CHAPTER 6
Integration

EXERCISE SET 6.1

1. Endpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$; using right endpoints,

$$A_n = \left[\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \cdots + \sqrt{\frac{n-1}{n}} + 1\right]\frac{1}{n}$$

n	2	5	10	50	100
A_n	0.853553	0.749739	0.710509	0.676095	0.671463

2. Endpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$; using right endpoints,

$$A_n = \left[\frac{n}{n+1} + \frac{n}{n+2} + \frac{n}{n+3} + \cdots + \frac{n}{2n-1} + \frac{1}{2}\right]\frac{1}{n}$$

n	2	5	10	50	100
A_n	0.583333	0.645635	0.668771	0.688172	0.690653

3. Endpoints $0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{(n-1)\pi}{n}, \pi$; using right endpoints,

$$A_n = [\sin(\pi/n) + \sin(2\pi/n) + \cdots + \sin(\pi(n-1)/n) + \sin\pi]\,\frac{\pi}{n}$$

n	2	5	10	50	100
A_n	1.57080	1.93376	1.98352	1.99935	1.99984

4. Endpoints $0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \ldots, \frac{(n-1)\pi}{2n}, \frac{\pi}{2}$; using right endpoints,

$$A_n = [\cos(\pi/2n) + \cos(2\pi/2n) + \cdots + \cos((n-1)\pi/2n) + \cos(\pi/2)]\,\frac{\pi}{2n}$$

n	2	5	10	50	100
A_n	0.555359	0.834683	0.919405	0.984204	0.992120

5. Endpoints $1, \frac{n+1}{n}, \frac{n+2}{n}, \ldots, \frac{2n-1}{n}, 2$; using right endpoints,

$$A_n = \left[\frac{n}{n+1} + \frac{n}{n+2} + \cdots + \frac{n}{2n-1} + \frac{1}{2}\right]\frac{1}{n}$$

n	2	5	10	50	100
A_n	0.583333	0.645635	0.668771	0.688172	0.690653

6. Endpoints $-\frac{\pi}{2}, -\frac{\pi}{2} + \frac{\pi}{n}, -\frac{\pi}{2} + \frac{2\pi}{n}, \ldots, -\frac{\pi}{2} + \frac{(n-1)\pi}{n}, \frac{\pi}{2}$; using right endpoints,

$$A_n = \left[\cos\left(-\frac{\pi}{2} + \frac{\pi}{n}\right) + \cos\left(-\frac{\pi}{2} + \frac{2\pi}{n}\right) + \cdots + \cos\left(-\frac{\pi}{2} + \frac{(n-1)\pi}{n}\right) + \cos\left(\frac{\pi}{2}\right)\right]\frac{\pi}{n}$$

n	2	5	10	50	100
A_n	1.99985	1.93376	1.98352	1.99936	1.99985

7. Endpoints $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1$; using right endpoints,

$$A_n = \left[\sqrt{1-\left(\frac{1}{n}\right)^2} + \sqrt{1-\left(\frac{2}{n}\right)^2} + \cdots + \sqrt{1-\left(\frac{n-1}{n}\right)^2} + 0\right]\frac{1}{n}$$

n	2	5	10	50	100
A_n	0.433013	0.659262	0.726130	0.774567	0.780106

8. Endpoints $-1, -1+\frac{2}{n}, -1+\frac{4}{n}, \ldots, -1+\frac{2(n-1)}{n}, 1$; using right endpoints,

$$A_n = \left[\sqrt{1-\left(\frac{n-2}{n}\right)^2} + \sqrt{1-\left(\frac{n-4}{n}\right)^2} + \cdots + \sqrt{1-\left(\frac{n-2}{n}\right)^2} + 0\right]\frac{2}{n}$$

n	2	5	10	50	100
A_n	1	1.423837	1.518524	1.566097	1.569136

9. $3(x-1)$ **10.** $5(x-2)$ **11.** $x(x+2)$ **12.** $\frac{3}{2}(x-1)^2$

13. $(x+3)(x-1)$ **14.** $\frac{3}{2}x(x-2)$

15. The area in Exercise 13 is always 3 less than the area in Exercise 11. The regions are identical except that the area in Exercise 11 has the extra trapezoid with vertices at $(0,0), (1,0), (0,2), (1,4)$ (with area 3).

16. **(a)** The region in question is a trapezoid, and the area of a trapezoid is $\frac{1}{2}(h_1+h_2)w$.

(b) From Part (a), $A'(x) = \frac{1}{2}[f(a)+f(x)] + (x-a)\frac{1}{2}f'(x)$

$$= \frac{1}{2}[f(a)+f(x)] + (x-a)\frac{1}{2}\frac{f(x)-f(a)}{x-a} = f(x)$$

17. B is also the area between the graph of $f(x)=\sqrt{x}$ and the interval $[0,1]$ on the $y-$axis, so $A+B$ is the area of the square.

18. If the plane is rotated about the line $y=x$ then A becomes B and vice versa.

EXERCISE SET 6.2

1. **(a)** $\int \frac{x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}+C$ **(b)** $\int (x+1)e^x dx = xe^x + C$

2. **(a)** $\frac{d}{dx}(\sin x - x\cos x + C) = \cos x - \cos x + x\sin x = x\sin x$

(b) $\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}+C\right) = \frac{\sqrt{1-x^2}+x^2/\sqrt{1-x^2}}{1-x^2} = \frac{1}{(1-x^2)^{3/2}}$

3. $\frac{d}{dx}\left[\sqrt{x^3+5}\right] = \frac{3x^2}{2\sqrt{x^3+5}}$ so $\int \frac{3x^2}{2\sqrt{x^3+5}}dx = \sqrt{x^3+5}+C$

4. $\dfrac{d}{dx}\left[\dfrac{x}{x^2+3}\right] = \dfrac{3-x^2}{(x^2+3)^2}$ so $\displaystyle\int \frac{3-x^2}{(x^2+3)^2}dx = \frac{x}{x^2+3} + C$

5. $\dfrac{d}{dx}\left[\sin\left(2\sqrt{x}\right)\right] = \dfrac{\cos\left(2\sqrt{x}\right)}{\sqrt{x}}$ so $\displaystyle\int \frac{\cos\left(2\sqrt{x}\right)}{\sqrt{x}}dx = \sin\left(2\sqrt{x}\right) + C$

6. $\dfrac{d}{dx}[\sin x - x\cos x] = x\sin x$ so $\displaystyle\int x\sin x\,dx = \sin x - x\cos x + C$

7. **(a)** $x^9/9 + C$ **(b)** $\dfrac{7}{12}x^{12/7} + C$ **(c)** $\dfrac{2}{9}x^{9/2} + C$

8. **(a)** $\dfrac{3}{5}x^{5/3} + C$ **(b)** $-\dfrac{1}{5}x^{-5} + C = -\dfrac{1}{5x^5} + C$ **(c)** $8x^{1/8} + C$

9. **(a)** $\displaystyle\frac{1}{2}\int x^{-3}dx = -\frac{1}{4}x^{-2} + C$ **(b)** $u^4/4 - u^2 + 7u + C$

10. $\dfrac{3}{5}x^{5/3} - 5x^{4/5} + 4x + C$

11. $\displaystyle\int(x^{-3} + x^{1/2} - 3x^{1/4} + x^2)dx = -\frac{1}{2}x^{-2} + \frac{2}{3}x^{3/2} - \frac{12}{5}x^{5/4} + \frac{1}{3}x^3 + C$

12. $\displaystyle\int(7y^{-3/4} - y^{1/3} + 4y^{1/2})dy = 28y^{1/4} - \frac{3}{4}y^{4/3} + \frac{8}{3}y^{3/2} + C$

13. $\displaystyle\int(x + x^4)dx = x^2/2 + x^5/5 + C$

14. $\displaystyle\int(4 + 4y^2 + y^4)dy = 4y + \frac{4}{3}y^3 + \frac{1}{5}y^5 + C$

15. $\displaystyle\int x^{1/3}(4 - 4x + x^2)dx = \int(4x^{1/3} - 4x^{4/3} + x^{7/3})dx = 3x^{4/3} - \frac{12}{7}x^{7/3} + \frac{3}{10}x^{10/3} + C$

16. $\displaystyle\int(2 - x + 2x^2 - x^3)dx = 2x - \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4 + C$

17. $\displaystyle\int(x + 2x^{-2} - x^{-4})dx = x^2/2 - 2/x + 1/(3x^3) + C$

18. $\displaystyle\int(t^{-3} - 2)dt = -\frac{1}{2}t^{-2} - 2t + C$

19. $\displaystyle\int\left[\frac{2}{x} + 3e^x\right]dx = 2\ln|x| + 3e^x + C$

20. $\displaystyle\int\left[\frac{1}{2}t^{-1} - \sqrt{2}e^t\right]dt = \frac{1}{2}\ln|t| - \sqrt{2}e^t + C$

21. $-4\cos x + 2\sin x + C$ **22.** $4\tan x - \csc x + C$

23. $\displaystyle\int(\sec^2 x + \sec x\tan x)dx = \tan x + \sec x + C$

24. $\int(\sec x\tan x+1)dx=\sec x+x+C$

25. $\int\frac{\sec\theta}{\cos\theta}\,d\theta=\int\sec^2\theta\,d\theta=\tan\theta+C$

26. $\int\sin y\,dy=-\cos y+C$

27. $\int\sec x\tan x\,dx=\sec x+C$

28. $\int(\phi+2\csc^2\phi)d\phi=\phi^2/2-2\cot\phi+C$

29. $\int(1+\sin\theta)d\theta=\theta-\cos\theta+C$

30. $\int\frac{2\sin x\cos x}{\cos x}dx=2\int\sin x\,dx=-2\cos x+C$

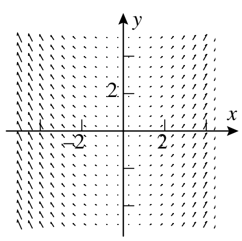
31. $\int\left[\frac{1}{2\sqrt{1-x^2}}-\frac{3}{1+x^2}\right]dx=\frac{1}{2}\sin^{-1}x-3\tan^{-1}x+C$

32. $\int\left[\frac{4}{x\sqrt{x^2-1}}+\frac{1+x+x^3}{1+x^2}\right]dx=4\sec^{-1}x+\int\left(x+\frac{1}{x^2+1}\right)dx=4\sec^{-1}x+\frac{1}{2}x^2+\tan^{-1}x+C$

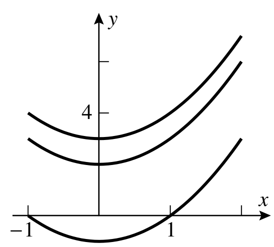
33. $\int\frac{1-\sin x}{1-\sin^2 x}dx=\int\frac{1-\sin x}{\cos^2 x}dx=\int\left(\sec^2 x-\sec x\tan x\right)dx=\tan x-\sec x+C$

34. $\int\frac{1}{1+\cos 2x}dx=\int\frac{1}{2\cos^2 x}dx=\int\frac{1}{2}\sec^2 x\,dx=\frac{1}{2}\tan x+C$

35. **(a)**

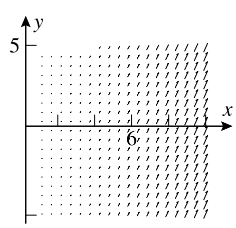


(b)

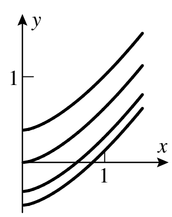


(c) $f(x)=x^2/2-1$

36. **(a)**

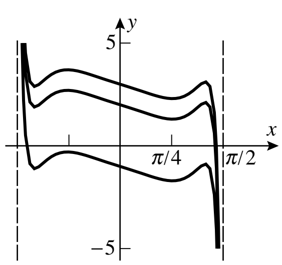


(b)

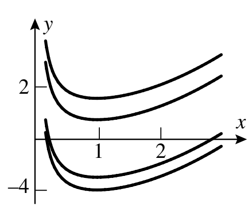


(c) $y=(e^x+1)/2$

37.



38.



39. $f'(x)=m=-\sin x$ so $f(x)=\int(-\sin x)dx=\cos x+C$; $f(0)=2=1+C$ so $C=1$, $f(x)=\cos x+1$

40. $f'(x) = m = (x+1)^2$, so $f(x) = \int (x+1)^2 dx = \frac{1}{3}(x+1)^3 + C$;
$f(-2) = 8 = \frac{1}{3}(-2+1)^3 + C = -\frac{1}{3} + C, = 8 + \frac{1}{3} = \frac{25}{3},\ f(x) = \frac{1}{3}(x+1)^3 + \frac{25}{3}$

41. **(a)** $y(x) = \int x^{1/3} dx = \frac{3}{4}x^{4/3} + C,\ y(1) = \frac{3}{4} + C = 2, C = \frac{5}{4};\ y(x) = \frac{3}{4}x^{4/3} + \frac{5}{4}$

(b) $y(t) = \int (\sin t + 1)\, dt = -\cos t + t + C,\ y\left(\frac{\pi}{3}\right) = -\frac{1}{2} + \frac{\pi}{3} + C = 1/2,\ C = 1 - \frac{\pi}{3}$;
$y(t) = -\cos t + t + 1 - \frac{\pi}{3}$

(c) $y(x) = \int (x^{1/2} + x^{-1/2}) dx = \frac{2}{3}x^{3/2} + 2x^{1/2} + C,\ y(1) = 0 = \frac{8}{3} + C,\ C = -\frac{8}{3}$,
$y(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} - \frac{8}{3}$

42. **(a)** $y(x) = \int \left(\frac{1}{8}x^{-3}\right) dx = -\frac{1}{16}x^{-2} + C,\ y(1) = 0 = -\frac{1}{16} + C,\ C = \frac{1}{16};\ y(x) = -\frac{1}{16}x^{-2} + \frac{1}{16}$

(b) $y(t) = \int (\sec^2 t - \sin t)\, dt = \tan t + \cos t + C,\ y(\frac{\pi}{4}) = 1 = 1 + \frac{\sqrt{2}}{2} + C,\ C = -\frac{\sqrt{2}}{2}$;
$y(t) = \tan t + \cos t - \frac{\sqrt{2}}{2}$

(c) $y(x) = \int x^{7/2} dx = \frac{2}{9}x^{9/2} + C,\ y(0) = 0 = C,\ C = 0;\ y(x) = \frac{2}{9}x^{9/2}$

43. **(a)** $y = \int 4e^x\, dx = 4e^x + C, 1 = y(0) = 4 + C, C = -3, y = 4e^x - 3$

(b) $y(t) = \int t^{-1} dt = \ln|t| + C,\ y(-1) = C = 5,\ C = 5;\ y(t) = \ln|t| + 5$

44. **(a)** $y = \int \frac{3}{\sqrt{1-t^2}}\, dt = 3\sin^{-1} t + C, y\left(\frac{\sqrt{3}}{2}\right) = 0 = \pi + C, C = -\pi, y = 3\sin^{-1} t - \pi$

(b) $\frac{dy}{dx} = 1 - \frac{2}{x^2+1}, y = \int \left[1 - \frac{2}{x^2+1}\right] dx = x - 2\tan^{-1} x + C$,
$y(1) = \frac{\pi}{2} = 1 - 2\frac{\pi}{4} + C, C = \pi - 1, y = x - 2\tan^{-1} x + \pi - 1$

45. $f'(x) = \frac{2}{3}x^{3/2} + C_1;\ f(x) = \frac{4}{15}x^{5/2} + C_1 x + C_2$

46. $f'(x) = x^2/2 + \sin x + C_1$, use $f'(0) = 2$ to get $C_1 = 2$ so $f'(x) = x^2/2 + \sin x + 2$,
$f(x) = x^3/6 - \cos x + 2x + C_2$, use $f(0) = 1$ to get $C_2 = 2$ so $f(x) = x^3/6 - \cos x + 2x + 2$

47. $dy/dx = 2x + 1, y = \int (2x+1) dx = x^2 + x + C;\ y = 0$ when $x = -3$
so $(-3)^2 + (-3) + C = 0, C = -6$ thus $y = x^2 + x - 6$

48. $dy/dx = x^2, y = \int x^2 dx = x^3/3 + C;\ y = 2$ when $x = -1$ so $(-1)^3/3 + C = 2, C = 7/3$
thus $y = x^3/3 + 7/3$

49. $dy/dx = \int 6x\,dx = 3x^2 + C_1$. The slope of the tangent line is -3 so $dy/dx = -3$ when $x = 1$. Thus $3(1)^2 + C_1 = -3$, $C_1 = -6$ so $dy/dx = 3x^2 - 6$, $y = \int (3x^2 - 6)dx = x^3 - 6x + C_2$. If $x = 1$, then $y = 5 - 3(1) = 2$ so $(1)^2 - 6(1) + C_2 = 2, C_2 = 7$ thus $y = x^3 - 6x + 7$.

