 0}\frac{\ln(e^2 + \Delta x) - 2}{\Delta x} = \left.\frac{d}{dx}(\ln x)\right|_{x=e^2} = \left.\frac{1}{x}\right|_{x=e^2} = e^{-2}$

(b) $f(w) = 2^w$; $f'(1) = \lim_{w\to 1}\frac{2^w - 2}{w - 1} = \left.\frac{d}{dw}(2^w)\right|_{w=1} = \left.2^w\ln 2\right|_{w=1} = 2\ln 2$

EXERCISE SET 4.4

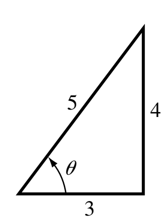
1. **(a)** $-\pi/2$ **(b)** π **(c)** $-\pi/4$ **(d)** 0

2. **(a)** $\pi/3$ **(b)** $\pi/3$ **(c)** $\pi/4$ **(d)** $2\pi/3$

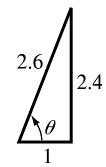
3. $\theta = -\pi/3$; $\cos\theta = 1/2$, $\tan\theta = -\sqrt{3}$, $\cot\theta = -1/\sqrt{3}$, $\sec\theta = 2$, $\csc\theta = -2/\sqrt{3}$

4. $\theta = \pi/3$; $\sin\theta = \sqrt{3}/2$, $\tan\theta = \sqrt{3}$, $\cot\theta = 1/\sqrt{3}$, $\sec\theta = 2$, $\csc\theta = 2/\sqrt{3}$

5. $\tan\theta = 4/3$, $0 < \theta < \pi/2$; use the triangle shown to get $\sin\theta = 4/5$, $\cos\theta = 3/5$, $\cot\theta = 3/4$, $\sec\theta = 5/3$, $\csc\theta = 5/4$



6. $\sec\theta = 2.6, 0 < \theta < \pi/2$; use the triangle shown to get
$\sin\theta = 2.4/2.6 = 12/13, \cos\theta = 1/2.6 = 5/13,$
$\tan\theta = 2.4 = 12/5, \cot\theta = 5/12, \csc\theta = 13/12$

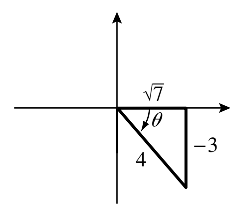


7. **(a)** $\pi/7$
(b) $\sin^{-1}(\sin\pi) = \sin^{-1}(\sin 0) = 0$
(c) $\sin^{-1}(\sin(5\pi/7)) = \sin^{-1}(\sin(2\pi/7)) = 2\pi/7$
(d) Note that $\pi/2 < 630 - 200\pi < \pi$ so
$\sin(630) = \sin(630 - 200\pi) = \sin(\pi - (630 - 200\pi)) = \sin(201\pi - 630)$ where
$0 < 201\pi - 630 < \pi/2$; $\sin^{-1}(\sin 630) = \sin^{-1}(\sin(201\pi - 630)) = 201\pi - 630$.

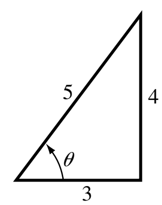
8. **(a)** $\pi/7$
(b) π
(c) $\cos^{-1}(\cos(12\pi/7)) = \cos^{-1}(\cos(2\pi/7)) = 2\pi/7$
(d) Note that $-\pi/2 < 200 - 64\pi < 0$ so $\cos(200) = \cos(200 - 64\pi) = \cos(64\pi - 200)$ where
$0 < 64\pi - 200 < \pi/2$; $\cos^{-1}(\cos 200) = \cos^{-1}(\cos(64\pi - 200)) = 64\pi - 200$.

9. **(a)** $0 \le x \le \pi$ **(b)** $-1 \le x \le 1$
(c) $-\pi/2 < x < \pi/2$ **(d)** $-\infty < x < +\infty$

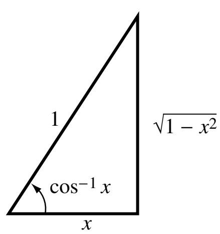
10. Let $\theta = \sin^{-1}(-3/4)$ then $\sin\theta = -3/4$, $-\pi/2 < \theta < 0$ and
(see figure) $\sec\theta = 4/\sqrt{7}$



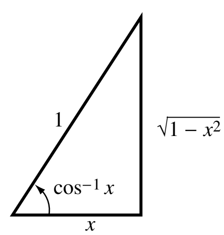
11. Let $\theta = \cos^{-1}(3/5)$, $\sin 2\theta = 2\sin\theta\cos\theta = 2(4/5)(3/5) = 24/25$



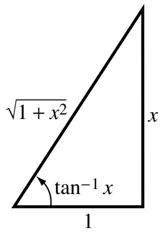
12. **(a)** $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$



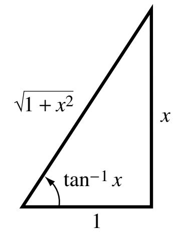
(b) $\tan(\cos^{-1} x) = \dfrac{\sqrt{1 - x^2}}{x}$



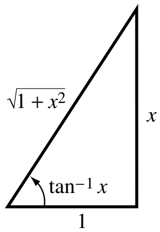
(c) $\csc(\tan^{-1} x) = \dfrac{\sqrt{1+x^2}}{x}$



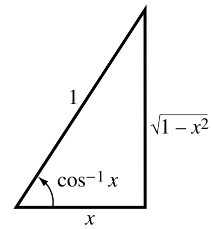
(d) $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{1+x^2}}$



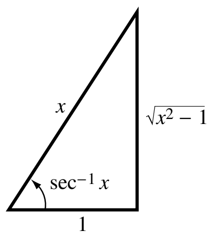
13. **(a)** $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$



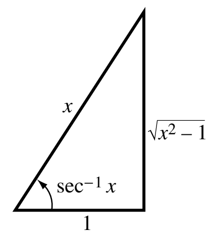
(b) $\tan(\cos^{-1} x) = \dfrac{\sqrt{1-x^2}}{x}$



(c) $\sin(\sec^{-1} x) = \dfrac{\sqrt{x^2-1}}{x}$



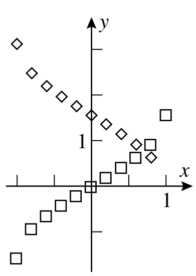
(d) $\cot(\sec^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$



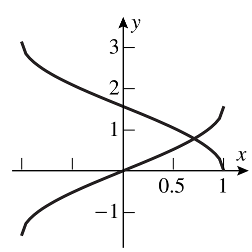
14. **(a)**

x	-1.00	-0.80	-0.6	-0.40	-0.20	0.00	0.20	0.40	0.60	0.80	1.00
$\sin^{-1} x$	-1.57	-0.93	-0.64	-0.41	-0.20	0.00	0.20	0.41	0.64	0.93	1.57
$\cos^{-1} x$	3.14	2.50	2.21	1.98	1.77	1.57	1.37	1.16	0.93	0.64	0.00

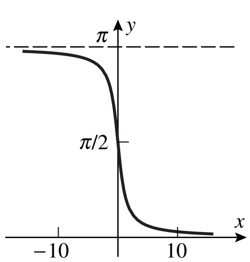
(b)

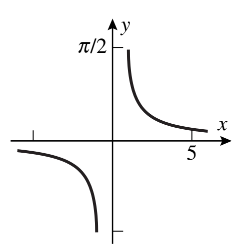


(c)



15. **(a)**





(b) The domain of $\cot^{-1} x$ is $(-\infty, +\infty)$, the range is $(0, \pi)$; the domain of $\csc^{-1} x$ is $(-\infty, -1] \cup [1, +\infty)$, the range is $[-\pi/2, 0) \cup (0, \pi/2]$.

16. **(a)** $y = \cot^{-1} x$; if $x > 0$ then $0 < y < \pi/2$ and $x = \cot y$, $\tan y = 1/x$, $y = \tan^{-1}(1/x)$;
if $x < 0$ then $\pi/2 < y < \pi$ and $x = \cot y = \cot(y - \pi), \tan(y - \pi) = 1/x, y = \pi + \tan^{-1} \dfrac{1}{x}$

(b) $y = \sec^{-1} x$, $x = \sec y$, $\cos y = 1/x$, $y = \cos^{-1}(1/x)$

(c) $y = \csc^{-1} x$, $x = \csc y$, $\sin y = 1/x$, $y = \sin^{-1}(1/x)$

17. **(a)** $55.0°$ **(b)** $33.6°$ **(c)** $25.8°$

18. **(a)** Let $x = f(y) = \cot y, 0 < y < \pi, -\infty < x < +\infty$. Then f is differentiable and one-to-one and $f'(f^{-1}(x)) = \cot(\cot^{-1} x)\cos(\cot^{-1} x) = -x\dfrac{\sqrt{x^2+1}}{x} = -\sqrt{x^2+1} \neq 0$, and

$$\frac{d}{dx}[\cot^{-1} x]\bigg|_{x=0} = \lim_{x\to 0} \frac{1}{f'(f^{-1}(x))} = -\lim_{x\to 0} \sqrt{x^2+1} = -1.$$

(b) If $x \neq 0$ then, from Exercise 16(a),
$\dfrac{d}{dx}\cot^{-1} x = \dfrac{d}{dx}\tan^{-1}\dfrac{1}{x} = -\dfrac{1}{x^2}\dfrac{1}{\sqrt{1+(1/x)^2}} = -\dfrac{1}{\sqrt{x^2+1}}$. For $x = 0$, Part (a) shows the same; thus for $-\infty < x < +\infty$, $\dfrac{d}{dx}[\cot^{-1} x] = -\dfrac{1}{\sqrt{x^2+1}}$.

(c) For $-\infty < u < +\infty$, by the chain rule it follows that $\dfrac{d}{dx}[\cot^{-1} u] = -\dfrac{1}{\sqrt{u^2+1}}\dfrac{du}{dx}$.

19. **(a)** By the chain rule, $\dfrac{d}{dx}[\csc^{-1} x] = -\dfrac{1}{x^2}\dfrac{1}{\sqrt{1-(1/x)^2}} = \dfrac{-1}{|x|\sqrt{x^2-1}}$

(b) By the chain rule, $\dfrac{d}{dx}[\csc^{-1} u] = \dfrac{du}{dx}\dfrac{d}{du}[\csc^{-1} u] = \dfrac{-1}{|u|\sqrt{u^2-1}}\dfrac{du}{dx}$

20. **(a)** $x = \pi - \sin^{-1}(0.37) \approx 2.7626$ rad **(b)** $\theta = 180° + \sin^{-1}(0.61) \approx 217.6°$

21. **(a)** $x = \pi + \cos^{-1}(0.85) \approx 3.6964$ rad **(b)** $\theta = -\cos^{-1}(0.23) \approx -76.7°$

22. **(a)** $x = \tan^{-1}(3.16) - \pi \approx -1.8773$ **(b)** $\theta = 180° - \tan^{-1}(0.45) \approx 155.8°$

23. **(a)** $\dfrac{1}{\sqrt{1-x^2/9}}(1/3) = 1/\sqrt{9-x^2}$ **(b)** $-2/\sqrt{1-(2x+1)^2}$

24. **(a)** $2x/(1+x^4)$ **(b)** $-\dfrac{1}{1+x}\left(\dfrac{1}{2}x^{-1/2}\right) = -\dfrac{1}{2(1+x)\sqrt{x}}$

25. **(a)** $\dfrac{1}{|x|^7\sqrt{x^{14}-1}}(7x^6) = \dfrac{7}{|x|\sqrt{x^{14}-1}}$ **(b)** $-1/\sqrt{e^{2x}-1}$

26. **(a)** $y = 1/\tan x = \cot x$, $dy/dx = -\csc^2 x$

(b) $y = (\tan^{-1} x)^{-1}$, $dy/dx = -(\tan^{-1} x)^{-2}\left(\dfrac{1}{1+x^2}\right)$

27. **(a)** $\dfrac{1}{\sqrt{1-1/x^2}}(-1/x^2) = -\dfrac{1}{|x|\sqrt{x^2-1}}$ **(b)** $\dfrac{\sin x}{\sqrt{1-\cos^2 x}} = \dfrac{\sin x}{|\sin x|} = \begin{cases} 1, & \sin x > 0 \\ -1, & \sin x < 0 \end{cases}$

28. **(a)** $-\dfrac{1}{(\cos^{-1}x)\sqrt{1-x^2}}$ **(b)** $-\dfrac{1}{2\sqrt{\cot^{-1}x}(1+x^2)}$

29. **(a)** $\dfrac{e^x}{|x|\sqrt{x^2-1}}+e^x\sec^{-1}x$ **(b)** $\dfrac{3x^2(\sin^{-1}x)^2}{\sqrt{1-x^2}}+2x(\sin^{-1}x)^3$

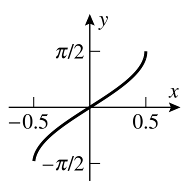
30. **(a)** 0 **(b)** 0

31. $x^3+x\tan^{-1}y=e^y$, $3x^2+\dfrac{x}{1+y^2}y'+\tan^{-1}y=e^yy'$, $y'=\dfrac{(3x^2+\tan^{-1}y)(1+y^2)}{(1+y^2)e^y-x}$

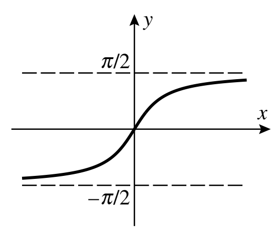
32. $\sin^{-1}(xy)=\cos^{-1}(x-y)$, $\dfrac{1}{\sqrt{1-x^2y^2}}(xy'+y)=-\dfrac{1}{\sqrt{1-(x-y)^2}}(1-y')$,

$y'=\dfrac{y\sqrt{1-(x-y)^2}+\sqrt{1-x^2y^2}}{\sqrt{1-x^2y^2}-x\sqrt{1-(x-y)^2}}$

33. **(a)**



(b)



34. **(a)** $\sin^{-1}0.9>1$, so it is not in the domain of $\sin^{-1}x$
(b) $-1\le\sin^{-1}x\le 1$ is necessary, or $-0.841471\le x\le 0.841471$

35. **(b)** $\theta=\sin^{-1}\dfrac{R}{R+h}=\sin^{-1}\dfrac{6378}{16,378}\approx 23°$

36. **(a)** If $\gamma=90°$, then $\sin\gamma=1$, $\sqrt{1-\sin^2\phi\sin^2\gamma}=\sqrt{1-\sin^2\phi}=\cos\phi$,
$D=\tan\phi\tan\lambda=(\tan 23.45°)(\tan 65°)\approx 0.93023374$ so $h\approx 21.1$ hours.
(b) If $\gamma=270°$, then $\sin\gamma=-1$, $D=-\tan\phi\tan\lambda\approx-0.93023374$ so $h\approx 2.9$ hours.