50. $dT/dx = C_1$, $T = C_1 x + C_2$; $T = 25$ when $x = 0$ so $C_2 = 25$, $T = C_1 x + 25$. $T = 85$ when $x = 50$ so $50C_1 + 25 = 85$, $C_1 = 1.2$, $T = 1.2x + 25$

51. **(a)** $F'(x) = G'(x) = 3x + 4$
(b) $F(0) = 16/6 = 8/3$, $G(0) = 0$, so $F(0) - G(0) = 8/3$
(c) $F(x) = (9x^2 + 24x + 16)/6 = 3x^2/2 + 4x + 8/3 = G(x) + 8/3$

52. **(a)** $F'(x) = G'(x) = 10x/(x^2 + 5)^2$
(b) $F(0) = 0$, $G(0) = -1$, so $F(0) - G(0) = 1$
(c) $F(x) = \dfrac{x^2}{x^2 + 5} = \dfrac{(x^2 + 5) - 5}{x^2 + 5} = 1 - \dfrac{5}{x^2 + 5} = G(x) + 1$

53. $\int (\sec^2 x - 1)dx = \tan x - x + C$

54. $\int (\csc^2 x - 1)dx = -\cot x - x + C$

55. **(a)** $\frac{1}{2}\int (1 - \cos x)dx = \frac{1}{2}(x - \sin x) + C$ **(b)** $\frac{1}{2}\int (1 + \cos x)\,dx = \frac{1}{2}(x + \sin x) + C$

56. **(a)** $F'(x) = G'(x) = f(x)$, where $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

(b) $G(x) - F(x) = \begin{cases} 2, & x > 0 \\ 3, & x < 0 \end{cases}$ so $G(x) \neq F(x)$ plus a constant

(c) no, because $(-\infty, 0) \cup (0, +\infty)$ is not an interval

57. $v = \dfrac{1087}{2\sqrt{273}}\int T^{-1/2}\,dT = \dfrac{1087}{\sqrt{273}}T^{1/2} + C$, $v(273) = 1087 = 1087 + C$ so $C = 0$, $v = \dfrac{1087}{\sqrt{273}}T^{1/2}$ ft/s

EXERCISE SET 6.3

1. **(a)** $\int u^{23}du = u^{24}/24 + C = (x^2 + 1)^{24}/24 + C$

(b) $-\int u^3 du = -u^4/4 + C = -(\cos^4 x)/4 + C$

(c) $2\int \sin u\,du = -2\cos u + C = -2\cos\sqrt{x} + C$

(d) $\frac{3}{8}\int u^{-1/2}du = \frac{3}{4}u^{1/2} + C = \frac{3}{4}\sqrt{4x^2 + 5} + C$

2. **(a)** $\frac{1}{4}\int \sec^2 u\,du = \frac{1}{4}\tan u + C = \frac{1}{4}\tan(4x + 1) + C$

(b) $\frac{1}{4}\int u^{1/2}du = \frac{1}{6}u^{3/2} + C = \frac{1}{6}(1 + 2y^2)^{3/2} + C$

(c) $\frac{1}{\pi}\int u^{1/2}du = \frac{2}{3\pi}u^{3/2} + C = \frac{2}{3\pi}\sin^{3/2}(\pi\theta) + C$

(d) $\int u^{4/5}du = \frac{5}{9}u^{9/5} + C = \frac{5}{9}(x^2 + 7x + 3)^{9/5} + C$

3. **(a)** $-\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cot^2 x + C$

(b) $\int u^9 du = \frac{1}{10}u^{10} + C = \frac{1}{10}(1+\sin t)^{10} + C$

(c) $\frac{1}{2}\int \cos u\,du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin 2x + C$

(d) $\frac{1}{2}\int \sec^2 u\,du = \frac{1}{2}\tan u + C = \frac{1}{2}\tan x^2 + C$

4. **(a)** $\int (u-1)^2 u^{1/2} du = \int (u^{5/2} - 2u^{3/2} + u^{1/2})du = \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C$

$$= \frac{2}{7}(1+x)^{7/2} - \frac{4}{5}(1+x)^{5/2} + \frac{2}{3}(1+x)^{3/2} + C$$

(b) $\int \csc^2 u\,du = -\cot u + C = -\cot(\sin x) + C$

(c) $\int \sin u\,du = -\cos u + C = -\cos(x-\pi) + C$

(d) $\int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{x^5+1} + C$

5. **(a)** $\int \frac{1}{u}du = \ln|u| + C = \ln|\ln x| + C$

(b) $-\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{-5x} + C$

(c) $-\frac{1}{3}\int \frac{1}{u}du = -\frac{1}{3}\ln|u| + C = -\frac{1}{3}\ln|1+\cos 3\theta| + C$

(d) $\int \frac{du}{u} = \ln u + C = \ln(1+e^x) + C$

6. **(a)** $u = x^3, \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1}(x^3) + C$

(b) $u = \ln x, \int \frac{1}{\sqrt{1-u^2}}du = \sin^{-1}(\ln x) + C$

(c) $u = 3x, \int \frac{1}{u\sqrt{u^2-1}}du = \sec^{-1}(3x) + C$

(d) $u = \sqrt{x}, 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}(\sqrt{x}) + C$

7. $u = 2-x^2,\ du = -2x\,dx;\ -\frac{1}{2}\int u^3 du = -u^4/8 + C = -(2-x^2)^4/8 + C$

8. $u = 3x-1,\ du = 3dx;\ \frac{1}{3}\int u^5 du = \frac{1}{18}u^6 + C = \frac{1}{18}(3x-1)^6 + C$

9. $u = 8x,\ du = 8dx;\ \frac{1}{8}\int \cos u\,du = \frac{1}{8}\sin u + C = \frac{1}{8}\sin 8x + C$

10. $u = 3x,\ du = 3dx;\ \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C$

11. $u = 4x,\ du = 4dx;\ \dfrac{1}{4}\displaystyle\int \sec u \tan u\, du = \dfrac{1}{4}\sec u + C = \dfrac{1}{4}\sec 4x + C$

12. $u = 5x,\ du = 5dx;\ \dfrac{1}{5}\displaystyle\int \sec^2 u\, du = \dfrac{1}{5}\tan u + C = \dfrac{1}{5}\tan 5x + C$

13. $u = 2x,\ du = 2dx;\ \dfrac{1}{2}\displaystyle\int e^u\, du = \dfrac{1}{2}e^u + C = \dfrac{1}{2}e^{2x} + C$

14. $u = 2x,\ du = 2dx;\ \dfrac{1}{2}\displaystyle\int \dfrac{1}{u}du = \dfrac{1}{2}\ln|u| + C = \dfrac{1}{2}\ln|2x| + C$

15. $u = 2x,\ \dfrac{1}{2}\displaystyle\int \dfrac{1}{\sqrt{1-u^2}}du = \dfrac{1}{2}\sin^{-1}(2x) + C$

16. $u = 4x,\ \dfrac{1}{4}\displaystyle\int \dfrac{1}{1+u^2}du = \dfrac{1}{4}\tan^{-1}(4x) + C$

17. $u = 7t^2 + 12,\ du = 14t\,dt;\ \dfrac{1}{14}\displaystyle\int u^{1/2}du = \dfrac{1}{21}u^{3/2} + C = \dfrac{1}{21}(7t^2+12)^{3/2} + C$

18. $u = 4 - 5x^2,\ du = -10x\,dx;\ -\dfrac{1}{10}\displaystyle\int u^{-1/2}du = -\dfrac{1}{5}u^{1/2} + C = -\dfrac{1}{5}\sqrt{4-5x^2} + C$

19. $u = x^3 + 1,\ du = 3x^2dx;\ \dfrac{1}{3}\displaystyle\int u^{-1/2}du = \dfrac{2}{3}u^{1/2} + C = \dfrac{2}{3}\sqrt{x^3+1} + C$

20. $u = 1 - 3x,\ du = -3dx;\ -\dfrac{1}{3}\displaystyle\int u^{-2}du = \dfrac{1}{3}u^{-1} + C = \dfrac{1}{3}(1-3x)^{-1} + C$

21. $u = 4x^2 + 1,\ du = 8x\,dx;\ \dfrac{1}{8}\displaystyle\int u^{-3}du = -\dfrac{1}{16}u^{-2} + C = -\dfrac{1}{16}(4x^2+1)^{-2} + C$

22. $u = 3x^2,\ du = 6x\,dx;\ \dfrac{1}{6}\displaystyle\int \cos u\, du = \dfrac{1}{6}\sin u + C = \dfrac{1}{6}\sin(3x^2) + C$

23. $u = \sin x,\ du = \cos x\,dx;\ \displaystyle\int e^u\, du = e^u + C = e^{\sin x} + C$

24. $u = x^4,\ du = 4x^3dx;\ \dfrac{1}{4}\displaystyle\int e^u\, du = \dfrac{1}{4}e^u + C = \dfrac{1}{4}e^{x^4} + C$

25. $u = -2x^3,\ du = -6x^2,\ -\dfrac{1}{6}\displaystyle\int e^u du = -\dfrac{1}{6}e^u + C = -\dfrac{1}{6}e^{-2x^3} + C$

26. $u = e^x - e^{-x},\ du = (e^x + e^{-x})dx,\ \displaystyle\int \dfrac{1}{u}du = \ln|u| + C = \ln\left|e^x - e^{-x}\right| + C$

27. $u = e^x,\ \displaystyle\int \dfrac{1}{1+u^2}du = \tan^{-1}(e^x) + C$

28. $u = t^2,\ \dfrac{1}{2}\displaystyle\int \dfrac{1}{u^2+1}du = \dfrac{1}{2}\tan^{-1}(t^2) + C$

29. $u = 5/x,\ du = -(5/x^2)dx;\ -\dfrac{1}{5}\displaystyle\int \sin u\, du = \dfrac{1}{5}\cos u + C = \dfrac{1}{5}\cos(5/x) + C$

30. $u = \sqrt{x},\ du = \dfrac{1}{2\sqrt{x}}dx;\ \ 2\displaystyle\int \sec^2 u\, du = 2\tan u + C = 2\tan\sqrt{x} + C$

31. $u = x^3,\ du = 3x^2dx;\ \ \frac{1}{3}\int \sec^2 u\,du = \frac{1}{3}\tan u + C = \frac{1}{3}\tan(x^3) + C$

32. $u = \cos 2t,\ du = -2\sin 2t\,dt;\ -\frac{1}{2}\int u^3 du = -\frac{1}{8}u^4 + C = -\frac{1}{8}\cos^4 2t + C$

33. $u = \sin 3t,\ du = 3\cos 3t\,dt;\ \frac{1}{3}\int u^5 du = \frac{1}{18}u^6 + C = \frac{1}{18}\sin^6 3t + C$

34. $u = 5 + \cos 2\theta,\ du = -2\sin 2\theta\,d\theta;\ -\frac{1}{2}\int u^{-3}du = \frac{1}{4}u^{-2} + C = \frac{1}{4}(5+\cos 2\theta)^{-2} + C$

35. $u = 2 - \sin 4\theta,\ du = -4\cos 4\theta\,d\theta;\ -\frac{1}{4}\int u^{1/2}du = -\frac{1}{6}u^{3/2} + C = -\frac{1}{6}(2-\sin 4\theta)^{3/2} + C$

36. $u = \tan 5x,\ du = 5\sec^2 5x\,dx;\ \frac{1}{5}\int u^3 du = \frac{1}{20}u^4 + C = \frac{1}{20}\tan^4 5x + C$

37. $u = \tan x,\ \int \frac{1}{\sqrt{1-u^2}}\,du = \sin^{-1}(\tan x) + C$

38. $u = \cos\theta,\ -\int \frac{1}{u^2+1}du = -\tan^{-1}(\cos\theta) + C$

39. $u = \sec 2x,\ du = 2\sec 2x\tan 2x\,dx;\ \frac{1}{2}\int u^2 du = \frac{1}{6}u^3 + C = \frac{1}{6}\sec^3 2x + C$

40. $u = \sin\theta,\ du = \cos\theta\,d\theta;\ \int \sin u\,du = -\cos u + C = -\cos(\sin\theta) + C$

41. $\int e^{-x}dx;\ u = -x,\ du = -dx;\ -\int e^u du = -e^u + C = -e^{-x} + C$

42. $\int e^{x/2}dx;\ u = x/2,\ du = dx/2;\ 2\int e^u du = 2e^u + C = 2e^{x/2} + C = 2\sqrt{e^x} + C$

43. $u = \sqrt{y+1},\ du = \frac{1}{2\sqrt{y+1}}dy,\ 2\int e^u du = 2e^u + C = 2e^{\sqrt{y+1}} + C$

44. $u = \sqrt{y},\ du = \frac{1}{2\sqrt{y}}dy,\ 2\int \frac{1}{e^u}du = 2\int e^{-u}du = -2e^{-u} + C = -2e^{-\sqrt{y}} + C$

45. $u = x - 3,\ x = u + 3,\ dx = du$

$$\int (u+3)u^{1/2}du = \int (u^{3/2} + 3u^{1/2})du = \frac{2}{5}u^{5/2} + 2u^{3/2} + C = \frac{2}{5}(x-3)^{5/2} + 2(x-3)^{3/2} + C$$

46. $u = y + 1,\ y = u - 1,\ dy = du$

$$\int \frac{u-1}{u^{1/2}}du = \int (u^{1/2} - u^{-1/2})du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(y+1)^{3/2} - 2(y+1)^{1/2} + C$$

47. $\int \sin^2 2\theta \sin 2\theta\,d\theta = \int (1 - \cos^2 2\theta)\sin 2\theta\,d\theta;\ u = \cos 2\theta,\ du = -2\sin 2\theta\,d\theta,$

$$-\frac{1}{2}\int (1-u^2)du = -\frac{1}{2}u + \frac{1}{6}u^3 + C = -\frac{1}{2}\cos 2\theta + \frac{1}{6}\cos^3 2\theta + C$$

48. $\sec^2 3\theta = \tan^2 3\theta + 1, u = 3\theta, du = 3d\theta$

$$\int \sec^4 3\theta \, d\theta = \frac{1}{3}\int (\tan^2 u + 1)\sec^2 u \, du = \frac{1}{9}\tan^3 u + \frac{1}{3}\tan u + C = \frac{1}{9}\tan^3 3\theta + \frac{1}{3}\tan 3\theta + C$$

49. $\int \left(1 + \frac{1}{t}\right) dt = t + \ln|t| + C$

50. $e^{2\ln x} = e^{\ln x^2} = x^2,\ x > 0$, so $\int e^{2\ln x} dx = \int x^2 dx = \frac{1}{3}x^3 + C$

51. $\ln(e^x) + \ln(e^{-x}) = \ln(e^x e^{-x}) = \ln 1 = 0$ so $\int [\ln(e^x) + \ln(e^{-x})]dx = C$

52. $\int \frac{\cos x}{\sin x} dx;\ u = \sin x,\ du = \cos x dx;\ \int \frac{1}{u} du = \ln|u| + C = \ln|\sin x| + C$

53. **(a)** $\sin^{-1}(x/3) + C$ **(b)** $(1/\sqrt{5})\tan^{-1}(x/\sqrt{5}) + C$
(c) $(1/\sqrt{\pi})\sec^{-1}(x/\sqrt{\pi}) + C$

54. **(a)** $u = e^x,\ \int \frac{1}{4+u^2} du = \frac{1}{2}\tan^{-1}(e^x/2) + C$

(b) $u = 2x,\ \frac{1}{2}\int \frac{1}{\sqrt{9-u^2}} du = \frac{1}{2}\sin^{-1}(2x/3) + C,$

(c) $u = \sqrt{5}y,\ \int \frac{1}{u\sqrt{u^2-3}} du = \frac{1}{\sqrt{3}}\sec^{-1}(\sqrt{5}y/\sqrt{3}) + C$

55. $u = a + bx, du = bdx,$

$$\int (a+bx)^n dx = \frac{1}{b}\int u^n du = \frac{(a+bx)^{n+1}}{b(n+1)} + C$$

56. $u = a + bx,\ du = b\,dx,\ dx = \frac{1}{b}du$

$$\frac{1}{b}\int u^{1/n} du = \frac{n}{b(n+1)}u^{(n+1)/n} + C = \frac{n}{b(n+1)}(a+bx)^{(n+1)/n} + C$$

57. $u = \sin(a+bx),\ du = b\cos(a+bx)dx$

$$\frac{1}{b}\int u^n du = \frac{1}{b(n+1)}u^{n+1} + C = \frac{1}{b(n+1)}\sin^{n+1}(a+bx) + C$$

59. **(a)** with $u = \sin x,\ du = \cos x\,dx;\ \int u\,du = \frac{1}{2}u^2 + C_1 = \frac{1}{2}\sin^2 x + C_1;$

with $u = \cos x,\ du = -\sin x\,dx;\ -\int u\,du = -\frac{1}{2}u^2 + C_2 = -\frac{1}{2}\cos^2 x + C_2$

(b) because they differ by a constant:

$$\left(\frac{1}{2}\sin^2 x + C_1\right) - \left(-\frac{1}{2}\cos^2 x + C_2\right) = \frac{1}{2}(\sin^2 x + \cos^2 x) + C_1 - C_2 = 1/2 + C_1 - C_2$$

60. **(a)** First method: $\int (25x^2 - 10x + 1)dx = \frac{25}{3}x^3 - 5x^2 + x + C_1;$

second method: $\frac{1}{5}\int u^2 du = \frac{1}{15}u^3 + C_2 = \frac{1}{15}(5x-1)^3 + C_2$

(b) $\frac{1}{15}(5x-1)^3 + C_2 = \frac{1}{15}(125x^3 - 75x^2 + 15x - 1) + C_2 = \frac{25}{3}x^3 - 5x^2 + x - \frac{1}{15} + C_2$; the answers differ by a constant.

61. $y(x) = \int \sqrt{3x+1}dx = \frac{2}{9}(3x+1)^{3/2} + C$,

$y(1) = \frac{16}{9} + C = 5$, $C = \frac{29}{9}$ so $y(x) = \frac{2}{9}(3x+1)^{3/2} + \frac{29}{9}$

62. $y(x) = \int (6 - 5\ \sin 2x)dx = 6x + \frac{5}{2}\cos 2x + C$,

$y(0) = \frac{5}{2} + C = 3$, $C = \frac{1}{2}$ so $y(x) = 6x + \frac{5}{2}\cos 2x + \frac{1}{2}$

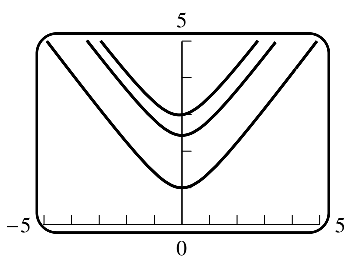
63. $y(t) = \int 2e^{-t}\,dt = -2e^{-t} + C,\ y(1) = -\frac{2}{e} + C = 3 - \frac{2}{e}, C = 3;\ y(t) = -2e^{-t} + 3$

64. $y = \int \frac{dx}{100 + 4x^2},\ u = x/5, dx = 5\,du,$

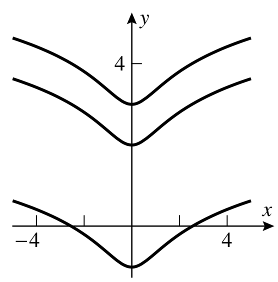
$y = \frac{1}{20}\int \frac{du}{1+u^2} = \frac{1}{20}\tan^{-1} u + C = \frac{1}{20}\tan^{-1}\left(\frac{x}{5}\right) + C;\ y(-5) = \frac{3\pi}{80} = \frac{1}{20}\left(-\frac{\pi}{4}\right) + C,$

$C = \frac{\pi}{20},\ y = \frac{1}{20}\tan^{-1}\left(\frac{x}{5}\right) + \frac{\pi}{20}$

65.



66.



67. $f'(x) = m = \sqrt{3x+1},\ f(x) = \int (3x+1)^{1/2}dx = \frac{2}{9}(3x+1)^{3/2} + C$

$f(0) = 1 = \frac{2}{9} + C,\ C = \frac{7}{9}$, so $f(x) = \frac{2}{9}(3x+1)^{3/2} + \frac{7}{9}$

68. $p(t) = \int (4 + 0.15t)^{3/2}dt = \frac{8}{3}(4+0.15t)^{5/2} + C;\ p(0) = 100{,}000 = \frac{8}{3}4^{5/2} + C = \frac{256}{3} + C,$

$C = 100{,}000 - \frac{256}{3} \approx 99{,}915, p(t) \approx \frac{8}{3}(4+0.15t)^{5/2} + 99{,}915, p(5) \approx \frac{8}{3}(4.75)^{5/2} + 99{,}915 \approx 100{,}046$

69. $u = a\sin\theta, du = a\cos\theta\,d\theta;\quad \int \frac{du}{\sqrt{a^2 - u^2}} = a\theta + C = \sin^{-1}\frac{u}{a} + C$

70. If $u > 0$ then $u = a\sec\theta, du = a\sec\theta\tan\theta\,d\theta,\ \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\theta = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$

EXERCISE SET 6.4

1. (a) $1+8+27=36$ (b) $5+8+11+14+17=55$
(c) $20+12+6+2+0+0=40$ (d) $1+1+1+1+1+1=6$
(e) $1-2+4-8+16=11$ (f) $0+0+0+0+0+0=0$

2. (a) $1+0-3+0=-2$ (b) $1-1+1-1+1-1=0$
(c) $\underbrace{\pi^2+\pi^2+\cdots+\pi^2}_{\text{(14 terms)}}=14\pi^2$ (d) $2^4+2^5+2^6=112$
(e) $\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{5}+\sqrt{6}$
(f) $1-1+1-1+1-1+1-1+1-1+1=1$

3. $\sum_{k=1}^{10} k$ **4.** $\sum_{k=1}^{20} 3k$ **5.** $\sum_{k=1}^{10} 2k$

6. $\sum_{k=1}^{8}(2k-1)$ **7.** $\sum_{k=1}^{6}(-1)^{k+1}(2k-1)$ **8.** $\sum_{k=1}^{5}(-1)^{k+1}\frac{1}{k}$

9. (a) $\sum_{k=1}^{50} 2k$ (b) $\sum_{k=1}^{50}(2k-1)$

10. (a) $\sum_{k=1}^{5}(-1)^{k+1}a_k$ (b) $\sum_{k=0}^{5}(-1)^{k+1}b_k$ (c) $\sum_{k=0}^{n}a_kx^k$ (d) $\sum_{k=0}^{5}a^{5-k}b^k$

11. $\frac{1}{2}(100)(100+1)=5050$

12. $7\sum_{k=1}^{100}k+\sum_{k=1}^{100}1=\frac{7}{2}(100)(101)+100=35{,}450$

13. $\frac{1}{6}(20)(21)(41)=2870$

14. $\sum_{k=1}^{20}k^2-\sum_{k=1}^{3}k^2=2870-14=2856$

15. $$\sum_{k=1}^{30}k(k^2-4)=\sum_{k=1}^{30}(k^3-4k)=\sum_{k=1}^{30}k^3-4\sum_{k=1}^{30}k=\frac{1}{4}(30)^2(31)^2-4\cdot\frac{1}{2}(30)(31)=214{,}365$$

16. $$\sum_{k=1}^{6}k-\sum_{k=1}^{6}k^3=\frac{1}{2}(6)(7)-\frac{1}{4}(6)^2(7)^2=-420$$

17. $$\sum_{k=1}^{n}\frac{3k}{n}=\frac{3}{n}\sum_{k=1}^{n}k=\frac{3}{n}\cdot\frac{1}{2}n(n+1)=\frac{3}{2}(n+1)$$

18. $$\sum_{k=1}^{n-1}\frac{k^2}{n}=\frac{1}{n}\sum_{k=1}^{n-1}k^2=\frac{1}{n}\cdot\frac{1}{6}(n-1)(n)(2n-1)=\frac{1}{6}(n-1)(2n-1)$$

19. $$\sum_{k=1}^{n-1}\frac{k^3}{n^2}=\frac{1}{n^2}\sum_{k=1}^{n-1}k^3=\frac{1}{n^2}\cdot\frac{1}{4}(n-1)^2n^2=\frac{1}{4}(n-1)^2$$

20. $$\sum_{k=1}^{n}\left(\frac{5}{n}-\frac{2k}{n}\right)=\frac{5}{n}\sum_{k=1}^{n}1-\frac{2}{n}\sum_{k=1}^{n}k=\frac{5}{n}(n)-\frac{2}{n}\cdot\frac{1}{2}n(n+1)=4-n$$

22. $\dfrac{n(n+1)}{2} = 465$, $n^2 + n - 930 = 0$, $(n+31)(n-30) = 0$, $n = 30$.

23. $\dfrac{1+2+3+\cdots+n}{n^2} = \displaystyle\sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}\cdot\frac{1}{2}n(n+1) = \frac{n+1}{2n}$; $\displaystyle\lim_{n\to+\infty} \frac{n+1}{2n} = \frac{1}{2}$

24. $\dfrac{1^2+2^2+3^2+\cdots+n^2}{n^3} = \displaystyle\sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) = \frac{(n+1)(2n+1)}{6n^2}$;

$\displaystyle\lim_{n\to+\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to+\infty} \frac{1}{6}(1+1/n)(2+1/n) = \frac{1}{3}$

25. $\displaystyle\sum_{k=1}^{n} \frac{5k}{n^2} = \frac{5}{n^2}\sum_{k=1}^{n} k = \frac{5}{n^2}\cdot\frac{1}{2}n(n+1) = \frac{5(n+1)}{2n}$; $\displaystyle\lim_{n\to+\infty} \frac{5(n+1)}{2n} = \frac{5}{2}$

26. $\displaystyle\sum_{k=1}^{n-1} \frac{2k^2}{n^3} = \frac{2}{n^3}\sum_{k=1}^{n-1} k^2 = \frac{2}{n^3}\cdot\frac{1}{6}(n-1)(n)(2n-1) = \frac{(n-1)(2n-1)}{3n^2}$;

$\displaystyle\lim_{n\to+\infty} \frac{(n-1)(2n-1)}{3n^2} = \lim_{n\to+\infty} \frac{1}{3}(1-1/n)(2-1/n) = \frac{2}{3}$

27. **(a)** $\displaystyle\sum_{j=0}^{5} 2^j$ **(b)** $\displaystyle\sum_{j=1}^{6} 2^{j-1}$ **(c)** $\displaystyle\sum_{j=2}^{7} 2^{j-2}$

28. **(a)** $\displaystyle\sum_{k=1}^{5} (k+4)2^{k+8}$ **(b)** $\displaystyle\sum_{k=9}^{13} (k-4)2^k$

29. Endpoints $2, 3, 4, 5, 6$; $\Delta x = 1$;

(a) Left endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 7 + 10 + 13 + 16 = 46$

(b) Midpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 8.5 + 11.5 + 14.5 + 17.5 = 52$

(c) Right endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 10 + 13 + 16 + 19 = 58$

30. Endpoints $1, 3, 5, 7, 9$, $\Delta x = 2$;

(a) Left endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\right)2 = \frac{352}{105}$

(b) Midpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right)2 = \frac{25}{12}$

(c) Right endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right)2 = \frac{496}{315}$

31. Endpoints: $0, \pi/4, \pi/2, 3\pi/4, \pi$; $\Delta x = \pi/4$

(a) Left endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(1 + \sqrt{2}/2 + 0 - \sqrt{2}/2\right)(\pi/4) = \pi/4$

(b) Midpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = [\cos(\pi/8) + \cos(3\pi/8) + \cos(5\pi/8) + \cos(7\pi/8)](\pi/4)$

$= [\cos(\pi/8) + \cos(3\pi/8) - \cos(3\pi/8) - \cos(\pi/8)](\pi/4) = 0$

(c) Right endpoints: $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(\sqrt{2}/2 + 0 - \sqrt{2}/2 - 1\right)(\pi/4) = -\pi/4$

32. Endpoints $-1, 0, 1, 2, 3; \Delta x = 1$

(a) $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = -3 + 0 + 1 + 0 = -2$

(b) $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = -\frac{5}{4} + \frac{3}{4} + \frac{3}{4} + \frac{15}{4} = 4$

(c) $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = 0 + 1 + 0 - 3 = -2$

33. **(a)** 0.718771403, 0.705803382, 0.698172179
(b) 0.668771403, 0.680803382, 0.688172179
(c) 0.692835360, 0.693069098, 0.693134682

34. **(a)** 0.761923639, 0.712712753, 0.684701150
(b) 0.584145862, 0.623823864, 0.649145594
(c) 0.663501867, 0.665867079, 0.666538346

35. **(a)** 4.884074734, 5.115572731, 5.248762738
(b) 5.684074734, 5.515572731, 5.408762738
(c) 5.34707029, 5.338362719, 5.334644416

36. **(a)** 0.919403170, 0.960215997, 0.984209789
(b) 1.076482803, 1.038755813, 1.015625715
(c) 1.001028824, 1.000257067, 1.000041125

37. $\Delta x = \dfrac{3}{n}$, $x_k^* = 1 + \dfrac{3}{n}k$; $f(x_k^*)\Delta x = \dfrac{1}{2}x_k^*\Delta x = \dfrac{1}{2}\left(1 + \dfrac{3}{n}k\right)\dfrac{3}{n} = \dfrac{3}{2}\left[\dfrac{1}{n} + \dfrac{3}{n^2}k\right]$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{3}{2}\left[\sum_{k=1}^{n}\frac{1}{n} + \sum_{k=1}^{n}\frac{3}{n^2}k\right] = \frac{3}{2}\left[1 + \frac{3}{n^2}\cdot\frac{1}{2}n(n+1)\right] = \frac{3}{2}\left[1 + \frac{3}{2}\frac{n+1}{n}\right]$$

$$A = \lim_{n\to+\infty}\frac{3}{2}\left[1 + \frac{3}{2}\left(1 + \frac{1}{n}\right)\right] = \frac{3}{2}\left(1 + \frac{3}{2}\right) = \frac{15}{4}$$

38. $\Delta x = \dfrac{5}{n}$, $x_k^* = 0 + k\dfrac{5}{n}$; $f(x_k^*)\Delta x = (5 - x_k^*)\Delta x = \left(5 - \dfrac{5}{n}k\right)\dfrac{5}{n} = \dfrac{25}{n} - \dfrac{25}{n^2}k$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\frac{25}{n} - \frac{25}{n^2}\sum_{k=1}^{n}k = 25 - \frac{25}{n^2}\cdot\frac{1}{2}n(n+1) = 25 - \frac{25}{2}\left(\frac{n+1}{n}\right)$$

$$A = \lim_{n\to+\infty}\left[25 - \frac{25}{2}\left(1 + \frac{1}{n}\right)\right] = 25 - \frac{25}{2} = \frac{25}{2}$$

39. $\Delta x = \dfrac{3}{n}, x_k^* = 0 + k\dfrac{3}{n}; f(x_k^*)\Delta x = \left(9 - 9\dfrac{k^2}{n^2}\right)\dfrac{3}{n}$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left(9 - 9\frac{k^2}{n^2}\right)\frac{3}{n} = \frac{27}{n}\sum_{k=1}^{n}\left(1 - \frac{k^2}{n^2}\right) = 27 - \frac{27}{n^3}\sum_{k=1}^{n} k^2$$

$$A = \lim_{n\to+\infty}\left[27 - \frac{27}{n^3}\sum_{k=1}^{n} k^2\right] = 27 - 27\left(\frac{1}{3}\right) = 18$$

40. $\Delta x = \dfrac{3}{n},\ x_k^* = k\dfrac{3}{n}$

$$f(x_k^*)\Delta x = \left[4 - \frac{1}{4}(x_k^*)^2\right]\Delta x = \left[4 - \frac{1}{4}\frac{9k^2}{n^2}\right]\frac{3}{n} = \frac{12}{n} - \frac{27k^2}{4n^3}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\frac{12}{n} - \frac{27}{4n^3}\sum_{k=1}^{n} k^2$$

$$= 12 - \frac{27}{4n^3}\cdot\frac{1}{6}n(n+1)(2n+1) = 12 - \frac{9}{8}\frac{(n+1)(2n+1)}{n^2}$$

$$A = \lim_{n\to+\infty}\left[12 - \frac{9}{8}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] = 12 - \frac{9}{8}(1)(2) = 39/4$$