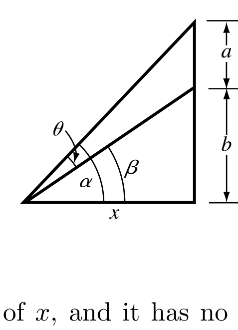
37. $\sin 2\theta=gR/v^2=(9.8)(18)/(14)^2=0.9$, $2\theta=\sin^{-1}(0.9)$ or $2\theta=180°-\sin^{-1}(0.9)$ so $\theta=\frac{1}{2}\sin^{-1}(0.9)\approx 32°$ or $\theta=90°-\frac{1}{2}\sin^{-1}(0.9)\approx 58°$. The ball will have a lower parabolic trajectory for $\theta=32°$ and hence will result in the shorter time of flight.

38. $4^2=2^2+3^2-2(2)(3)\cos\theta$, $\cos\theta=-1/4$, $\theta=\cos^{-1}(-1/4)\approx 104°$

39. $y=0$ when $x^2=6000v^2/g$, $x=10v\sqrt{60/g}=1000\sqrt{30}$ for $v=400$ and $g=32$;
$\tan\theta=3000/x=3/\sqrt{30}$, $\theta=\tan^{-1}(3/\sqrt{30})\approx 29°$.

40. **(a)** $\theta=\alpha-\beta$, $\cot\alpha=\dfrac{x}{a+b}$ and $\cot\beta=\dfrac{x}{b}$ so

$$\theta=\cot^{-1}\frac{x}{a+b}-\cot^{-1}\left(\frac{x}{b}\right)$$



(b) $$\frac{d\theta}{dx}=-\frac{1}{a+b}\left(\frac{1}{1+x^2/(a+b)^2}\right)-\frac{1}{b}\frac{1}{1+(x/b)^2}$$
$$=-\frac{a+b}{(a+b)^2+x^2}-\frac{b}{b^2+x^2}$$

which is negative for all x. Thus θ is a decreasing function of x, and it has no maximum since $\lim_{x\to 0^+}\theta=+\infty$.

41. **(a)** Let $\theta = \sin^{-1}(-x)$ then $\sin\theta = -x$, $-\pi/2 \le \theta \le \pi/2$. But $\sin(-\theta) = -\sin\theta$ and $-\pi/2 \le -\theta \le \pi/2$ so $\sin(-\theta) = -(-x) = x$, $-\theta = \sin^{-1} x$, $\theta = -\sin^{-1} x$.

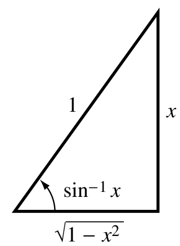
(b) proof is similar to that in Part (a)

42. **(a)** Let $\theta = \cos^{-1}(-x)$ then $\cos\theta = -x$, $0 \le \theta \le \pi$. But $\cos(\pi - \theta) = -\cos\theta$ and $0 \le \pi - \theta \le \pi$ so $\cos(\pi - \theta) = x$, $\pi - \theta = \cos^{-1} x$, $\theta = \pi - \cos^{-1} x$

(b) Let $\theta = \sec^{-1}(-x)$ for $x \ge 1$; then $\sec\theta = -x$ and $\pi/2 < \theta \le \pi$. So $0 \le \pi - \theta < \pi/2$ and $\pi - \theta = \sec^{-1}\sec(\pi - \theta) = \sec^{-1}(-\sec\theta) = \sec^{-1} x$, or $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

43. **(a)** $\sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$ (see figure)

(b) $\sin^{-1} x + \cos^{-1} x = \pi/2$; $\cos^{-1} x = \pi/2 - \sin^{-1} x = \pi/2 - \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$



44. $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$,

$$\tan(\tan^{-1} x + \tan^{-1} y) = \frac{\tan(\tan^{-1} x) + \tan(\tan^{-1} y)}{1 - \tan(\tan^{-1} x)\tan(\tan^{-1} y)} = \frac{x+y}{1-xy}$$

so $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\dfrac{x+y}{1-xy}$

45. **(a)** $\tan^{-1}\dfrac{1}{2} + \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1/2 + 1/3}{1 - (1/2)(1/3)} = \tan^{-1} 1 = \pi/4$

(b) $2\tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{3} = \tan^{-1}\dfrac{1/3 + 1/3}{1 - (1/3)(1/3)} = \tan^{-1}\dfrac{3}{4}$,

$2\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{1}{7} = \tan^{-1}\dfrac{3/4 + 1/7}{1 - (3/4)(1/7)} = \tan^{-1} 1 = \pi/4$

46. $\sin(\sec^{-1} x) = \sin(\cos^{-1}(1/x)) = \sqrt{1 - \left(\dfrac{1}{x}\right)^2} = \dfrac{\sqrt{x^2-1}}{|x|}$

EXERCISE SET 4.5

1. **(a)** $\displaystyle\lim_{x\to 2}\frac{x^2-4}{x^2+2x-8} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x+4)(x-2)} = \lim_{x\to 2}\frac{x+2}{x+4} = \frac{2}{3}$

(b) $\displaystyle\lim_{x\to+\infty}\frac{2x-5}{3x+7} = \frac{2 - \lim_{x\to+\infty}\frac{5}{x}}{3 + \lim_{x\to+\infty}\frac{7}{x}} = \frac{2}{3}$

2. **(a)** $\displaystyle\frac{\sin x}{\tan x} = \sin x\frac{\cos x}{\sin x} = \cos x$ so $\displaystyle\lim_{x\to 0}\frac{\sin x}{\tan x} = \lim_{x\to 0}\cos x = 1$

(b) $\displaystyle\frac{x^2-1}{x^3-1} = \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)} = \frac{x+1}{x^2+x+1}$ so $\displaystyle\lim_{x\to 1}\frac{x^2-1}{x^3-1} = \frac{2}{3}$

3. $\lim_{x\to 1} \dfrac{1/x}{1} = 1$

4. $\lim_{x\to 0} \dfrac{2\cos 2x}{5\cos 5x} = 2/5$

5. $\lim_{x\to 0} \dfrac{e^x}{\cos x} = 1$

6. $\lim_{x\to 3} \dfrac{1}{6x-13} = 1/5$

7. $\lim_{\theta\to 0} \dfrac{\sec^2\theta}{1} = 1$

8. $\lim_{t\to 0} \dfrac{te^t + e^t}{-e^t} = -1$

9. $\lim_{x\to \pi^+} \dfrac{\cos x}{1} = -1$

10. $\lim_{x\to 0^+} \dfrac{\cos x}{2x} = +\infty$

11. $\lim_{x\to +\infty} \dfrac{1/x}{1} = 0$

12. $\lim_{x\to +\infty} \dfrac{3e^{3x}}{2x} = \lim_{x\to +\infty} \dfrac{9e^{3x}}{2} = +\infty$

13. $\lim_{x\to 0^+} \dfrac{-\csc^2 x}{1/x} = \lim_{x\to 0^+} \dfrac{-x}{\sin^2 x} = \lim_{x\to 0^+} \dfrac{-1}{2\sin x\cos x} = -\infty$

14. $\lim_{x\to 0^+} \dfrac{-1/x}{(-1/x^2)e^{1/x}} = \lim_{x\to 0^+} \dfrac{x}{e^{1/x}} = 0$

15. $\lim_{x\to +\infty} \dfrac{100x^{99}}{e^x} = \lim_{x\to +\infty} \dfrac{(100)(99)x^{98}}{e^x} = \cdots = \lim_{x\to +\infty} \dfrac{(100)(99)(98)\cdots(1)}{e^x} = 0$

16. $\lim_{x\to 0^+} \dfrac{\cos x/\sin x}{\sec^2 x/\tan x} = \lim_{x\to 0^+} \cos^2 x = 1$

17. $\lim_{x\to 0} \dfrac{2/\sqrt{1-4x^2}}{1} = 2$

18. $\lim_{x\to 0} \dfrac{1 - \dfrac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \dfrac{1}{3(1+x^2)} = \dfrac{1}{3}$