41. $\Delta x = \dfrac{4}{n},\ x_k^* = 2 + k\dfrac{4}{n}$

$$f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \left[2 + \frac{4}{n}k\right]^3\frac{4}{n} = \frac{32}{n}\left[1 + \frac{2}{n}k\right]^3 = \frac{32}{n}\left[1 + \frac{6}{n}k + \frac{12}{n^2}k^2 + \frac{8}{n^3}k^3\right]$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{32}{n}\left[\sum_{k=1}^{n} 1 + \frac{6}{n}\sum_{k=1}^{n} k + \frac{12}{n^2}\sum_{k=1}^{n} k^2 + \frac{8}{n^3}\sum_{k=1}^{n} k^3\right]$$

$$= \frac{32}{n}\left[n + \frac{6}{n}\cdot\frac{1}{2}n(n+1) + \frac{12}{n^2}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{8}{n^3}\cdot\frac{1}{4}n^2(n+1)^2\right]$$

$$= 32\left[1 + 3\frac{n+1}{n} + 2\frac{(n+1)(2n+1)}{n^2} + 2\frac{(n+1)^2}{n^2}\right]$$

$$A = \lim_{n\to+\infty} 32\left[1 + 3\left(1 + \frac{1}{n}\right) + 2\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 2\left(1 + \frac{1}{n}\right)^2\right]$$

$$= 32[1 + 3(1) + 2(1)(2) + 2(1)^2] = 320$$

42. $\Delta x = \dfrac{2}{n},\ x_k^* = -3 + k\dfrac{2}{n};\ f(x_k^*)\Delta x = [1 - (x_k^*)^3]\Delta x = \left[1 - \left(-3 + \dfrac{2}{n}k\right)^3\right]\dfrac{2}{n}$

$$= \frac{2}{n}\left[28 - \frac{54}{n}k + \frac{36}{n^2}k^2 - \frac{8}{n^3}k^3\right]$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{2}{n}\left[28n - 27(n+1) + 6\frac{(n+1)(2n+1)}{n} - 2\frac{(n+1)^2}{n}\right]$$

$$A = \lim_{n\to+\infty} 2\left[28 - 27\left(1 + \frac{1}{n}\right) + 6\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) - 2\left(1 + \frac{1}{n}\right)^2\right]$$

$$= 2(28 - 27 + 12 - 2) = 22$$

43. $\Delta x = \dfrac{3}{n},\ x_k^* = 1 + (k-1)\dfrac{3}{n}$

$$f(x_k^*)\Delta x = \frac{1}{2}x_k^*\Delta x = \frac{1}{2}\left[1 + (k-1)\frac{3}{n}\right]\frac{3}{n} = \frac{1}{2}\left[\frac{3}{n} + (k-1)\frac{9}{n^2}\right]$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{2}\left[\sum_{k=1}^{n}\frac{3}{n} + \frac{9}{n^2}\sum_{k=1}^{n}(k-1)\right] = \frac{1}{2}\left[3 + \frac{9}{n^2}\cdot\frac{1}{2}(n-1)n\right] = \frac{3}{2} + \frac{9}{4}\frac{n-1}{n}$$

$$A = \lim_{n\to+\infty}\left[\frac{3}{2} + \frac{9}{4}\left(1 - \frac{1}{n}\right)\right] = \frac{3}{2} + \frac{9}{4} = \frac{15}{4}$$

44. $\Delta x = \dfrac{5}{n},\ x_k^* = \dfrac{5}{n}(k-1)$

$$f(x_k^*)\Delta x = (5 - x_k^*)\Delta x = \left[5 - \frac{5}{n}(k-1)\right]\frac{5}{n} = \frac{25}{n} - \frac{25}{n^2}(k-1)$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{25}{n}\sum_{k=1}^{n} 1 - \frac{25}{n^2}\sum_{k=1}^{n}(k-1) = 25 - \frac{25}{2}\frac{n-1}{n}$$

$$A = \lim_{n\to+\infty}\left[25 - \frac{25}{2}\left(1 - \frac{1}{n}\right)\right] = 25 - \frac{25}{2} = \frac{25}{2}$$

45. $\Delta x = \dfrac{3}{n}, x_k^* = 0 + (k-1)\dfrac{3}{n}; f(x_k^*)\Delta x = (9 - 9\dfrac{(k-1)^2}{n^2})\dfrac{3}{n}$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\left[9 - 9\frac{(k-1)^2}{n^2}\right]\frac{3}{n} = \frac{27}{n}\sum_{k=1}^{n}\left(1 - \frac{(k-1)^2}{n^2}\right) = 27 - \frac{27}{n^3}\sum_{k=1}^{n}k^2 + \frac{54}{n^3}\sum_{k=1}^{n}k - \frac{27}{n^2}$$

$$A = \lim_{n\to+\infty} = 27 - 27\left(\frac{1}{3}\right) + 0 + 0 = 18$$

46. $\Delta x = \dfrac{3}{n},\ x_k^* = (k-1)\dfrac{3}{n}$

$$f(x_k^*)\Delta x = \left[4 - \frac{1}{4}(x_k^*)^2\right]\Delta x = \left[4 - \frac{1}{4}\frac{9(k-1)^2}{n^2}\right]\frac{3}{n} = \frac{12}{n} - \frac{27k^2}{4n^3} + \frac{27k}{2n^3} - \frac{27}{4n^3}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n}\frac{12}{n} - \frac{27}{4n^3}\sum_{k=1}^{n}k^2 + \frac{27}{2n^3}\sum_{k=1}^{n}k - \frac{27}{4n^3}\sum_{k=1}^{n}1$$

$$= 12 - \frac{27}{4n^3}\cdot\frac{1}{6}n(n+1)(2n+1) + \frac{27}{2n^3}\frac{n(n+1)}{2} - \frac{27}{4n^2}$$

$$= 12 - \frac{9}{8}\frac{(n+1)(2n+1)}{n^2} + \frac{27}{4n} + \frac{27}{4n^2} - \frac{27}{4n^2}$$

$$A = \lim_{n\to+\infty}\left[12 - \frac{9}{8}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] + 0 + 0 - 0 = 12 - \frac{9}{8}(1)(2) = 39/4$$

47. $\Delta x = \dfrac{1}{n}, x_k^* = \dfrac{2k-1}{2n}$

$$f(x_k^*)\Delta x = \frac{(2k-1)^2}{(2n)^2}\frac{1}{n} = \frac{k^2}{n^3} - \frac{k}{n^3} + \frac{1}{4n^3}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{n^3}\sum_{k=1}^{n}k^2 - \frac{1}{n^3}\sum_{k=1}^{n}k + \frac{1}{4n^3}\sum_{k=1}^{n}1$$

Using Theorem 6.4.4,

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{1}{3} + 0 + 0 = \frac{1}{3}$$

48. $\Delta x = \dfrac{2}{n}, x_k^* = -1 + \dfrac{2k-1}{n}$

$$f(x_k^*)\Delta x = \left(-1 + \frac{2k-1}{n}\right)^2 \frac{2}{n} = \frac{8k^2}{n^3} - \frac{8k}{n^3} + \frac{2}{n^3} - \frac{2}{n}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{8}{n^3}\sum_{k=1}^{n} k^2 - \frac{8}{n^3}\sum_{k=1}^{n} k + \frac{2}{n^2} - 2$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{8}{3} + 0 + 0 - 2 = \frac{2}{3}$$

49. $\Delta x = \dfrac{2}{n}, x_k^* = -1 + \dfrac{2k}{n}$

$$f(x_k^*)\Delta x = \left(-1 + \frac{2k}{n}\right)\frac{2}{n} = -\frac{2}{n} + 4\frac{k}{n^2}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = -2 + \frac{4}{n^2}\sum_{k=1}^{n} k = -2 + \frac{4}{n^2}\frac{n(n+1)}{2} = -2 + 2 + \frac{2}{n}$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = 0$$

The area below the x-axis cancels the area above the x-axis.

50. $\Delta x = \dfrac{3}{n}, x_k^* = -1 + \dfrac{3k}{n}$

$$f(x_k^*)\Delta x = \left(-1 + \frac{3k}{n}\right)\frac{3}{n} = -\frac{3}{n} + \frac{9}{n^2}k$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = -3 + \frac{9}{n^2}\frac{n(n+1)}{2}$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = -3 + \frac{9}{2} + 0 = \frac{3}{2}$$

The area below the x-axis cancels the area above the x-axis that lies to the right of the line $x = 1$; the remaining area is a trapezoid of width 1 and heights 1, 2, hence its area is $\dfrac{1+2}{2} = \dfrac{3}{2}$

51. $\Delta x = \dfrac{2}{n}, x_k^* = \dfrac{2k}{n}$

$$f(x_k^*) = \left[\left(\frac{2k}{n}\right)^2 - 1\right]\frac{2}{n} = \frac{8k^2}{n^3} - \frac{2}{n}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{8}{n^3}\sum_{k=1}^{n} k^2 - \frac{2}{n}\sum_{k=1}^{n} 1 = \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} - 2$$

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{16}{6} - 2 = \frac{2}{3}$$

52. $\Delta x = \dfrac{2}{n}, x_k^* = -1 + \dfrac{2k}{n}$

$$f(x_k^*)\Delta x = \left(-1 + \frac{2k}{n}\right)^3 \frac{2}{n} = -\frac{2}{n} + 12\frac{k}{n^2} - 24\frac{k^2}{n^3} + 16\frac{k^3}{n^4}$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = -2 + \frac{12}{n^2}\frac{n(n+1)}{2} - \frac{24}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{16}{n^4}\left(\frac{n(n+1)}{2}\right)^2$$

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*) = -2 + \frac{12}{2} - \frac{48}{6} + \frac{16}{2^2} = 0$$

53. $\Delta x = \dfrac{b-a}{n}$, $x_k^* = a + \dfrac{b-a}{n}(k-1)$

$$f(x_k^*)\Delta x = mx_k^*\Delta x = m\left[a + \frac{b-a}{n}(k-1)\right]\frac{b-a}{n} = m(b-a)\left[\frac{a}{n} + \frac{b-a}{n^2}(k-1)\right]$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = m(b-a)\left[a + \frac{b-a}{2}\cdot\frac{n-1}{n}\right]$$

$$A = \lim_{n\to+\infty} m(b-a)\left[a + \frac{b-a}{2}\left(1 - \frac{1}{n}\right)\right] = m(b-a)\frac{b+a}{2} = \frac{1}{2}m(b^2 - a^2)$$

54. $\Delta x = \dfrac{b-a}{n}, x_k^* = a + \dfrac{k}{n}(b-a)$

$$f(x_k^*)\Delta x = \frac{ma}{n}(b-a) + \frac{mk}{n^2}(b-a)^2$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = ma(b-a) + \frac{m}{n^2}(b-a)^2\frac{n(n+1)}{2}$$

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = ma(b-a) + \frac{m}{2}(b-a)^2 = m(b-a)\frac{a+b}{2}$$

55. **(a)** With x_k^* as the right endpoint, $\Delta x = \dfrac{b}{n}$, $x_k^* = \dfrac{b}{n}k$

$$f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \frac{b^4}{n^4}k^3,\ \sum_{k=1}^{n} f(x_k^*)\Delta x = \frac{b^4}{n^4}\sum_{k=1}^{n} k^3 = \frac{b^4}{4}\frac{(n+1)^2}{n^2}$$

$$A = \lim_{n\to+\infty}\frac{b^4}{4}\left(1 + \frac{1}{n}\right)^2 = b^4/4$$

(b) $\Delta x = \dfrac{b-a}{n}$, $x_k^* = a + \dfrac{b-a}{n}k$

$$f(x_k^*)\Delta x = (x_k^*)^3\Delta x = \left[a + \frac{b-a}{n}k\right]^3\frac{b-a}{n}$$

$$= \frac{b-a}{n}\left[a^3 + \frac{3a^2(b-a)}{n}k + \frac{3a(b-a)^2}{n^2}k^2 + \frac{(b-a)^3}{n^3}k^3\right]$$

$$\sum_{k=1}^{n} f(x_k^*)\Delta x = (b-a)\left[a^3 + \frac{3}{2}a^2(b-a)\frac{n+1}{n} + \frac{1}{2}a(b-a)^2\frac{(n+1)(2n+1)}{n^2}\right.$$
$$\left. + \frac{1}{4}(b-a)^3\frac{(n+1)^2}{n^2}\right]$$

$$A = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= (b-a)\left[a^3 + \frac{3}{2}a^2(b-a) + a(b-a)^2 + \frac{1}{4}(b-a)^3\right] = \frac{1}{4}(b^4 - a^4).$$

56. Let A be the area of the region under the curve and above the interval $0 \le x \le 1$ on the x-axis, and let B be the area of the region between the curve and the interval $0 \le y \le 1$ on the y-axis. Together A and B form the square of side 1, so $A + B = 1$.
But B can also be considered as the area between the curve $x = y^2$ and the interval $0 \le y \le 1$ on the y-axis. By Exercise 47 above, $B = \dfrac{1}{3}$, so $A = 1 - \dfrac{1}{3} = \dfrac{2}{3}$.

57. If $n = 2m$ then $2m + 2(m-1) + \cdots + 2 \cdot 2 + 2 = 2\sum_{k=1}^{m} k = 2 \cdot \dfrac{m(m+1)}{2} = m(m+1) = \dfrac{n^2 + 2n}{4}$;

if $n = 2m+1$ then $(2m+1) + (2m-1) + \cdots + 5 + 3 + 1 = \sum_{k=1}^{m+1} (2k-1)$

$$= 2\sum_{k=1}^{m+1} k - \sum_{k=1}^{m+1} 1 = 2 \cdot \frac{(m+1)(m+2)}{2} - (m+1) = (m+1)^2 = \frac{n^2 + 2n + 1}{4}$$

58. $50 \cdot 30 + 49 \cdot 29 + \cdots + 22 \cdot 2 + 21 \cdot 1 = \sum_{k=1}^{30} k(k+20) = \sum_{k=1}^{30} k^2 + 20\sum_{k=1}^{30} k = \dfrac{30 \cdot 31 \cdot 61}{6} + 20\dfrac{30 \cdot 31}{2} = 18{,}755$

59. both are valid

60. none is valid

61. $$\sum_{k=1}^{n}(a_k - b_k) = (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_n - b_n)$$
$$= (a_1 + a_2 + \cdots + a_n) - (b_1 + b_2 + \cdots + b_n) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

62. $\sum_{k=1}^{n} \left[(k+1)^4 - k^4\right] = (n+1)^4 - 1$ (telescoping sum), expand the quantity in brackets to get $\sum_{k=1}^{n} (4k^3 + 6k^2 + 4k + 1) = (n+1)^4 - 1$,

$$4\sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 = (n+1)^4 - 1$$

$$\sum_{k=1}^{n} k^3 = \frac{1}{4}\left[(n+1)^4 - 1 - 6\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1\right]$$
$$= \frac{1}{4}[(n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n]$$
$$= \frac{1}{4}(n+1)[(n+1)^3 - n(2n+1) - 2n - 1]$$
$$= \frac{1}{4}(n+1)(n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$$

63. **(a)** $\sum_{k=1}^{n} 1$ means add 1 to itself n times, which gives the result.

(b) $\dfrac{1}{n^2}\sum_{k=1}^{n} k = \dfrac{1}{n^2}\dfrac{n(n+1)}{2} = \dfrac{1}{2} + \dfrac{1}{2n}$, so $\lim_{n \to +\infty} \dfrac{1}{n^2}\sum_{k=1}^{n} k = \dfrac{1}{2}$

(c) $\frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2}$, so $\lim_{n\to+\infty}\frac{1}{n^3}\sum_{k=1}^{n} k^2 = \frac{1}{3}$

(d) $\frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}$, so $\lim_{n\to+\infty}\frac{1}{n^4}\sum_{k=1}^{n} k^3 = \frac{1}{4}$

EXERCISE SET 6.5

1. **(a)** $(4/3)(1) + (5/2)(1) + (4)(2) = 71/6$ **(b)** 2

2. **(a)** $(\sqrt{2}/2)(\pi/2) + (-1)(3\pi/4) + (0)(\pi/2) + (\sqrt{2}/2)(\pi/4) = 3(\sqrt{2}-2)\pi/8$
(b) $3\pi/4$

3. **(a)** $(-9/4)(1) + (3)(2) + (63/16)(1) + (-5)(3) = -117/16$
(b) 3

4. **(a)** $(-8)(2) + (0)(1) + (0)(1) + (8)(2) = 0$ **(b)** 2

5. $\int_{-1}^{2} x^2\,dx$

6. $\int_{1}^{2} x^3 dx$

7. $\int_{-3}^{3} 4x(1-3x)dx$

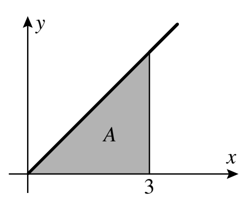
8. $\int_{0}^{\pi/2} \sin^2 x\,dx$

9. **(a)** $\lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} 2x_k^*\Delta x_k;\ a=1,\ b=2$ **(b)** $\lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} \frac{x_k^*}{x_k^*+1}\Delta x_k;\ a=0,\ b=1$

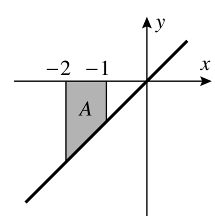
10. **(a)** $\lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} \sqrt{x_k^*}\,\Delta x_k,\ a=1,\ b=2$

(b) $\lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} (1+\cos x_k^*)\,\Delta x_k,\ a=-\pi/2,\ b=\pi/2$

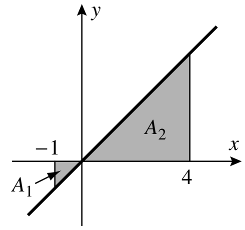
11. **(a)** $A = \frac{1}{2}(3)(3) = 9/2$



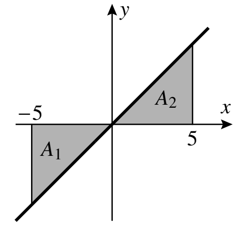
(b) $-A = -\frac{1}{2}(1)(1+2) = -3/2$



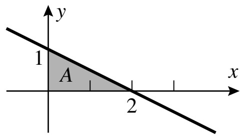
(c) $-A_1 + A_2 = -\frac{1}{2} + 8 = 15/2$