19. $\lim_{x\to +\infty} xe^{-x} = \lim_{x\to +\infty} \dfrac{x}{e^x} = \lim_{x\to +\infty} \dfrac{1}{e^x} = 0$

20. $\lim_{x\to \pi} (x-\pi)\tan(x/2) = \lim_{x\to \pi} \dfrac{x-\pi}{\cot(x/2)} = \lim_{x\to \pi} \dfrac{1}{-(1/2)\csc^2(x/2)} = -2$

21. $\lim_{x\to +\infty} x\sin(\pi/x) = \lim_{x\to +\infty} \dfrac{\sin(\pi/x)}{1/x} = \lim_{x\to +\infty} \dfrac{(-\pi/x^2)\cos(\pi/x)}{-1/x^2} = \lim_{x\to +\infty} \pi\cos(\pi/x) = \pi$

22. $\lim_{x\to 0^+} \tan x \ln x = \lim_{x\to 0^+} \dfrac{\ln x}{\cot x} = \lim_{x\to 0^+} \dfrac{1/x}{-\csc^2 x} = \lim_{x\to 0^+} \dfrac{-\sin^2 x}{x} = \lim_{x\to 0^+} \dfrac{-2\sin x\cos x}{1} = 0$

23. $\lim_{x\to (\pi/2)^-} \sec 3x\cos 5x = \lim_{x\to (\pi/2)^-} \dfrac{\cos 5x}{\cos 3x} = \lim_{x\to (\pi/2)^-} \dfrac{-5\sin 5x}{-3\sin 3x} = \dfrac{-5(+1)}{(-3)(-1)} = -\dfrac{5}{3}$

24. $\lim_{x\to \pi} (x-\pi)\cot x = \lim_{x\to \pi} \dfrac{x-\pi}{\tan x} = \lim_{x\to \pi} \dfrac{1}{\sec^2 x} = 1$

25. $y = (1-3/x)^x$, $\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \dfrac{\ln(1-3/x)}{1/x} = \lim_{x\to +\infty} \dfrac{-3}{1-3/x} = -3$, $\lim_{x\to +\infty} y = e^{-3}$

26. $y = (1+2x)^{-3/x}$, $\lim_{x\to 0} \ln y = \lim_{x\to 0} -\dfrac{3\ln(1+2x)}{x} = \lim_{x\to 0} -\dfrac{6}{1+2x} = -6$, $\lim_{x\to 0} y = e^{-6}$

27. $y = (e^x + x)^{1/x}$, $\lim_{x\to 0} \ln y = \lim_{x\to 0} \dfrac{\ln(e^x + x)}{x} = \lim_{x\to 0} \dfrac{e^x + 1}{e^x + x} = 2$, $\lim_{x\to 0} y = e^2$

28. $y = (1 + a/x)^{bx}$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \frac{b\ln(1+a/x)}{1/x} = \lim_{x\to+\infty} \frac{ab}{1+a/x} = ab$, $\lim_{x\to+\infty} y = e^{ab}$

29. $y = (2-x)^{\tan(\pi x/2)}$, $\lim_{x\to1} \ln y = \lim_{x\to1} \frac{\ln(2-x)}{\cot(\pi x/2)} = \lim_{x\to1} \frac{2\sin^2(\pi x/2)}{\pi(2-x)} = 2/\pi$, $\lim_{x\to1} y = e^{2/\pi}$

30. $y = [\cos(2/x)]^{x^2}$, $\lim_{x\to+\infty} \ln y = \lim_{x\to+\infty} \frac{\ln\cos(2/x)}{1/x^2} = \lim_{x\to+\infty} \frac{(-2/x^2)(-\tan(2/x))}{-2/x^3}$

$= \lim_{x\to+\infty} \frac{-\tan(2/x)}{1/x} = \lim_{x\to+\infty} \frac{(2/x^2)\sec^2(2/x)}{-1/x^2} = -2$, $\lim_{x\to+\infty} y = e^{-2}$

31. $\lim_{x\to0}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to0} \frac{x - \sin x}{x\sin x} = \lim_{x\to0} \frac{1-\cos x}{x\cos x + \sin x} = \lim_{x\to0} \frac{\sin x}{2\cos x - x\sin x} = 0$

32. $\lim_{x\to0} \frac{1-\cos 3x}{x^2} = \lim_{x\to0} \frac{3\sin 3x}{2x} = \lim_{x\to0} \frac{9}{2}\cos 3x = \frac{9}{2}$

33. $\lim_{x\to+\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \frac{1}{\sqrt{1+1/x}+1} = 1/2$

34. $\lim_{x\to0} \frac{e^x-1-x}{xe^x-x} = \lim_{x\to0} \frac{e^x-1}{xe^x+e^x-1} = \lim_{x\to0} \frac{e^x}{xe^x+2e^x} = 1/2$

35. $\lim_{x\to+\infty}[x - \ln(x^2+1)] = \lim_{x\to+\infty}[\ln e^x - \ln(x^2+1)] = \lim_{x\to+\infty} \ln\frac{e^x}{x^2+1}$,

$\lim_{x\to+\infty} \frac{e^x}{x^2+1} = \lim_{x\to+\infty} \frac{e^x}{2x} = \lim_{x\to+\infty} \frac{e^x}{2} = +\infty$ so $\lim_{x\to+\infty}[x - \ln(x^2+1)] = +\infty$

36. $\lim_{x\to+\infty} \ln\frac{x}{1+x} = \lim_{x\to+\infty} \ln\frac{1}{1/x+1} = \ln(1) = 0$

38. **(a)** $\lim_{x\to+\infty} \frac{\ln x}{x^n} = \lim_{x\to+\infty} \frac{1/x}{nx^{n-1}} = \lim_{x\to+\infty} \frac{1}{nx^n} = 0$

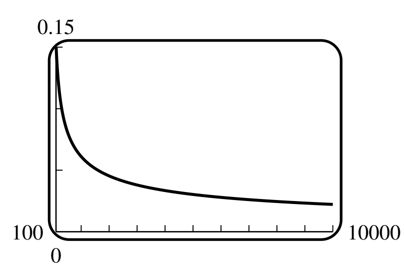
(b) $\lim_{x\to+\infty} \frac{x^n}{\ln x} = \lim_{x\to+\infty} \frac{nx^{n-1}}{1/x} = \lim_{x\to+\infty} nx^n = +\infty$

39. **(a)** L'Hôpital's Rule does not apply to the problem $\lim_{x\to1} \frac{3x^2-2x+1}{3x^2-2x}$ because it is not a $\frac{0}{0}$ form.