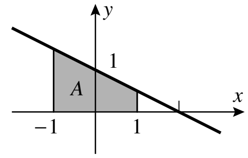
(d) $-A_1 + A_2 = 0$



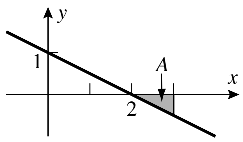
12. **(a)** $A = \frac{1}{2}(1)(2) = 1$



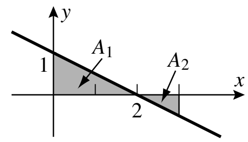
(b) $A = \frac{1}{2}(2)(3/2 + 1/2) = 2$



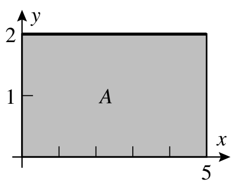
(c) $-A = -\frac{1}{2}(1/2)(1) = -1/4$



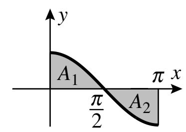
(d) $A_1 - A_2 = 1 - 1/4 = 3/4$



13. **(a)** $A = 2(5) = 10$

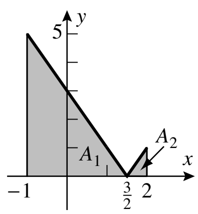


(b) 0; $A_1 = A_2$ by symmetry

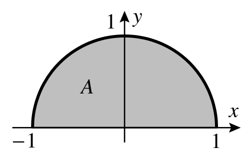


(c) $A_1 + A_2 = \frac{1}{2}(5)(5/2) + \frac{1}{2}(1)(1/2)$

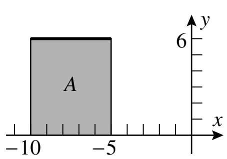
$= 13/2$



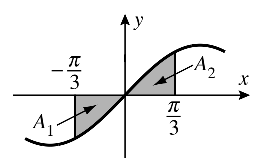
(d) $\frac{1}{2}[\pi(1)^2] = \pi/2$



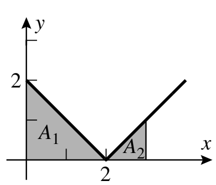
14. **(a)** $A = (6)(5) = 30$



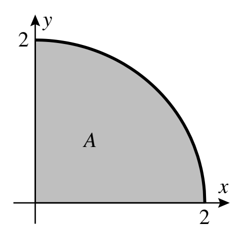
(b) $-A_1 + A_2 = 0$ because $A_1 = A_2$ by symmetry



(c) $A_1 + A_2 = \frac{1}{2}(2)(2) + \frac{1}{2}(1)(1) = 5/2$



(d) $\frac{1}{4}\pi(2)^2 = \pi$



15. **(a)** 0.8 **(b)** -2.6 **(c)** -1.8 **(d)** -0.3

16. **(a)** $\int_0^1 f(x)dx = \int_0^1 2xdx = x^2\Big]_0^1 = 1$

(b) $\int_{-1}^1 f(x)dx = \int_{-1}^1 2xdx = x^2\Big]_{-1}^1 = 1^2 - (-1)^2 = 0$

(c) $\int_1^{10} f(x)dx = \int_1^{10} 2dx = 2x\Big]_1^{10} = 18$

(d) $\int_{1/2}^5 f(x)dx = \int_{1/2}^1 2xdx + \int_1^5 2dx = x^2\Big]_{1/2}^1 + 2x\Big]_1^5 = 1^2 - (1/2)^2 + 2\cdot 5 - 2\cdot 1 = 3/4 + 8 = 35/4$

17. $\int_{-1}^2 f(x)dx + 2\int_{-1}^2 g(x)dx = 5 + 2(-3) = -1$

18. $3\int_1^4 f(x)dx - \int_1^4 g(x)dx = 3(2) - 10 = -4$

19. $\int_1^5 f(x)dx = \int_0^5 f(x)dx - \int_0^1 f(x)dx = 1 - (-2) = 3$

20. $\int_3^{-2} f(x)dx = -\int_{-2}^3 f(x)dx = -\left[\int_{-2}^1 f(x)dx + \int_1^3 f(x)dx\right] = -(2-6) = 4$

21. **(a)** $\int_0^1 xdx + 2\int_0^1 \sqrt{1-x^2}dx = 1/2 + 2(\pi/4) = (1+\pi)/2$

(b) $4\int_{-1}^3 dx - 5\int_{-1}^3 xdx = 4\cdot 4 - 5(-1/2 + (3\cdot 3)/2) = -4$

22. **(a)** $\int_{-3}^0 2dx + \int_{-3}^0 \sqrt{9-x^2}dx = 2\cdot 3 + (\pi(3)^2)/4 = 6 + 9\pi/4$

(b) $\int_{-2}^2 dx - 3\int_{-2}^2 |x|dx = 4\cdot 1 - 3(2)(2\cdot 2)/2 = -8$

23. **(a)** $\sqrt{x} > 0,\ 1 - x < 0$ on $[2,3]$ so the integral is negative

(b) $x^2 > 0,\ 3 - \cos x > 0$ for all x so the integral is positive

24. **(a)** $x^4 > 0,\ \sqrt{3-x} > 0$ on $[-3,-1]$ so the integral is positive

(b) $x^3 - 9 < 0,\ |x| + 1 > 0$ on $[-2,2]$ so the integral is negative

25. $\int_0^{10} \sqrt{25-(x-5)^2}dx = \pi(5)^2/2 = 25\pi/2$

26. $\int_0^3 \sqrt{9-(x-3)^2}dx = \pi(3)^2/4 = 9\pi/4$

27. $\int_0^1 (3x+1)dx = 5/2$

28. $\int_{-2}^2 \sqrt{4-x^2}dx = \pi(2)^2/2 = 2\pi$

29. **(a)** f is continuous on $[-1,1]$ so f is integrable there by Part (a) of Theorem 6.5.8

(b) $|f(x)| \le 1$ so f is bounded on $[-1,1]$, and f has one point of discontinuity, so by Part (b) of Theorem 6.5.8 f is integrable on $[-1,1]$

(c) f is not bounded on [-1,1] because $\lim_{x\to 0} f(x) = +\infty$, so f is not integrable on [0,1]

(d) $f(x)$ is discontinuous at the point $x = 0$ because $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist. f is continuous elsewhere. $-1 \le f(x) \le 1$ for x in $[-1,1]$ so f is bounded there. By Part (b), Theorem 6.5.8, f is integrable on $[-1,1]$.

30. Each subinterval of a partition of $[a,b]$ contains both rational and irrational numbers. If all x_k^* are chosen to be rational then

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \sum_{k=1}^{n}(1)\Delta x_k = \sum_{k=1}^{n}\Delta x_k = b-a \text{ so } \lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k = b-a.$$

If all x_k^* are irrational then $\lim_{\max \Delta x_k \to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k = 0$. Thus f is not integrable on $[a,b]$ because the preceding limits are not equal.

31. (a) Let $S_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ and $S = \int_a^b f(x)dx$ then $\sum_{k=1}^{n} cf(x_k^*)\Delta x_k = cS_n$ and we want to prove that $\lim_{\max \Delta x_k \to 0} cS_n = cS$. If $c = 0$ the result follows immediately, so suppose that $c \ne 0$ then for any $\epsilon > 0$, $|cS_n - cS| = |c||S_n - S| < \epsilon$ if $|S_n - S| < \epsilon/|c|$. But because f is integrable on $[a,b]$, there is a number $\delta > 0$ such that $|S_n - S| < \epsilon/|c|$ whenever $\max \Delta x_k < \delta$ so $|cS_n - cS| < \epsilon$ and hence $\lim_{\max \Delta x_k \to 0} cS_n = cS$.

(b) Let $R_n = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$, $S_n = \sum_{k=1}^{n} g(x_k^*)\Delta x_k$, $T_n = \sum_{k=1}^{n}[f(x_k^*) + g(x_k^*)]\Delta x_k$, $R = \int_a^b f(x)dx$, and $S = \int_a^b g(x)dx$ then $T_n = R_n + S_n$ and we want to prove that $\lim_{\max \Delta x_k \to 0} T_n = R + S$.
$|T_n - (R+S)| = |(R_n - R) + (S_n - S)| \le |R_n - R| + |S_n - S|$
so for any $\epsilon > 0$ $|T_n - (R+S)| < \epsilon$ if $|R_n - R| + |S_n - S| < \epsilon$.
Because f and g are integrable on $[a,b]$, there are numbers δ_1 and δ_2 such that
$|R_n - R| < \epsilon/2$ for $\max \Delta x_k < \delta_1$ and $|S_n - S| < \epsilon/2$ for $\max \Delta x_k < \delta_2$.
If $\delta = \min(\delta_1, \delta_2)$ then $|R_n - R| < \epsilon/2$ and $|S_n - S| < \epsilon/2$ for $\max \Delta x_k < \delta$ thus
$|R_n - R| + |S_n - S| < \epsilon$ and so $|T_n - (R+S)| < \epsilon$ for $\max \Delta x_k < \delta$ which shows that
$\lim_{\max \Delta x_k \to 0} T_n = R + S$.

32. For the smallest, find x_k^* so that $f(x_k^*)$ is minimum on each subinterval: $x_1^* = 1$, $x_2^* = 3/2$, $x_3^* = 3$ so $(2)(1) + (7/4)(2) + (4)(1) = 9.5$. For the largest, find x_k^* so that $f(x_k^*)$ is maximum on each subinterval: $x_1^* = 0$, $x_2^* = 3$, $x_3^* = 4$ so $(4)(1) + (4)(2) + (8)(1) = 20$.

33. $\Delta x_k = \dfrac{4k^2}{n^2} - \dfrac{4(k-1)^2}{n^2} = \dfrac{4}{n^2}(2k-1)$, $x_k^* = \dfrac{4k^2}{n^2}$,

$f(x_k^*) = \dfrac{2k}{n}$, $f(x_k^*)\Delta x_k = \dfrac{8k}{n^3}(2k-1) = \dfrac{8}{n^3}(2k^2 - k)$,

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \frac{8}{n^3}\sum_{k=1}^{n}(2k^2 - k) = \frac{8}{n^3}\left[\frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1)\right] = \frac{4}{3}\frac{(n+1)(4n-1)}{n^2},$$

$$\lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty}\frac{4}{3}\left(1 + \frac{1}{n}\right)\left(4 - \frac{1}{n}\right) = \frac{16}{3}.$$

34. For any partition of $[a,b]$ use the right endpoints to form the sum $\sum_{k=1}^{n} f(x_k^*)\Delta x_k$. Since $f(x_k^*) = 0$ for each k, the sum is zero and so is $\int_a^b f(x)\,dx = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x_k$.

35. With $f(x) = g(x)$ then $f(x) - g(x) = 0$ for $a < x \le b$. By Theorem 6.5.4(b)

$$\int_a^b f(x)\,dx = \int_a^b [(f(x) - g(x) + g(x)]dx = \int_a^b [f(x) - g(x)]dx + \int_a^b g(x)dx.$$

But the first term on the right hand side is zero (from Exercise 34), so

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$$

36. Choose any large positive integer N and any partition of $[0, a]$. Then choose x_1^* in the first interval so small that $f(x_1^*)\Delta x_1 > N$. For example choose $x_1^* < \Delta x_1/N$. Then with this partition and choice of x_1^*, $\sum_{k=1}^{n} f(x_k^*)\Delta x_k > f(x_1^*)\Delta x_1 > N$. This shows that the sum is dependent on partition and/or points, so Definition 6.5.1 is not satisfied.

EXERCISE SET 6.6

1. (a) $\int_0^2 (2-x)dx = (2x - x^2/2)\Big]_0^2 = 4 - 4/2 = 2$

(b) $\int_{-1}^1 2dx = 2x\Big]_{-1}^1 = 2(1) - 2(-1) = 4$

(c) $\int_1^3 (x+1)dx = (x^2/2 + x)\Big]_1^3 = 9/2 + 3 - (1/2 + 1) = 6$

2. (a) $\int_0^5 xdx = x^2/2\Big]_0^5 = 25/2$

(b) $\int_3^9 5dx = 5x\Big]_3^9 = 5(9) - 5(3) = 30$

(c) $\int_{-1}^2 (x+3)dx = (x^2/2 + 3x)\Big]_{-1}^2 = 4/2 + 6 - (1/2 - 3) = 21/2$

3. $\int_2^3 x^3 dx = x^4/4\Big]_2^3 = 81/4 - 16/4 = 65/4$

4. $\int_{-1}^1 x^4 dx = x^5/5\Big]_{-1}^1 = 1/5 - (-1)/5 = 2/5$

5. $\int_1^9 \sqrt{x}dx = \frac{2}{3}x^{3/2}\Big]_1^9 = \frac{2}{3}(27 - 1) = 52/3$

6. $\int_1^4 x^{-3/5}dx = \frac{5}{2}x^{2/5}\Big]_1^4 = \frac{5}{2}(4^{2/5} - 1)$

7. $\int_1^3 e^x dx = e^x\Big]_1^3 = e^3 - e$

8. $\int_1^5 \frac{1}{x}dx = \ln x\Big]_1^5 = \ln 5 - \ln 1 = \ln 5$

9. $\left(\frac{1}{3}x^3 - 2x^2 + 7x\right)\Big]_{-3}^0 = 48$

10. $\left(\frac{1}{2}x^2 + \frac{1}{5}x^5\right)\Big]_{-1}^2 = 81/10$

11. $\int_1^3 x^{-2}dx = -\frac{1}{x}\Big]_1^3 = 2/3$

12. $\int_1^2 x^{-6}dx = -\frac{1}{5x^5}\Big]_1^2 = 31/160$

13. $\frac{4}{5}x^{5/2}\Big]_4^9 = 844/5$

14. $\left(3x^{5/3} + \frac{4}{x}\right)\Big]_1^8 = 179/2$

15. $-\cos\theta\Big]_{-\pi/2}^{\pi/2} = 0$

16. $\tan\theta\Big]_{0}^{\pi/4} = 1$

17. $\sin x\Big]_{-\pi/4}^{\pi/4} = \sqrt{2}$

18. $\left(\frac{1}{2}x^2 - \sec x\right)\Big]_0^1 = 3/2 - \sec(1)$

19. $5e^x\Big]_{\ln 2}^{3} = 5e^3 - 5(2) = 5e^3 - 10$

20. $(\ln x)/2\Big]_{1/2}^{1} = (\ln 2)/2$

21. $\sin^{-1} x\Big]_0^{1/\sqrt{2}} = \sin^{-1}(1/\sqrt{2}) - \sin^{-1} 0 = \pi/4$

22. $\tan^{-1} x\Big]_{-1}^{1} = \tan^{-1} 1 - \tan^{-1}(-1) = \pi/4 - (-\pi/4) = \pi/2$

23. $\sec^{-1} x\Big]_{\sqrt{2}}^{2} = \sec^{-1} 2 - \sec^{-1}\sqrt{2} = \pi/3 - \pi/4 = \pi/12$

24. $-\sec^{-1} x\Big]_{-\sqrt{2}}^{-2/\sqrt{3}} = -\sec^{-1}(-2/\sqrt{3}) + \sec^{-1}(-\sqrt{2}) = -5\pi/6 + 3\pi/4 = -\pi/12$

25. $\left(6\sqrt{t} - \frac{10}{3}t^{3/2} + \frac{2}{\sqrt{t}}\right)\Big]_1^4 = -55/3$

26. $\left(8\sqrt{y} + \frac{4}{3}y^{3/2} - \frac{2}{3y^{3/2}}\right)\Big]_4^9 = 10819/324$

27. $\left(\frac{1}{2}x^2 - 2\cot x\right)\Big]_{\pi/6}^{\pi/2} = \pi^2/9 + 2\sqrt{3}$

28. $\left(a^{1/2}x - \frac{2}{3}x^{3/2}\right)\Big]_a^{4a} = -\frac{5}{3}a^{3/2}$

29. **(a)** $\int_0^{3/2}(3-2x)dx + \int_{3/2}^{2}(2x-3)dx = (3x - x^2)\Big]_0^{3/2} + (x^2 - 3x)\Big]_{3/2}^{2} = 9/4 + 1/4 = 5/2$

(b) $\int_0^{\pi/2}\cos x\,dx + \int_{\pi/2}^{3\pi/4}(-\cos x)dx = \sin x\Big]_0^{\pi/2} - \sin x\Big]_{\pi/2}^{3\pi/4} = 2 - \sqrt{2}/2$

30. **(a)** $\int_{-1}^{0}\sqrt{2-x}\,dx + \int_0^2\sqrt{2+x}\,dx = -\frac{2}{3}(2-x)^{3/2}\Big]_{-1}^{0} + \frac{2}{3}(2+x)^{3/2}\Big]_0^2$

$$= -\frac{2}{3}(2\sqrt{2} - 3\sqrt{3}) + \frac{2}{3}(8 - 2\sqrt{2}) = \frac{2}{3}(8 - 4\sqrt{2} + 3\sqrt{3})$$

(b) $\int_0^{\pi/6}(1/2 - \sin x)\,dx + \int_{\pi/6}^{\pi/2}(\sin x - 1/2)\,dx$

$$= (x/2 + \cos x)\Big]_0^{\pi/6} - (\cos x + x/2)\Big]_{\pi/6}^{\pi/2}$$

$$= (\pi/12 + \sqrt{3}/2) - 1 - \pi/4 + (\sqrt{3}/2 + \pi/12) = \sqrt{3} - \pi/12 - 1$$

31. **(a)** $\displaystyle\int_{-1}^{0}(1-e^x)dx+\int_0^1(e^x-1)dx = (x-e^x)\Big]_{-1}^{0}+(e^x-x)\Big]_0^1 = -1-(-1-e^{-1})+e-1-1 = e+1/e-2$

(b) $\displaystyle\int_1^2 \frac{2-x}{x}\,dx + \int_2^4 \frac{x-2}{x}\,dx = 2\ln x\Big]_1^2 - 1 + 2 - 2\ln x\Big]_2^4 = 2\ln 2 + 1 - 2\ln 4 + 2\ln 2 = 1$

32. **(a)** The function $f(x) = x^2 - 1 - \dfrac{15}{x^2+1}$ is an even function and changes sign at $x = 2$, thus

$$\int_{-3}^{3}|f(x)|\,dx = 2\int_0^3 |f(x)|\,dx = -2\int_0^2 f(x)\,dx + 2\int_2^3 f(x)\,dx$$

$$= \frac{28}{3} - 30\tan^{-1}(3) + 60\tan^{-1}(2)$$

(b) $\displaystyle\int_0^{\sqrt{3}/2}\left|\frac{1}{\sqrt{1-x^2}} - \sqrt{2}\right| dx = -\int_0^{\sqrt{2}/2}\left[\frac{1}{\sqrt{1-x^2}} - \sqrt{2}\right] dx + \int_{\sqrt{2}/2}^{\sqrt{3}/2}\left[\frac{1}{\sqrt{1-x^2}} - \sqrt{2}\right] dx$

$$= -2\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\right) + 1 = -2\frac{\pi}{4} + \frac{\pi}{3} - \frac{\sqrt{3}}{\sqrt{2}} + 2$$

$$= 2 - \frac{\sqrt{3}}{\sqrt{2}} - \frac{\pi}{6}$$

33. **(a)** 17/6 **(b)** $F(x) = \begin{cases} \dfrac{1}{2}x^2, & x \le 1 \\ \dfrac{1}{3}x^3 + \dfrac{1}{6}, & x > 1 \end{cases}$

34. **(a)** $\displaystyle\int_0^1 \sqrt{x}\,dx + \int_1^4 \frac{1}{x^2}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 - \frac{1}{x}\Big]_1^4 = 17/12$

(b) $F(x) = \begin{cases} \dfrac{2}{3}x^{3/2}, & x < 1 \\ -\dfrac{1}{x} + \dfrac{5}{3}, & x \ge 1 \end{cases}$

35. 0.665867079; $\displaystyle\int_1^3 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_1^3 = 2/3$

36. 1.000257067; $\displaystyle\int_0^{\pi/2} \sin x\,dx = -\cos x\Big]_0^{\pi/2} = 1$

37. 3.106017890; $\displaystyle\int_{-1}^{1} \sec^2 x\,dx = \tan x\Big|_{-1}^{1} = 2\tan 1 \approx 3.114815450$

38. 1.098242635; $\displaystyle\int_1^3 \frac{1}{x}\,dx = \ln x\Big]_1^3 = \ln 3 \approx 1.098612289$

39. $\displaystyle A = \int_0^3 (x^2+1)dx = \left(\frac{1}{3}x^3 + x\right)\Big]_0^3 = 12$

40. $\displaystyle A = \int_1^2 (-x^2+3x-2)dx = \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x\right)\Big]_1^2 = 1/6$

41. $A = \int_0^{2\pi/3} 3\sin x\,dx = -3\cos x\Big]_0^{2\pi/3} = 9/2$

42. $A = -\int_{-2}^{-1} x^3 dx = -\frac{1}{4}x^4\Big]_{-2}^{-1} = 15/4$

43. **(a)** $A = \int_0^{0.8} \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x\Big]_0^{0.8} = \sin^{-1}(0.8)$

(b) The calculator was in degree mode instead of radian mode; the correct answer is 0.93.

44. **(a)** the area is positive

(b) $\int_{-2}^{5}\left(\frac{1}{100}x^3 - \frac{1}{20}x^2 - \frac{1}{25}x + \frac{1}{5}\right)dx = \left(\frac{1}{400}x^4 - \frac{1}{60}x^3 - \frac{1}{50}x^2 + \frac{1}{5}x\right)\Big]_{-2}^{5} = \frac{343}{1200}$

45. **(a)** the area between the curve and the x-axis breaks into equal parts, one above and one below the x-axis, so the integral is zero

(b) $\int_{-1}^{1} x^3 dx = \frac{1}{4}x^4\Big]_{-1}^{1} = \frac{1}{4}(1^4 - (-1)^4) = 0;$

$\int_{-\pi/2}^{\pi/2} \sin x dx = -\cos x\Big]_{-\pi/2}^{\pi/2} = -\cos(\pi/2) + \cos(-\pi/2) = 0 + 0 = 0$

(c) The area on the left side of the y-axis is equal to the area on the right side, so $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

(d) $\int_{-1}^{1} x^2 dx = \frac{1}{3}x^3\Big]_{-1}^{1} = \frac{1}{3}(1^3 - (-1)^3) = \frac{2}{3} = 2\int_0^1 x^2 dx;$

$\int_{-\pi/2}^{\pi/2} \cos x dx = \sin x\Big]_{-\pi/2}^{\pi/2} = \sin(\pi/2) - \sin(-\pi/2) = 1 + 1 = 2 = 2\int_0^{\pi/2} \cos x dx$

46. The numerator is an odd function and the denominator is an even function, so the integrand is an odd function and the integral is zero.

47. **(a)** $x^3 + 1$ **(b)** $F(x) = \left(\frac{1}{4}t^4 + t\right)\Big]_1^x = \frac{1}{4}x^4 + x - \frac{5}{4}$; $F'(x) = x^3 + 1$

48. **(a)** $\cos 2x$ **(b)** $F(x) = \frac{1}{2}\sin 2t\Big]_{\pi/4}^{x} = \frac{1}{2}\sin 2x - \frac{1}{2}$, $F'(x) = \cos 2x$

49. **(a)** $\sin\sqrt{x}$ **(b)** e^{x^2}

50. **(a)** $\frac{1}{1+\sqrt{x}}$ **(b)** $\ln x$

51. $-\frac{x}{\cos x}$

52. $|u|$

53. $F'(x) = \sqrt{3x^2+1}$, $F''(x) = \frac{3x}{\sqrt{3x^2+1}}$

(a) 0 **(b)** $\sqrt{13}$ **(c)** $6/\sqrt{13}$

54. $F'(x) = \tan^{-1} x$, $F''(x) = \frac{1}{1+x^2}$

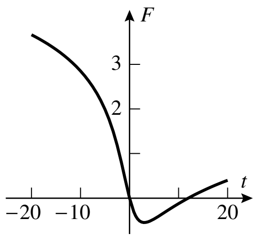
(a) 0 **(b)** $\pi/3$ **(c)** $1/4$

55. **(a)** $F'(x) = \dfrac{x-3}{x^2+7} = 0$ when $x = 3$, which is a relative minimum, and hence the absolute minimum, by the first derivative test.

(b) increasing on $[3, +\infty)$, decreasing on $(-\infty, 3]$

(c) $F''(x) = \dfrac{7+6x-x^2}{(x^2+7)^2} = \dfrac{(7-x)(1+x)}{(x^2+7)^2}$; concave up on $(-1, 7)$, concave down on $(-\infty, -1)$ and on $(7, +\infty)$

56.



57. **(a)** $(0, +\infty)$ because f is continuous there and 1 is in $(0, +\infty)$

(b) at $x = 1$ because $F(1) = 0$

58. **(a)** $(-3, 3)$ because f is continuous there and 1 is in $(-3, 3)$

(b) at $x = 1$ because $F(1) = 0$

59. **(a)** $f_{\text{ave}} = \dfrac{1}{9}\displaystyle\int_0^9 x^{1/2}dx = 2$; $\sqrt{x^*} = 2$, $x^* = 4$

(b) $f_{\text{ave}} = \dfrac{1}{3}\displaystyle\int_{-1}^2 (3x^2+2x+1)\,dx = \dfrac{1}{3}(x^3+x^2+x)\Big]_{-1}^2 = 5$; $3x^{*2} + 2x^* + 1 = 5$, with solutions $x^* = -(1/3)(1 \pm \sqrt{13})$, but only $x^* = -(1/3)(1-\sqrt{13})$ lies in the interval $[-1, 2]$.

60. **(a)** $f_{\text{ave}} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \sin x\,dx = 0$; $\sin x^* = 0$, $x^* = -\pi, 0, \pi$

(b) $f_{\text{ave}} = \dfrac{1}{2}\displaystyle\int_1^3 \frac{1}{x^2}dx = \frac{1}{3}$; $\dfrac{1}{(x^*)^2} = \dfrac{1}{3}$, $x^* = \sqrt{3}$

61. $\sqrt{2} \le \sqrt{x^3+2} \le \sqrt{29}$, so $3\sqrt{2} \le \displaystyle\int_0^3 \sqrt{x^3+2}dx \le 3\sqrt{29}$

62. Let $f(x) = x\sin x$, $f(0) = f(1) = 0$, $f'(x) = \sin x + x\cos x = 0$ when $x = -\tan x$, $x \approx 2.0288$, so f has an absolute maximum at $x \approx 2.0288$; $f(2.0288) \approx 1.8197$, so $0 \le x\sin x \le 1.82$ and $0 \le \displaystyle\int_0^{\pi} x\sin x dx \le 1.82\pi = 5.72$

63. **(a)** $\big[cF(x)\big]_a^b = cF(b) - cF(a) = c[F(b) - F(a)] = c\,\big[F(x)\big]_a^b$

(b) $\big[F(x) + G(x)\big]_a^b = [F(b) + G(b)] - [F(a) + G(a)]$
$= [F(b) - F(a)] + [G(b) - G(a)] = F(x)\big]_a^b + G(x)\big]_a^b$

(c) $\big[F(x) - G(x)\big]_a^b = [F(b) - G(b)] - [F(a) - G(a)]$
$= [F(b) - F(a)] - [G(b) - G(a)] = F(x)\big]_a^b - G(x)\big]_a^b$

64. Let f be continuous on a closed interval $[a,b]$ and let F be an antiderivative of f on $[a,b]$. By Theorem 5.8.2, $\dfrac{F(b)-F(a)}{b-a} = F'(x^*)$ for some x^* in (a,b). By Theorem 6.6.1, $\int_a^b f(x)\,dx = F(b)-F(a)$, i.e. $\int_a^b f(x)\,dx = F'(x^*)(b-a) = f(x^*)(b-a)$.

65. $\sum_{k=1}^{n} \dfrac{\pi}{4n}\sec^2\left(\dfrac{\pi k}{4n}\right) = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x)=\sec^2 x$, $x_k^* = \dfrac{\pi k}{4n}$ and $\Delta x = \dfrac{\pi}{4n}$ for $0 \le x \le \dfrac{\pi}{4}$.

Thus $\lim_{n\to+\infty} \sum_{k=1}^{n} \dfrac{\pi}{4n}\sec^2\left(\dfrac{\pi k}{4n}\right) \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1$

66. $\dfrac{n}{n^2+k^2} = \dfrac{1}{1+k^2/n^2}\dfrac{1}{n}$ so $\sum_{k=1}^{n}\dfrac{n}{n^2+k^2} = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = \dfrac{1}{1+x^2}$, $x_k^* = \dfrac{k}{n}$, and $\Delta x = \dfrac{1}{n}$ for $0 \le x \le 1$. Thus $\lim_{n\to+\infty}\sum_{k=1}^{n}\dfrac{n}{n^2+k^2} = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^1 \dfrac{1}{1+x^2}dx = \dfrac{\pi}{4}$.

EXERCISE SET 6.7

1. **(a)** the increase in height in inches, during the first ten years

(b) the change in the radius in centimeters, during the time interval $t=1$ to $t=2$ seconds

(c) the change in the speed of sound in ft/s, during an increase in temperature from $t = 32°$F to $t = 100°$F

(d) the displacement of the particle in cm, during the time interval $t = t_1$ to $t = t_2$ seconds

2. **(a)** $\int_0^1 V(t)dt$ gal

(b) the change $f(x_1) - f(x_2)$ in the values of f over the interval

3. **(a)** displ $= s(3) - s(0)$

$$= \int_0^3 v(t)dt = \int_0^2 (1-t)dt + \int_2^3 (t-3)dt = (t - t^2/2)\Big]_0^2 + (t^2/2 - 3t)\Big]_2^3 = -1/2;$$

$$\text{dist} = \int_0^3 |v(t)|dt = (t-t^2/2)\Big]_0^1 + (t^2/2 - t)\Big]_1^2 - (t^2/2-3t)\Big]_2^3 = 3/2$$

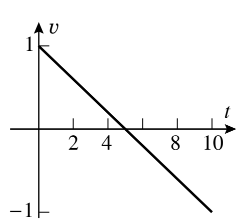
(b) displ $= s(3) - s(0)$

$$= \int_0^3 v(t)dt = \int_0^1 tdt + \int_1^2 dt + \int_2^3 (5-2t)dt = t^2/2\Big]_0^1 + t\Big]_1^2 + (5t-t^2)\Big]_2^3 = 3/2;$$

$$\text{dist} = \int_0^1 tdt + \int_1^2 dt + \int_2^{5/2}(5-2t)dt + \int_{5/2}^3 (2t-5)dt$$

$$= t^2/2\Big]_0^1 + t\Big]_1^2 + (5t-t^2)\Big]_2^{5/2} + (t^2-5t)\Big]_{5/2}^3 = 2$$

4.



5. **(a)** $v(t) = 20 + \int_0^t a(u)du$; add areas of the small blocks to get

$v(4) \approx 20 + 1.4 + 3.0 + 4.7 + 6.2 = 35.3$ m/s

(b) $v(6) = v(4) + \int_4^6 a(u)du \approx 35.3 + 7.5 + 8.6 = 51.4$ m/s

6. $a > 0$ and therefore (Theorem 6.5.6(a)) $v > 0$, so the particle is always speeding up for $0 < t < 10$

7. **(a)** $s(t) = \int (t^3 - 2t^2 + 1)dt = \frac{1}{4}t^4 - \frac{2}{3}t^3 + t + C$,

$s(0) = \frac{1}{4}(0)^4 - \frac{2}{3}(0)^3 + 0 + C = 1$, $C = 1$, $s(t) = \frac{1}{4}t^4 - \frac{2}{3}t^3 + t + 1$

(b) $v(t) = \int 4\cos 2t\, dt = 2\sin 2t + C_1$, $v(0) = 2\sin 0 + C_1 = -1$, $C_1 = -1$,

$v(t) = 2\sin 2t - 1$, $s(t) = \int (2\sin 2t - 1)dt = -\cos 2t - t + C_2$,

$s(0) = -\cos 0 - 0 + C_2 = -3$, $C_2 = -2$, $s(t) = -\cos 2t - t - 2$

8. **(a)** $s(t) = \int (1 + \sin t)dt = t - \cos t + C$, $s(0) = 0 - \cos 0 + C = -3$, $C = -2$, $s(t) = t - \cos t - 2$

(b) $v(t) = \int (t^2 - 3t + 1)dt = \frac{1}{3}t^3 - \frac{3}{2}t^2 + t + C_1$,

$v(0) = \frac{1}{3}(0)^3 - \frac{3}{2}(0)^2 + 0 + C_1 = 0$, $C_1 = 0$, $v(t) = \frac{1}{3}t^3 - \frac{3}{2}t^2 + t$,

$s(t) = \int \left(\frac{1}{3}t^3 - \frac{3}{2}t^2 + t\right) dt = \frac{1}{12}t^4 - \frac{1}{2}t^3 + \frac{1}{2}t^2 + C_2$,

$s(0) = \frac{1}{12}(0)^4 - \frac{1}{2}(0)^3 + \frac{1}{2}(0)^2 + C_2 = 0$, $C_2 = 0$, $s(t) = \frac{1}{12}t^4 - \frac{1}{2}t^3 + \frac{1}{2}t^2$

9. **(a)** $s(t) = \int (2t - 3)dt = t^2 - 3t + C$, $s(1) = (1)^2 - 3(1) + C = 5$, $C = 7$, $s(t) = t^2 - 3t + 7$

(b) $v(t) = \int \cos t dt = \sin t + C_1$, $v(\pi/2) = 2 = 1 + C_1$, $C_1 = 1$, $v(t) = \sin t + 1$,

$s(t) = \int (\sin t + 1)dt = -\cos t + t + C_2$, $s(\pi/2) = 0 = \pi/2 + C_2$, $C_2 = -\pi/2$,