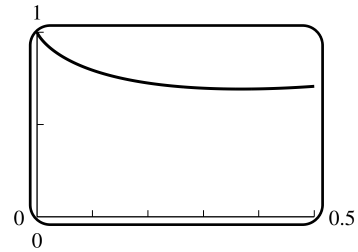
(b) $\lim_{x\to1} \frac{3x^2-2x+1}{3x^2-2x} = 2$

40. $\lim_{x\to1} \frac{4x^3-12x^2+12x-4}{4x^3-9x^2+6x-1} = \lim_{x\to1} \frac{12x^2-24x+12}{12x^2-18x+6} = \lim_{x\to1} \frac{24x-24}{24x-18} = 0$

41. $\lim_{x\to+\infty} \frac{1/(x\ln x)}{1/(2\sqrt{x})} = \lim_{x\to+\infty} \frac{2}{\sqrt{x}\ln x} = 0$



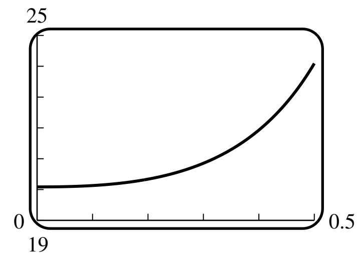
42. $y = x^x$, $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} -x = 0$, $\lim_{x\to 0^+} y = 1$



43. $y = (\sin x)^{3/\ln x}$,

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{3\ln\sin x}{\ln x} = \lim_{x\to 0^+} (3\cos x)\frac{x}{\sin x} = 3$,

$\lim_{x\to 0^+} y = e^3$

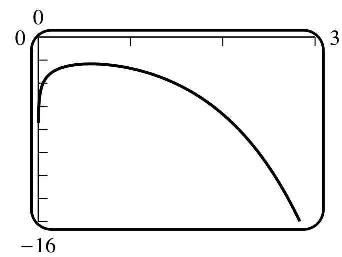


44. $\lim_{x\to \pi/2^-} \frac{4\sec^2 x}{\sec x \tan x} = \lim_{x\to \pi/2^-} \frac{4}{\sin x} = 4$

45. $\ln x - e^x = \ln x - \frac{1}{e^{-x}} = \frac{e^{-x}\ln x - 1}{e^{-x}}$;

$\lim_{x\to +\infty} e^{-x}\ln x = \lim_{x\to +\infty} \frac{\ln x}{e^x} = \lim_{x\to +\infty} \frac{1/x}{e^x} = 0$ by L'Hôpital's Rule,

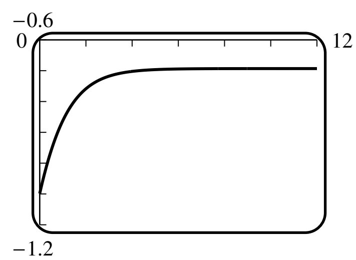
so $\lim_{x\to +\infty} [\ln x - e^x] = \lim_{x\to +\infty} \frac{e^{-x}\ln x - 1}{e^{-x}} = -\infty$



46. $\lim_{x\to +\infty} [\ln e^x - \ln(1+2e^x)] = \lim_{x\to +\infty} \ln \frac{e^x}{1+2e^x}$

$= \lim_{x\to +\infty} \ln \frac{1}{e^{-x}+2} = \ln \frac{1}{2}$;

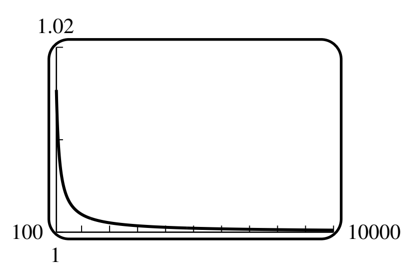
horizontal asymptote $y = -\ln 2$



47. $y = (\ln x)^{1/x}$,

$\lim_{x\to +\infty} \ln y = \lim_{x\to +\infty} \frac{\ln(\ln x)}{x} = \lim_{x\to +\infty} \frac{1}{x\ln x} = 0$;

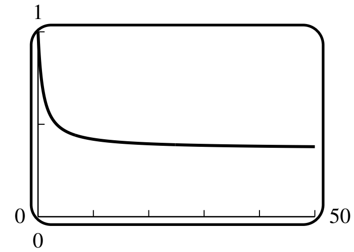
$\lim_{x\to +\infty} y = 1$, $y = 1$ is the horizontal asymptote



48. $y = \left(\dfrac{x+1}{x+2}\right)^x$, $\lim\limits_{x\to+\infty} \ln y = \lim\limits_{x\to+\infty} \dfrac{\ln \dfrac{x+1}{x+2}}{1/x}$

$$= \lim_{x\to+\infty} \frac{-x^2}{(x+1)(x+2)} = -1;$$

$\lim\limits_{x\to+\infty} y = e^{-1}$ is the horizontal asymptote



49. **(a)** 0 **(b)** $+\infty$ **(c)** 0 **(d)** $-\infty$ **(e)** $+\infty$ **(f)** $-\infty$

50. **(a)** Type 0^0; $y = x^{(\ln a)/(1+\ln x)}$; $\lim\limits_{x\to0^+} \ln y = \lim\limits_{x\to0^+} \dfrac{(\ln a)\ln x}{1+\ln x} = \lim\limits_{x\to0^+} \dfrac{(\ln a)/x}{1/x} = \lim\limits_{x\to0^+} \ln a = \ln a$, $\lim\limits_{x\to0^+} y = e^{\ln a} = a$

(b) Type ∞^0; same calculation as Part (a) with $x \to +\infty$

(c) Type 1^∞; $y = (x+1)^{(\ln a)/x}$, $\lim\limits_{x\to0} \ln y = \lim\limits_{x\to0} \dfrac{(\ln a)\ln(x+1)}{x} = \lim\limits_{x\to0} \dfrac{\ln a}{x+1} = \ln a$, $\lim\limits_{x\to0} y = e^{\ln a} = a$

51. $\lim\limits_{x\to+\infty} \dfrac{1+2\cos 2x}{1}$ does not exist, nor is it $\pm\infty$; $\lim\limits_{x\to+\infty} \dfrac{x+\sin 2x}{x} = \lim\limits_{x\to+\infty} \left(1 + \dfrac{\sin 2x}{x}\right) = 1$

52. $\lim\limits_{x\to+\infty} \dfrac{2-\cos x}{3+\cos x}$ does not exist, nor is it $\pm\infty$; $\lim\limits_{x\to+\infty} \dfrac{2x-\sin x}{3x+\sin x} = \lim\limits_{x\to+\infty} \dfrac{2-(\sin x)/x}{3+(\sin x)/x} = \dfrac{2}{3}$

53. $\lim\limits_{x\to+\infty} (2 + x\cos 2x + \sin 2x)$ does not exist, nor is it $\pm\infty$; $\lim\limits_{x\to+\infty} \dfrac{x(2+\sin 2x)}{x+1} = \lim\limits_{x\to+\infty} \dfrac{2+\sin 2x}{1+1/x}$, which does not exist because $\sin 2x$ oscillates between -1 and 1 as $x \to +\infty$

54. $\lim\limits_{x\to+\infty} \left(\dfrac{1}{x} + \dfrac{1}{2}\cos x + \dfrac{\sin x}{2x}\right)$ does not exist, nor is it $\pm\infty$;

$\lim\limits_{x\to+\infty} \dfrac{x(2+\sin x)}{x^2+1} = \lim\limits_{x\to+\infty} \dfrac{2+\sin x}{x+1/x} = 0$

55. $\lim\limits_{R\to0^+} \dfrac{\frac{Vt}{L}e^{-Rt/L}}{1} = \dfrac{Vt}{L}$

56. **(a)** $\lim\limits_{x\to\pi/2} (\pi/2 - x)\tan x = \lim\limits_{x\to\pi/2} \dfrac{\pi/2 - x}{\cot x} = \lim\limits_{x\to\pi/2} \dfrac{-1}{-\csc^2 x} = \lim\limits_{x\to\pi/2} \sin^2 x = 1$

(b) $\lim\limits_{x\to\pi/2} \left(\dfrac{1}{\pi/2 - x} - \tan x\right) = \lim\limits_{x\to\pi/2} \left(\dfrac{1}{\pi/2 - x} - \dfrac{\sin x}{\cos x}\right) = \lim\limits_{x\to\pi/2} \dfrac{\cos x - (\pi/2 - x)\sin x}{(\pi/2 - x)\cos x}$

$$= \lim_{x\to\pi/2} \frac{-(\pi/2 - x)\cos x}{-(\pi/2 - x)\sin x - \cos x}$$

$$= \lim_{x\to\pi/2} \frac{(\pi/2 - x)\sin x + \cos x}{-(\pi/2 - x)\cos x + 2\sin x} = 0$$