$s(t) = -\cos t + t - \pi/2$

10. **(a)** $s(t) = \int t^{2/3}dt = \frac{3}{5}t^{5/3} + C$, $s(8) = 0 = \frac{3}{5}32 + C$, $C = -\frac{96}{5}$, $s(t) = \frac{3}{5}t^{5/3} - \frac{96}{5}$

(b) $v(t) = \int \sqrt{t}dt = \frac{2}{3}t^{3/2} + C_1$, $v(4) = 1 = \frac{2}{3}8 + C_1$, $C_1 = -\frac{13}{3}$, $v(t) = \frac{2}{3}t^{3/2} - \frac{13}{3}$,

$s(t) = \int \left(\frac{2}{3}t^{3/2} - \frac{13}{3}\right) dt = \frac{4}{15}t^{5/2} - \frac{13}{3}t + C_2$, $s(4) = -5 = \frac{4}{15}32 - \frac{13}{3}4 + C_2 = -\frac{44}{5} + C_2$,

$C_2 = \frac{19}{5}$, $s(t) = \frac{4}{15}t^{5/2} - \frac{13}{3}t + \frac{19}{5}$

11. **(a)** displacement $= s(\pi/2) - s(0) = \int_0^{\pi/2} \sin t dt = -\cos t\Big]_0^{\pi/2} = 1$ m

distance $= \int_0^{\pi/2} |\sin t|dt = 1$ m

(b) displacement $= s(2\pi) - s(\pi/2) = \int_{\pi/2}^{2\pi} \cos t dt = \sin t \Big]_{\pi/2}^{2\pi} = -1$ m

distance $= \int_{\pi/2}^{2\pi} |\cos t| dt = -\int_{\pi/2}^{3\pi/2} \cos t dt + \int_{3\pi/2}^{2\pi} \cos t dt = 3$ m

12. (a) displacement $= s(6) - s(0) = \int_0^6 (2t-4)dt = (t^2-4t)\Big]_0^6 = 12$ m

distance $= \int_0^6 |2t-4|dt = \int_0^2 (4-2t)dt + \int_2^6 (2t-4)dt = (4t-t^2)\Big]_0^2 + (t^2-4t)\Big]_2^6 = 20$ m

(b) displacement $= \int_0^5 |t-3|dt = \int_0^3 -(t-3)dt + \int_3^5 (t-3)dt = 13/2$ m

distance $= \int_0^5 |t-3|dt = 13/2$ m

13. (a) $v(t) = t^3 - 3t^2 + 2t = t(t-1)(t-2)$

displacement $= \int_0^3 (t^3 - 3t^2 + 2t)dt = 9/4$ m

distance $= \int_0^3 |v(t)|dt = \int_0^1 v(t)dt + \int_1^2 -v(t)dt + \int_2^3 v(t)dt = 11/4$ m

(b) displacement $= \int_0^3 (\sqrt{t} - 2)dt = 2\sqrt{3} - 6$ m

distance $= \int_0^3 |v(t)|dt = -\int_0^3 v(t)dt = 6 - 2\sqrt{3}$ m

14. (a) displacement $= \int_1^3 (\frac{1}{2} - \frac{1}{t^2})dt = 1/3$ m

distance $= \int_1^3 |v(t)|dt = -\int_1^{\sqrt{2}} v(t)dt + \int_{\sqrt{2}}^3 v(t)dt = 10/3 - 2\sqrt{2}$ m

(b) displacement $= \int_4^9 3t^{-1/2}dt = 6$ m

distance $= \int_4^9 |v(t)|dt = \int_4^9 v(t)dt = 6$ m

15. $v(t) = -2t + 3$

displacement $= \int_1^4 (-2t+3)dt = -6$ m

distance $= \int_1^4 |-2t+3|dt = \int_1^{3/2} (-2t+3)dt + \int_{3/2}^4 (2t-3)dt = 13/2$ m

16. $v(t) = \frac{1}{2}t^2 - 2t$

displacement $= \int_1^5 \left(\frac{1}{2}t^2 - 2t\right) dt = -10/3$ m

distance $= \int_1^5 \left|\frac{1}{2}t^2 - 2t\right| dt = \int_1^4 -\left(\frac{1}{2}t^2 - 2t\right) dt + \int_4^5 \left(\frac{1}{2}t^2 - 2t\right) dt = 17/3$ m

17. $$v(t) = \frac{2}{5}\sqrt{5t+1} + \frac{8}{5}$$

$$\text{displacement} = \int_0^3 \left(\frac{2}{5}\sqrt{5t+1} + \frac{8}{5}\right) dt = \frac{4}{75}(5t+1)^{3/2} + \frac{8}{5}t\Big]_0^3 = 204/25 \text{ m}$$

$$\text{distance} = \int_0^3 |v(t)|dt = \int_0^3 v(t)dt = 204/25 \text{ m}$$

18. $$v(t) = -\cos t + 2$$

$$\text{displacement} = \int_{\pi/4}^{\pi/2} (-\cos t + 2)dt = (\pi + \sqrt{2} - 2)/2 \text{ m}$$

$$\text{distance} = \int_{\pi/4}^{\pi/2} |-\cos t + 2|dt = \int_{\pi/4}^{\pi/2} (-\cos t + 2)dt = (\pi + \sqrt{2} - 2)/2 \text{ m}$$

19. **(a)** $s = \int \sin \frac{1}{2}\pi t\, dt = -\frac{2}{\pi}\cos\frac{1}{2}\pi t + C$

$s = 0$ when $t = 0$ which gives $C = \frac{2}{\pi}$ so $s = -\frac{2}{\pi}\cos\frac{1}{2}\pi t + \frac{2}{\pi}$.

$a = \frac{dv}{dt} = \frac{\pi}{2}\cos\frac{1}{2}\pi t$. When $t = 1 : s = 2/\pi,\ v = 1,\ |v| = 1,\ a = 0$.

(b) $v = -3\int t\, dt = -\frac{3}{2}t^2 + C_1$, $v = 0$ when $t = 0$ which gives $C_1 = 0$ so $v = -\frac{3}{2}t^2$

$s = -\frac{3}{2}\int t^2 dt = -\frac{1}{2}t^3 + C_2$, $s = 1$ when $t = 0$ which gives $C_2 = 1$ so $s = -\frac{1}{2}t^3 + 1$.

When $t = 1 : s = 1/2,\ v = -3/2,\ |v| = 3/2,\ a = -3$.

20. **(a)** negative, because v is decreasing

(b) speeding up when $av > 0$, so $2 < t < 5$; slowing down when $1 < t < 2$

(c) negative, because the area between the graph of $v(t)$ and the t-axis appears to be greater where $v < 0$ compared to where $v > 0$

21. $A = A_1 + A_2 = \int_0^1 (1 - x^2)dx + \int_1^3 (x^2 - 1)dx = 2/3 + 20/3 = 22/3$

22. $A = A_1 + A_2 = \int_0^{\pi} \sin x dx - \int_{\pi}^{3\pi/2} \sin x dx = 2 + 1 = 3$

23. $$A = A_1 + A_2 = \int_{-1}^0 \left[1 - \sqrt{x+1}\right] dx + \int_0^1 [\sqrt{x+1} - 1]\, dx$$

$$= \left(x - \frac{2}{3}(x+1)^{3/2}\right)\Big]_{-1}^0 + \left(\frac{2}{3}(x+1)^{3/2} - x\right)\Big]_0^1 = -\frac{2}{3} + 1 + \frac{4\sqrt{2}}{3} - 1 - \frac{2}{3} = 4\frac{\sqrt{2}-1}{3}$$

24. $$A = A_1 + A_2 = \int_{1/2}^1 \frac{1 - x^2}{x^2}\, dx + \int_1^2 \frac{x^2 - 1}{x^2}\, dx = \left(-\frac{1}{x} - x\right)\Big|_{1/2}^1 + \left(x + \frac{1}{x}\right)\Big|_1^2$$

$$= -2 + 2 + \frac{1}{2} + 2 + \frac{1}{2} - 2 = 1$$

25. $A = A_1 + A_2 = \int_{-1}^{0} (1 - e^x)dx + \int_0^1 (e^x - 1)dx = 1/e + e - 2$

26. $A = A_1 + A_2 = \int_{1/2}^{1} \frac{1-x}{x}dx + \int_1^2 \frac{x-1}{x}dx = -\left(\frac{1}{2} - \ln 2\right) + (1 - \ln 2) = 1/2$

27. By inspection the velocity is positive for $t > 0$, and during the first second the particle is at most $5/2$ cm from the starting position. For $T > 1$ the displacement of the particle during the time interval $[0, T]$ is given by

$$\int_0^T v(t)\,dt = 5/2 + \int_1^T (6\sqrt{t} - 1/t)\,dt = 5/2 + (4t^{3/2} - \ln t)\Big]_1^T = -3/2 + 4T^{3/2} - \ln T,$$

and the displacement equals 4 cm if $4T^{3/2} - \ln T = 11/2, T \approx 1.272$ s

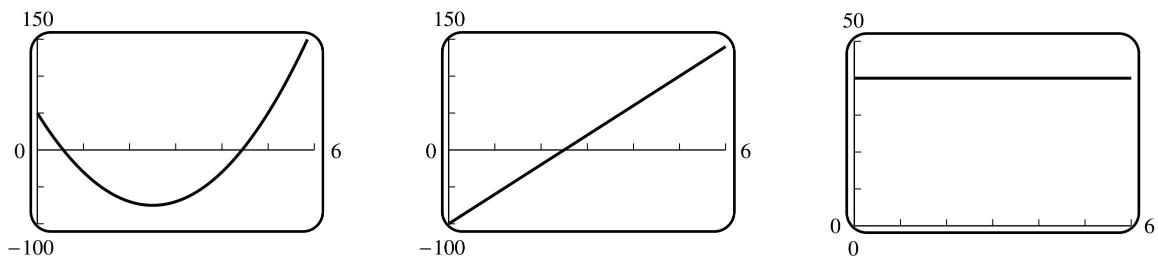
28. The displacement of the particle during the time interval $[0, T]$ is given by

$\int_0^T v(t)dt = 3\tan^{-1} T - 0.25T^2$. The particle is 2 cm from its starting position when

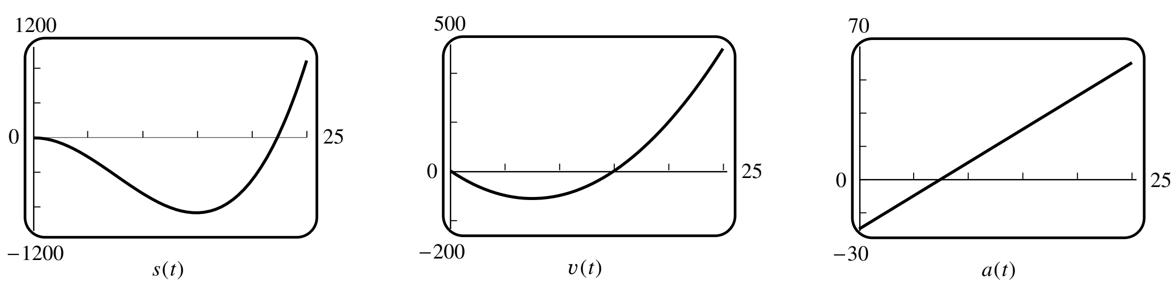
$3\tan^{-1} T - 0.25T^2 = 2$ or when $3\tan^{-1} T - 0.25T^2 = -2$; solve for T to get

$T = 0.90$, 2.51, and 4.95 s.

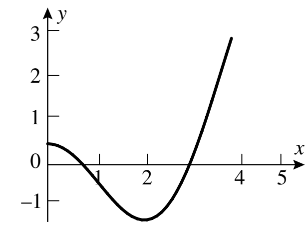
29. $s(t) = \frac{20}{3}t^3 - 50t^2 + 50t + s_0$, $s(0) = 0$ gives $s_0 = 0$, so $s(t) = \frac{20}{3}t^3 - 50t^2 + 50t$, $a(t) = 40t - 100$



30. $v(t) = 2t^2 - 30t + v_0$, $v(0) = 3 = v_0$, so $v(t) = 2t^2 - 30t + 3$, $s(t) = \frac{2}{3}t^3 - 15t^2 + 3t + s_0$, $s(0) = -5 = s_0$, so $s(t) = \frac{2}{3}t^3 - 15t^2 + 3t - 5$

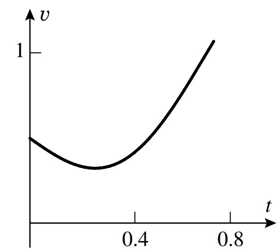


31. **(a)** From the graph the velocity is at first positive, but then turns negative, then positive again. The displacement, which is the cumulative area from $x = 0$ to $x = 5$, starts positive, turns negative, and then turns positive again.



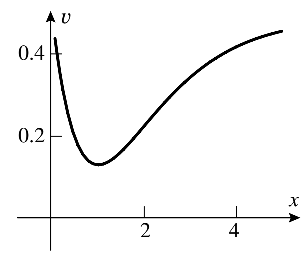
(b) $\text{displ} = 5/2 - \sin 5 + 5\cos 5$

32. **(a)** If $t_0 < 1$ then the area between the velocity curve and the t-axis, between $t = 0$ and $t = t_0$, will always be positive, so the displacement will be positive.



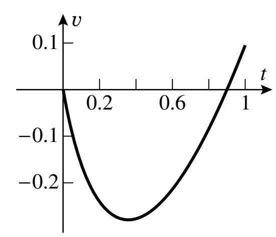
(b) $\text{displ} = \dfrac{\pi^2 + 4}{2\pi^2}$

33. **(a)** From the graph the velocity is positive, so the displacement is always increasing and is therefore positive.



(b) $s(t) = t/2 + (t+1)e^{-t}$

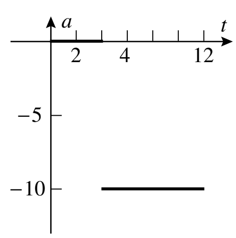
34. **(a)** If $t_0 < 1$ then the area between the velocity curve and the t-axis, between $t = 0$ and $t = t_0$, will always be negative, so the displacement will be negative.

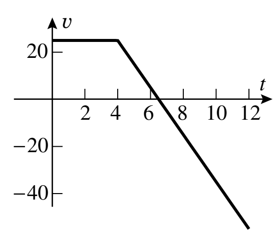


(b) $s(t) = \left(\dfrac{t^2}{2} - \dfrac{1}{200}\right)\ln(t + 0.1) - \dfrac{t^2}{4} + \dfrac{t}{20} - \dfrac{1}{200}\ln 10$

35. **(a)** $a(t) = \begin{cases} 0, & t < 4 \\ -10, & t > 4 \end{cases}$

(b) $v(t) = \begin{cases} 25, & t < 4 \\ 65 - 10t, & t > 4 \end{cases}$





(c) $x(t) = \begin{cases} 25t, & t < 4 \\ 65t - 5t^2 - 80, & t > 4 \end{cases}$, so $x(8) = 120$, $x(12) = -20$

(d) $x(6.5) = 131.25$

36. **(a)** From (9) $t = \dfrac{v - v_0}{a}$; from that and (8)

$s - s_0 = v_0 \dfrac{v - v_0}{a} + \dfrac{1}{2} a \dfrac{(v - v_0)^2}{a^2}$; multiply through by a to get

$a(s - s_0) = v_0(v - v_0) + \dfrac{1}{2}(v - v_0)^2 = (v - v_0)\left[v_0 + \dfrac{1}{2}(v - v_0)\right] = \dfrac{1}{2}(v^2 - v_0^2)$. Thus

$a = \dfrac{v^2 - v_0^2}{2(s - s_0)}$.

(b) Put the last result of Part (a) into the first equation of Part (a) to obtain
$t = \dfrac{v - v_0}{a} = (v - v_0)\dfrac{2(s - s_0)}{v^2 - v_0^2} = \dfrac{2(s - s_0)}{v + v_0}$.

(c) From (9) $v_0 = v - at$; use this in (8) to get
$s - s_0 = (v - at)t + \dfrac{1}{2}at^2 = vt - \dfrac{1}{2}at^2$
This expression contains no v_0 terms and so differs from (8).

37. **(a)** $a = -1$ mi/h/s $= -22/15$ ft/s^2 **(b)** $a = 30$ km/h/min $= 1/7200$ km/s^2

38. Take $t = 0$ when deceleration begins, then $a = -10$ so $v = -10t + C_1$, but $v = 88$ when $t = 0$ which gives $C_1 = 88$ thus $v = -10t + 88$, $t \ge 0$

(a) $v = 45$ mi/h $= 66$ ft/s, $66 = -10t + 88$, $t = 2.2$ s

(b) $v = 0$ (the car is stopped) when $t = 8.8$ s

$s = \int v\,dt = \int (-10t + 88)dt = -5t^2 + 88t + C_2$, and taking $s = 0$ when $t = 0$, $C_2 = 0$ so $s = -5t^2 + 88t$. At $t = 8.8$, $s = 387.2$. The car travels 387.2 ft before coming to a stop.