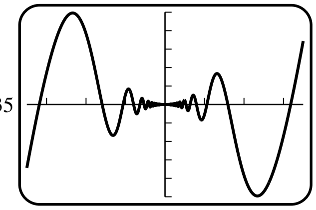
(c) $1/(\pi/2 - 1.57) \approx 1255.765849, \tan 1.57 \approx 1255.765592$;
$1/(\pi/2 - 1.57) - \tan 1.57 \approx 0.000265$

57. **(b)** $\lim_{x\to+\infty} x(k^{1/x}-1) = \lim_{t\to 0^+}\frac{k^t-1}{t} = \lim_{t\to 0^+}\frac{(\ln k)k^t}{1} = \ln k$

(c) $\ln 0.3 = -1.20397$, $1024\left(\sqrt[1024]{0.3}-1\right) = -1.20327$;
$\ln 2 = 0.69315$, $1024\left(\sqrt[1024]{2}-1\right) = 0.69338$

58. **(a)** No; $\sin(1/x)$ oscillates as $x \to 0$.

(b)



(c) For the limit as $x \to 0^+$ use the Squeezing Theorem together with the inequalities $-x^2 \le x^2\sin(1/x) \le x^2$. For $x \to 0^-$ do the same; thus $\lim_{x\to 0} f(x) = 0$.

59. If $k \ne -1$ then $\lim_{x\to 0}(k+\cos \ell x) = k+1 \ne 0$, so $\lim_{x\to 0}\frac{k+\cos\ell x}{x^2} = \pm\infty$. Hence $k=-1$, and by the rule

$$\lim_{x\to 0}\frac{-1+\cos\ell x}{x^2} = \lim_{x\to 0}\frac{-\ell\sin\ell x}{2x} = \lim_{x\to 0}\frac{-\ell^2\cos\ell x}{2} = -\frac{\ell^2}{2} = -4 \text{ if } \ell = \pm 2\sqrt{2}.$$

60. **(a)** Apply the rule to get $\lim_{x\to 0}\frac{-\cos(1/x)+2x\sin(1/x)}{\cos x}$ which does not exist (nor is it $\pm\infty$).

(b) Rewrite as $\lim_{x\to 0}\left[\frac{x}{\sin x}\right][x\sin(1/x)]$, but $\lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{1}{\cos x} = 1$ and $\lim_{x\to 0} x\sin(1/x) = 0$, thus $\lim_{x\to 0}\left[\frac{x}{\sin x}\right][x\sin(1/x)] = (1)(0) = 0$

61. $\lim_{x\to 0^+}\frac{\sin(1/x)}{(\sin x)/x}$, $\lim_{x\to 0^+}\frac{\sin x}{x} = 1$ but $\lim_{x\to 0^+}\sin(1/x)$ does not exist because $\sin(1/x)$ oscillates between -1 and 1 as $x \to +\infty$, so $\lim_{x\to 0^+}\frac{x\sin(1/x)}{\sin x}$ does not exist.

CHAPTER 4 SUPPLEMENTARY EXERCISES

1. **(a)** $f(g(x)) = x$ for all x in the domain of g, and $g(f(x)) = x$ for all x in the domain of f.

(b) They are reflections of each other through the line $y = x$.

(c) The domain of one is the range of the other and vice versa.

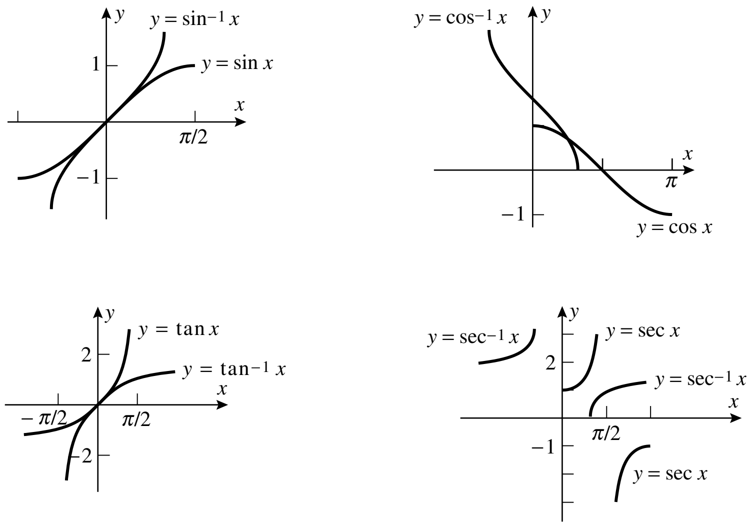
(d) The equation $y = f(x)$ can always be solved for x as a function of y. Functions with no inverses include $y = x^2$, $y = \sin x$.

(e) Yes, g is continuous; this is evident from the statement about the graphs in Part (b) above.

(f) Yes, g must be differentiable (where $f' \ne 0$); this can be inferred from the graphs. Note that if $f' = 0$ at a point then g' cannot exist (infinite slope).

2. **(a)** For $\sin x$, $-\pi/2 \le x \le \pi/2$; for $\cos x$, $0 \le x \le \pi$; for $\tan x$, $-\pi/2 < x < \pi/2$; for $\sec x$, $0 \le x < \pi/2$ or $\pi/2 < x \le \pi$.

(b)



3. **(a)** $x = f(y) = 8y^3 - 1$; $y = f^{-1}(x) = \left(\dfrac{x+1}{8}\right)^{1/3} = \dfrac{1}{2}(x+1)^{1/3}$

(b) $f(x) = (x-1)^2$; f does not have an inverse because f is not one-to-one, for example $f(0) = f(2) = 1$.

(c) $x = f(y) = (e^y)^2 + 1$; $y = f^{-1}(x) = \ln\sqrt{x-1} = \frac{1}{2}\ln(x-1)$

(d) $x = f(y) = \dfrac{y+2}{y-1}$; $y = f^{-1}(x) = \dfrac{x+2}{x-1}$

4. $f'(x) = \dfrac{ad-bc}{(cx+d)^2}$; if $ad - bc = 0$ then the function represents a horizontal line, no inverse. If $ad - bc \neq 0$ then $f'(x) > 0$ or $f'(x) < 0$ so f is invertible. If $x = f(y) = \dfrac{ay+b}{cy+d}$ then $y = f^{-1}(x) = \dfrac{b - xd}{xc - a}$.

5. $3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1) = 3\ln e^{2x} + 3\ln(e^x)^3 + 2\cdot 1 = 3(2x) + (3\cdot 3)x + 2 = 15x + 2$

6. Draw equilateral triangles of sides 5, 12, 13, and 3, 4, 5. Then $\sin[\cos^{-1}(4/5)] = 3/5$, $\sin[\cos^{-1}(5/13)] = 12/13$, $\cos[\sin^{-1}(4/5)] = 3/5$, $\cos[\sin^{-1}(5/13)] = 12/13$

(a)
$$\begin{aligned}\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)] &= \cos(\cos^{-1}(4/5))\cos(\sin^{-1}(5/13)) \\ &\quad - \sin(\cos^{-1}(4/5))\sin(\sin^{-1}(5/13)) \\ &= \frac{4}{5}\frac{12}{13} - \frac{3}{5}\frac{5}{13} = \frac{33}{65}.\end{aligned}$$

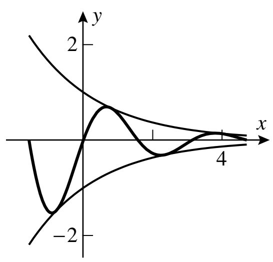
(b)
$$\begin{aligned}\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)] &= \sin(\sin^{-1}(4/5))\cos(\cos^{-1}(5/13)) \\ &\quad + \cos(\sin^{-1}(4/5))\sin(\cos^{-1}(5/13)) \\ &= \frac{4}{5}\frac{5}{13} + \frac{3}{5}\frac{12}{13} = \frac{56}{65}.\end{aligned}$$

7. **(a)** $f'(x) = -3/(x+1)^2$. If $x = f(y) = 3/(y+1)$ then $y = f^{-1}(x) = (3/x) - 1$, so $\dfrac{d}{dx}f^{-1}(x) = -\dfrac{3}{x^2}$; and $\dfrac{1}{f'(f^{-1}(x))} = -\dfrac{(f^{-1}(x)+1)^2}{3} = -\dfrac{(3/x)^2}{3} = -\dfrac{3}{x^2}$.

(b) $f(x) = e^{x/2}$, $f'(x) = \frac{1}{2}e^{x/2}$. If $x = f(y) = e^{y/2}$ then $y = f^{-1}(x) = 2\ln x$, so $\dfrac{d}{dx}f^{-1}(x) = \dfrac{2}{x}$; and $\dfrac{1}{f'(f^{-1}(x))} = 2e^{-f^{-1}(x)/2} = 2e^{-\ln x} = 2x^{-1} = \dfrac{2}{x}$

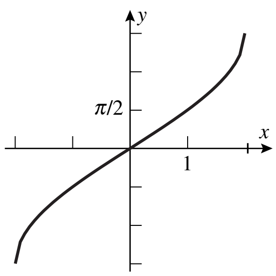
8. $Y = \ln(Ce^{kt}) = \ln C + \ln e^{kt} = \ln C + kt$, a line with slope k and Y-intercept $\ln C$

9. **(a)**

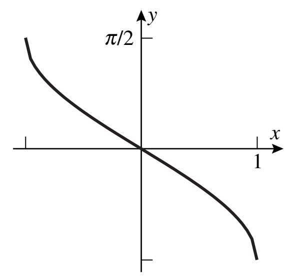


(b) The curve $y = e^{-x/2}\sin 2x$ has x-intercepts at $x = -\pi/2, 0, \pi/2, \pi, 3\pi/2$. It intersects the curve $y = e^{-x/2}$ at $x = \pi/4, 5\pi/4$, and it intersects the curve $y = -e^{-x/2}$ at $x = -\pi/4, 3\pi/4$.

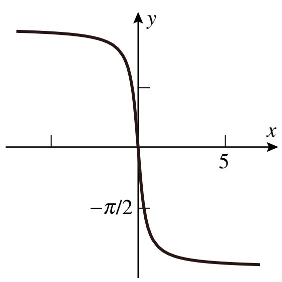
10. **(a)**



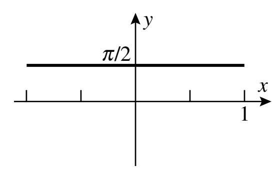
(b)



(c)



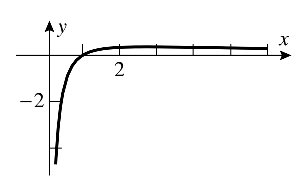
(d)



11. **(a)** The function $\ln x - x^{0.2}$ is negative at $x = 1$ and positive at $x = 4$, so it must be zero in between (IVT).

(b) $x = 3.654$

12. **(a)** If $x^k = e^x$ then $k\ln x = x$, or $\dfrac{\ln x}{x} = \dfrac{1}{k}$. The steps are reversible.



(b) By zooming it is seen that the maximum value of y is approximately 0.368 (actually, $1/e$), so there are two distinct solutions of $x^k = e^x$ whenever $k > 1/0.368 \approx 2.717$.

(c) $x \approx 1.155$

13. $\ln y = \ln 5000 + 1.07x$; $\dfrac{dy/dx}{y} = 1.07$, or $\dfrac{dy}{dx} = 1.07y$

14. $\ln y = 2x\ln 3 + 7x\ln 5$; $\dfrac{dy/dx}{y} = 2\ln 3 + 7\ln 5$, or $\dfrac{dy}{dx} = (2\ln 3 + 7\ln 5)y$

15. **(a)** $y = x^3 + 1$ so $y' = 3x^2$.

(b) $y' = \dfrac{abe^{-x}}{(1+be^{-x})^2}$

(c) $y = \dfrac{1}{2}\ln x + \dfrac{1}{3}\ln(x+1) - \ln\sin x + \ln\cos x$, so

$$y' = \frac{1}{2x} + \frac{1}{3(x+1)} - \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{5x+3}{6x(x+1)} - \cot x - \tan x.$$

(d) $\ln y = \dfrac{\ln(1+x)}{x}$, $\dfrac{y'}{y} = \dfrac{x/(1+x) - \ln(1+x)}{x^2} = \dfrac{1}{x(1+x)} - \dfrac{\ln(1+x)}{x^2}$,

$$\frac{dy}{dx} = \frac{1}{x}(1+x)^{(1/x)-1} - \frac{(1+x)^{(1/x)}}{x^2}\ln(1+x)$$

(e) $\ln y = e^x \ln x$, $\dfrac{y'}{y} = e^x\left(\dfrac{1}{x} + \ln x\right)$, $\dfrac{dy}{dx} = x^{e^x} e^x\left(\dfrac{1}{x} + \ln x\right) = e^x\left[x^{e^x-1} + x^{e^x}\ln x\right]$

(f) $y = \ln\dfrac{(1+e^x+e^{2x})}{(1-e^x)(1+e^x+e^{2x})} = -\ln(1-e^x)$, $\dfrac{dy}{dx} = \dfrac{e^x}{1-e^x}$

16. $y' = ae^{ax}\sin bx + be^{ax}\cos bx$ and $y'' = (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx$, so $y'' - 2ay' + (a^2+b^2)y$
$= (a^2-b^2)e^{ax}\sin bx + 2abe^{ax}\cos bx - 2a(ae^{ax}\sin bx + be^{ax}\cos bx) + (a^2+b^2)e^{ax}\sin bx = 0.$

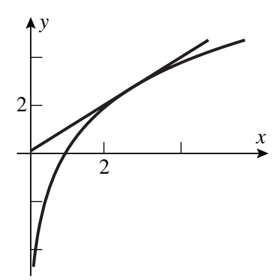
17. $\sin(\tan^{-1}x) = x/\sqrt{1+x^2}$ and $\cos(\tan^{-1}x) = 1/\sqrt{1+x^2}$, and $y' = \dfrac{1}{1+x^2}$, $y'' = \dfrac{-2x}{(1+x^2)^2}$, hence

$$y'' + 2\sin y\cos^3 y = \frac{-2x}{(1+x^2)^2} + 2\frac{x}{\sqrt{1+x^2}}\frac{1}{(1+x^2)^{3/2}} = 0.$$

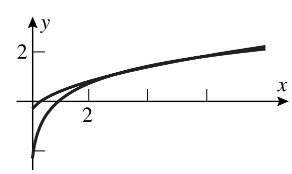
18. **(a)** Find x when $y = 5\cdot 12 = 60$ in. Since $y = \log x$, $x = 10^y = 10^{60}$ in. This is approximately 2.68×10^{42} light-years, so even in astronomical terms it is a fabulously long distance.

(b) Find x when $y = 100(5280)(12)$ in. Since $y = 10^x$, $x = \log y = 6.80$ in or 0.57 ft, approximately.

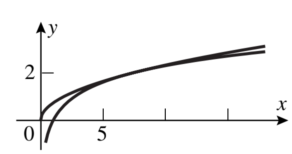
19. Set $y = \log_b x$ and solve $y' = 1$: $y' = \dfrac{1}{x\ln b} = 1$ so $x = \dfrac{1}{\ln b}$. The curves intersect when (x, x) lies on the graph of $y = \log_b x$, so $x = \log_b x$. From Formula (9), Section 4.2, $\log_b x = \dfrac{\ln x}{\ln b}$ from which $\ln x = 1$, $x = e$, $\ln b = 1/e$, $b = e^{1/e} \approx 1.4447$.



20. **(a)** Find the point of intersection: $f(x) = \sqrt{x} + k = \ln x$. The slopes are equal, so $m_1 = \dfrac{1}{x} = m_2 = \dfrac{1}{2\sqrt{x}}$, $\sqrt{x} = 2$, $x = 4$. Then $\ln 4 = \sqrt{4} + k$, $k = \ln 4 - 2$.



(b) Since the slopes are equal $m_1 = \dfrac{k}{2\sqrt{x}} = m_2 = \dfrac{1}{x}$, so $k\sqrt{x} = 2$. At the point of intersection $k\sqrt{x} = \ln x$, $2 = \ln x$, $x = e^2$, $k = 2/e$.



21. Solve $\dfrac{dy}{dt} = 3\dfrac{dx}{dt}$ given $y = x\ln x$. Then $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = (1+\ln x)\dfrac{dx}{dt}$, so $1+\ln x = 3$, $\ln x = 2$, $x = e^2$.

22. Let $P(x_0, y_0)$ be a point on $y = e^{3x}$ then $y_0 = e^{3x_0}$. $dy/dx = 3e^{3x}$ so $m_{\tan} = 3e^{3x_0}$ at P and an equation of the tangent line at P is $y - y_0 = 3e^{3x_0}(x - x_0)$, $y - e^{3x_0} = 3e^{3x_0}(x - x_0)$. If the line passes through the origin then $(0,0)$ must satisfy the equation so $-e^{3x_0} = -3x_0e^{3x_0}$ which gives $x_0 = 1/3$ and thus $y_0 = e$. The point is $(1/3, e)$.

23. $\dfrac{dk}{dT} = k_0 \exp\left[-\dfrac{q(T-T_0)}{2T_0T}\right]\left(-\dfrac{q}{2T^2}\right) = -\dfrac{qk_0}{2T^2}\exp\left[-\dfrac{q(T-T_0)}{2T_0T}\right]$

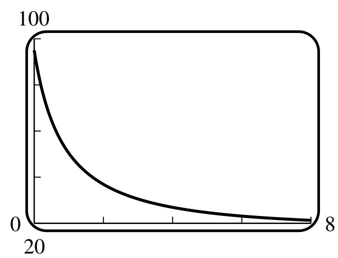
24. $\beta = 10\log I - 10\log I_0$, $\dfrac{d\beta}{dI} = \dfrac{10}{I\ln 10}$

(a) $\left.\dfrac{d\beta}{dI}\right]_{I=10I_0} = \dfrac{1}{I_0\ln 10}$ db/W/m^2

(b) $\left.\dfrac{d\beta}{dI}\right]_{I=100I_0} = \dfrac{1}{10I_0\ln 10}$ db/W/m^2

(c) $\left.\dfrac{d\beta}{dI}\right]_{I=100I_0} = \dfrac{1}{100I_0\ln 10}$ db/W/m^2

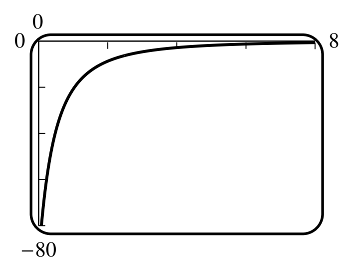
25. **(a)**



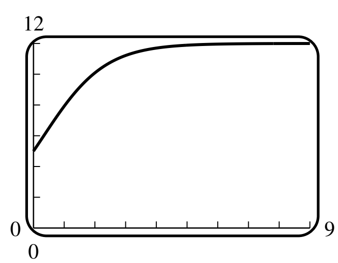
(b) as t tends to $+\infty$, the population tends to 19

$$\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty}\frac{95}{5-4e^{-t/4}} = \frac{95}{5-4\displaystyle\lim_{t\to+\infty} e^{-t/4}} = \frac{95}{5} = 19$$

(c) the rate of population growth tends to zero

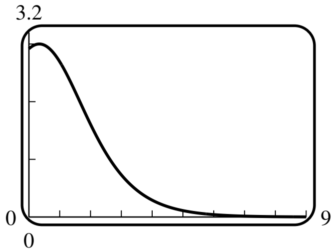


26. **(a)**

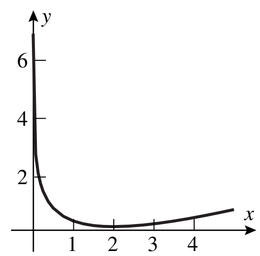


(b) P tends to 12 as t gets large; $\displaystyle\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty}\frac{60}{5+7e^{-t}} = \frac{60}{5+7\displaystyle\lim_{t\to+\infty} e^{-t}} = \frac{60}{5} = 12$

(c) the rate of population growth tends to zero



27. **(b)**



(c) $\dfrac{dy}{dx} = \dfrac{1}{2} - \dfrac{1}{x}$ so $\dfrac{dy}{dx} < 0$ at $x = 1$ and $\dfrac{dy}{dx} > 0$ at $x = e$

(d) The slope is a continuous function which goes from a negative value to a positive value; therefore it must take the value zero in between, by the Intermediate Value Theorem.

(e) $\dfrac{dy}{dx} = 0$ when $x = 2$

28. In the case $+\infty - (-\infty)$ the limit is $+\infty$; in the case $-\infty - (+\infty)$ the limit is $-\infty$, because large positive (negative) quantities are added to large positive (negative) quantities. The cases $+\infty - (+\infty)$ and $-\infty - (-\infty)$ are indeterminate; large numbers of opposite sign are subtracted, and more information about the sizes is needed.

29. **(a)** when the limit takes the form $0/0$ or ∞/∞

(b) Not necessarily; only if $\lim_{x\to a} f(x) = 0$. Consider $g(x) = x$; $\lim_{x\to 0} g(x) = 0$. For $f(x)$ choose $\cos x$, x^2, and $|x|^{1/2}$. Then: $\lim_{x\to 0} \dfrac{\cos x}{x}$ does not exist, $\lim_{x\to 0} \dfrac{x^2}{x} = 0$, and $\lim_{x\to 0} \dfrac{|x|^{1/2}}{x^2} = +\infty$.

30. **(a)** $\lim_{x\to+\infty}(e^x - x^2) = \lim_{x\to+\infty} x^2(e^x/x^2 - 1)$, but $\lim_{x\to+\infty}\dfrac{e^x}{x^2} = \lim_{x\to+\infty}\dfrac{e^x}{2x} = \lim_{x\to+\infty}\dfrac{e^x}{2} = +\infty$
so $\lim_{x\to+\infty}(e^x/x^2 - 1) = +\infty$ and thus $\lim_{x\to+\infty} x^2(e^x/x^2 - 1) = +\infty$

(b) $\lim_{x\to 1}\dfrac{\ln x}{x^4 - 1} = \lim_{x\to 1}\dfrac{1/x}{4x^3} = \dfrac{1}{4}$; $\lim_{x\to 1}\sqrt{\dfrac{\ln x}{x^4 - 1}} = \sqrt{\lim_{x\to 1}\dfrac{\ln x}{x^4 - 1}} = \dfrac{1}{2}$

(c) $\lim_{x\to 0} a^x \ln a = \ln a$