39. $a = a_0$ ft/s^2, $v = a_0 t + v_0 = a_0 t + 132$ ft/s, $s = a_0 t^2/2 + 132t + s_0 = a_0 t^2/2 + 132t$ ft; $s = 200$ ft when $v = 88$ ft/s. Solve $88 = a_0 t + 132$ and $200 = a_0 t^2/2 + 132t$ to get $a_0 = -\dfrac{121}{5}$ when $t = \dfrac{20}{11}$, so $s = -12.1t^2 + 132t$, $v = -\dfrac{121}{5}t + 132$.

(a) $a_0 = -\dfrac{121}{5}$ ft/s^2 **(b)** $v = 55$ mi/h $= \dfrac{242}{3}$ ft/s when $t = \dfrac{70}{33}$ s

(c) $v = 0$ when $t = \dfrac{60}{11}$ s

40. $dv/dt = 3$, $v = 3t + C_1$, but $v = v_0$ when $t = 0$ so $C_1 = v_0$, $v = 3t + v_0$. From $ds/dt = v = 3t + v_0$ we get $s = 3t^2/2 + v_0 t + C_2$ and, with $s = 0$ when $t = 0$, $C_2 = 0$ so $s = 3t^2/2 + v_0 t$. $s = 40$ when $t = 4$ thus $40 = 3(4)^2/2 + v_0(4)$, $v_0 = 4$ m/s

41. Suppose $s = s_0 = 0$, $v = v_0 = 0$ at $t = t_0 = 0$; $s = s_1 = 120$, $v = v_1$ at $t = t_1$; and $s = s_2$, $v = v_2 = 12$ at $t = t_2$. From Exercise 36(a),

$2.6 = a = \dfrac{v_1^2 - v_0^2}{2(s_1 - s_0)}$, $v_1^2 = 2as_1 = 5.2(120) = 624$. Applying the formula again,

$-1.5 = a = \dfrac{v_2^2 - v_1^2}{2(s_2 - s_1)}$, $v_2^2 = v_1^2 - 3(s_2 - s_1)$, so

$s_2 = s_1 - (v_2^2 - v_1^2)/3 = 120 - (144 - 624)/3 = 280$ m.

42. $a(t) = \begin{cases} 4, & t < 2 \\ 0, & t > 2 \end{cases}$, so, with $v_0 = 0$, $v(t) = \begin{cases} 4t, & t < 2 \\ 8, & t > 2 \end{cases}$ and,

since $s_0 = 0$, $s(t) = \begin{cases} 2t^2, & t < 2 \\ 8t - 8, & t > 2 \end{cases}$ $s = 100$ when $8t - 8 = 100$, $t = 108/8 = 13.5$ s

43. The truck's velocity is $v_T = 50$ and its position is $s_T = 50t + 5000$. The car's acceleration is $a_C = 2$, so $v_C = 2t$, $s_C = t^2$ (initial position and initial velocity of the car are both zero). $s_T = s_C$ when $50t + 5000 = t^2$, $t^2 - 50t - 5000 = (t+50)(t-100) = 0$, $t = 100$ s and $s_C = s_T = t^2 = 10{,}000$ ft.

44. Let $t = 0$ correspond to the time when the leader is 100 m from the finish line; let $s = 0$ correspond to the finish line. Then $v_C = 12$, $s_C = 12t - 115$; $a_L = 0.5$ for $t > 0$, $v_L = 0.5t + 8$, $s_L = 0.25t^2 + 8t - 100$. $s_C = 0$ at $t = 115/12 \approx 9.58$ s, and $s_L = 0$ at $t = -16 + 4\sqrt{41} \approx 9.61$, so the challenger wins.

45. $s = 0$ and $v = 112$ when $t = 0$ so $v(t) = -32t + 112$, $s(t) = -16t^2 + 112t$

(a) $v(3) = 16$ ft/s, $v(5) = -48$ ft/s

(b) $v = 0$ when the projectile is at its maximum height so $-32t + 112 = 0$, $t = 7/2$ s, $s(7/2) = -16(7/2)^2 + 112(7/2) = 196$ ft.

(c) $s = 0$ when it reaches the ground so $-16t^2 + 112t = 0$, $-16t(t - 7) = 0$, $t = 0, 7$ of which $t = 7$ is when it is at ground level on its way down. $v(7) = -112$, $|v| = 112$ ft/s.

46. $s = 112$ when $t = 0$ so $s(t) = -16t^2 + v_0 t + 112$. But $s = 0$ when $t = 2$ thus $-16(2)^2 + v_0(2) + 112 = 0$, $v_0 = -24$ ft/s.

47. **(a)** $s(t) = 0$ when it hits the ground, $s(t) = -16t^2 + 16t = -16t(t - 1) = 0$ when $t = 1$ s.

(b) The projectile moves upward until it gets to its highest point where $v(t) = 0$, $v(t) = -32t + 16 = 0$ when $t = 1/2$ s.

48. **(a)** $s(t) = 0$ when the rock hits the ground, $s(t) = -16t^2 + 555 = 0$ when $t = \sqrt{555}/4$ s

(b) $v(t) = -32t$, $v(\sqrt{555}/4) = -8\sqrt{555}$, the speed at impact is $8\sqrt{555}$ ft/s

49. **(a)** $s(t) = 0$ when the package hits the ground, $s(t) = -16t^2 + 20t + 200 = 0$ when $t = (5 + 5\sqrt{33})/8$ s

(b) $v(t) = -32t + 20$, $v[(5 + 5\sqrt{33})/8] = -20\sqrt{33}$, the speed at impact is $20\sqrt{33}$ ft/s

50. **(a)** $s(t) = 0$ when the stone hits the ground, $s(t) = -16t^2 - 96t + 112 = -16(t^2 + 6t - 7) = -16(t + 7)(t - 1) = 0$ when $t = 1$ s

(b) $v(t) = -32t - 96$, $v(1) = -128$, the speed at impact is 128 ft/s

51. $s(t) = -4.9t^2 + 49t + 150$ and $v(t) = -9.8t + 49$

(a) the projectile reaches its maximum height when $v(t) = 0$, $-9.8t + 49 = 0$, $t = 5$ s

(b) $s(5) = -4.9(5)^2 + 49(5) + 150 = 272.5$ m

(c) the projectile reaches its starting point when $s(t) = 150$, $-4.9t^2 + 49t + 150 = 150$, $-4.9t(t - 10) = 0$, $t = 10$ s

(d) $v(10) = -9.8(10) + 49 = -49$ m/s

(e) $s(t) = 0$ when the projectile hits the ground, $-4.9t^2 + 49t + 150 = 0$ when (use the quadratic formula) $t \approx 12.46$ s

(f) $v(12.46) = -9.8(12.46) + 49 \approx -73.1$, the speed at impact is about 73.1 m/s

52. take $s = 0$ at the water level and let h be the height of the bridge, then $s = h$ and $v = 0$ when $t = 0$ so $s(t) = -16t^2 + h$

(a) $s = 0$ when $t = 4$ thus $-16(4)^2 + h = 0$, $h = 256$ ft

(b) First, find how long it takes for the stone to hit the water (find t for $s = 0$) : $-16t^2 + h = 0$, $t = \sqrt{h}/4$. Next, find how long it takes the sound to travel to the bridge: this time is $h/1080$ because the speed is constant at 1080 ft/s. Finally, use the fact that the total of these two times must be 4 s: $\dfrac{h}{1080} + \dfrac{\sqrt{h}}{4} = 4$, $h + 270\sqrt{h} = 4320$, $h + 270\sqrt{h} - 4320 = 0$, and by the quadratic formula $\sqrt{h} = \dfrac{-270 \pm \sqrt{(270)^2 + 4(4320)}}{2}$, reject the negative value to get $\sqrt{h} \approx 15.15$, $h \approx 229.5$ ft.

53. $g = 9.8/6 = 4.9/3$ m/s^2, so $v = -(4.9/3)t$, $s = -(4.9/6)t^2 + 5$, $s = 0$ when $t = \sqrt{30/4.9}$ and $v = -(4.9/3)\sqrt{30/4.9} \approx -4.04$, so the speed of the module upon landing is 4.04 m/s

54. $s(t) = -\frac{1}{2}gt^2 + v_0t$; $s = 1000$ when $v = 0$, so $0 = v = -gt + v_0$, $t = v_0/g$,
$1000 = s(v_0/g) = -\frac{1}{2}g(v_0/g)^2 + v_0(v_0/g) = \frac{1}{2}v_0^2/g$, so $v_0^2 = 2000g$, $v_0 = \sqrt{2000g}$.
The initial velocity on the Earth would have to be $\sqrt{6}$ times faster than that on the Moon.

55. $f_{\text{ave}} = \dfrac{1}{3-1}\displaystyle\int_1^3 3x\,dx = \dfrac{3}{4}x^2\Big]_1^3 = 6$

56. $f_{\text{ave}} = \dfrac{1}{2-(-1)}\displaystyle\int_{-1}^2 x^2\,dx = \dfrac{1}{9}x^3\Big]_{-1}^2 = 1$

57. $f_{\text{ave}} = \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi \sin x\,dx = -\dfrac{1}{\pi}\cos x\Big]_0^\pi = 2/\pi$

58. $f_{\text{ave}} = \dfrac{1}{\pi - 0}\displaystyle\int_0^\pi \cos x\,dx = \dfrac{1}{\pi}\sin x\Big]_0^\pi = 0$

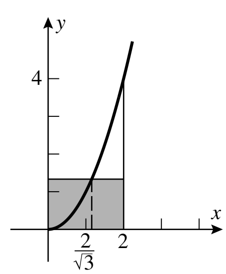
59. $f\text{ave} = \dfrac{1}{e-1}\displaystyle\int_1^e \dfrac{1}{x}\,dx = \dfrac{1}{e-1}(\ln e - \ln 1) = \dfrac{1}{e-1}$

60. $f\text{ave} = \dfrac{1}{\ln 5 - (-1)}\displaystyle\int_{-1}^{\ln 5} e^x dx = \dfrac{1}{\ln 5 + 1}(5 - e^{-1}) = \dfrac{5 - e^{-1}}{1 + \ln 5}$

61. **(a)** $f_{\text{ave}} = \dfrac{1}{2-0}\displaystyle\int_0^2 x^2 dx = 4/3$

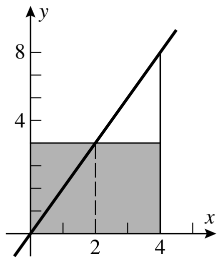
(b) $(x^*)^2 = 4/3, x^* = \pm 2/\sqrt{3}$, but only $2/\sqrt{3}$ is in $[0, 2]$

(c)



62. **(a)** $f_{\text{ave}} = \dfrac{1}{4-0}\displaystyle\int_0^4 2x\,dx = 4$

(b) $2x^* = 4$, $x^* = 2$

(c) 

63. **(a)** $v_{\text{ave}} = \dfrac{1}{4-1}\displaystyle\int_1^4 (3t^3+2)dt = \dfrac{1}{3}\dfrac{789}{4} = \dfrac{263}{4}$

(b) $v_{\text{ave}} = \dfrac{s(4)-s(1)}{4-1} = \dfrac{100-7}{3} = 31$

64. **(a)** $a_{\text{ave}} = \dfrac{1}{5-0}\displaystyle\int_0^5 (t+1)dt = 7/2$

(b) $a_{\text{ave}} = \dfrac{v(\pi/4)-v(0)}{\pi/4-0} = \dfrac{\sqrt{2}/2-1}{\pi/4} = (2\sqrt{2}-4)/\pi$

65. time to fill tank = (volume of tank)/(rate of filling) = $[\pi(3)^2 5]/(1) = 45\pi$, weight of water in tank at time $t = (62.4)$ (rate of filling)(time) $= 62.4t$,

$\text{weight}_{\text{ave}} = \dfrac{1}{45\pi}\displaystyle\int_0^{45\pi} 62.4t\,dt = 1404\pi$ lb

66. **(a)** If x is the distance from the cooler end, then the temperature is $T(x) = (15+1.5x)^\circ$ C, and $T_{\text{ave}} = \dfrac{1}{10-0}\displaystyle\int_0^{10}(15+1.5x)dx = 22.5^\circ$ C

(b) By the Mean-Value Theorem for Integrals there exists x^* in $[0,10]$ such that $f(x^*) = \dfrac{1}{10-0}\displaystyle\int_0^{10}(15+1.5x)dx = 22.5$, $15+1.5x^* = 22.5$, $x^* = 5$

67. **(a)** amount of water = (rate of flow)(time) = $4t$ gal, total amount $= 4(30) = 120$ gal

(b) amount of water $= \displaystyle\int_0^{60}(4+t/10)dt = 420$ gal

(c) amount of water $= \displaystyle\int_0^{120}(10+\sqrt{t})dt = 1200+160\sqrt{30} \approx 2076.36$ gal

68. **(a)** The maximum value of R occurs at 4:30 P.M. when $t = 0$.

(b) $\displaystyle\int_0^{60} 100(1-0.0001t^2)dt = 5280$ cars

69. **(a)** $\displaystyle\int_a^b [f(x)-f_{\text{ave}}]\,dx = \int_a^b f(x)dx - \int_a^b f_{\text{ave}}dx = \int_a^b f(x)dx - f_{\text{ave}}(b-a) = 0$

because $f_{\text{ave}}(b-a) = \displaystyle\int_a^b f(x)dx$

(b) no, because if $\displaystyle\int_a^b [f(x)-c]dx = 0$ then $\displaystyle\int_a^b f(x)dx - c(b-a) = 0$ so

$c = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = f_{\text{ave}}$ is the only value

EXERCISE SET 6.8

1. **(a)** $\int_1^3 u^7 du$ **(b)** $-\frac{1}{2}\int_7^4 u^{1/2} du$ **(c)** $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin u\, du$ **(d)** $\int_{-3}^{0} (u+5)u^{20} du$

2. **(a)** $\frac{1}{2}\int_{-3}^{7} u^8\, du$ **(b)** $\int_{3/2}^{5/2} \frac{1}{\sqrt{u}}\, du$

(c) $\int_0^1 u^2 du$ **(d)** $\frac{1}{2}\int_3^4 (u-3)u^{1/2} du$

3. **(a)** $\frac{1}{2}\int_{-1}^{1} e^u\, du$ **(b)** $\int_1^2 u\, du$

4. **(a)** $\int_{\pi/4}^{\pi/3} \sqrt{u}\, du$ **(b)** $\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$

5. $u = 2x+1$, $\frac{1}{2}\int_1^3 u^4 du = \frac{1}{10}u^5\Big]_1^3 = 121/5$, or $\frac{1}{10}(2x+1)^5\Big]_0^1 = 121/5$

6. $u = 4x-2$, $\frac{1}{4}\int_2^6 u^3 du = \frac{1}{16}u^4\Big]_2^6 = 80$, or $\frac{1}{16}(4x-2)^4\Big]_1^2 = 80$

7. $u = 1-2x$, $-\frac{1}{2}\int_3^1 u^3 du = -\frac{1}{8}u^4\Big]_3^1 = 10$, or $-\frac{1}{8}(1-2x)^4\Big]_{-1}^{0} = 10$

8. $u = 4-3x$, $-\frac{1}{3}\int_1^{-2} u^8 du = -\frac{1}{27}u^9\Big]_1^{-2} = 19$, or $-\frac{1}{27}(4-3x)^9\Big]_1^2 = 19$

9. $u = 1+x$, $\int_1^9 (u-1)u^{1/2} du = \int_1^9 (u^{3/2} - u^{1/2}) du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\Big]_1^9 = 1192/15$,
or $\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2}\Big]_0^8 = 1192/15$

10. $u = 4-x$, $\int_9^4 (u-4)u^{1/2} du = \int_9^4 (u^{3/2} - 4u^{1/2}) du = \frac{2}{5}u^{5/2} - \frac{8}{3}u^{3/2}\Big]_9^4 = -506/15$
or $\frac{2}{5}(4-x)^{5/2} - \frac{8}{3}(4-x)^{3/2}\Big]_{-5}^{0} = -506/15$

11. $u = x/2$, $8\int_0^{\pi/4} \sin u\, du = -8\cos u\Big]_0^{\pi/4} = 8 - 4\sqrt{2}$, or $-8\cos(x/2)\Big]_0^{\pi/2} = 8 - 4\sqrt{2}$

12. $u = 3x$, $\frac{2}{3}\int_0^{\pi/2} \cos u\, du = \frac{2}{3}\sin u\Big]_0^{\pi/2} = 2/3$, or $\frac{2}{3}\sin 3x\Big]_0^{\pi/6} = 2/3$

13. $u = x^2+2$, $\frac{1}{2}\int_6^3 u^{-3} du = -\frac{1}{4u^2}\Big]_6^3 = -1/48$, or $-\frac{1}{4}\frac{1}{(x^2+2)^2}\Big]_{-2}^{-1} = -1/48$

14. $u = \frac{1}{4}x - \frac{1}{4}$, $4\int_{-\pi/4}^{\pi/4} \sec^2 u\, du = 4\tan u\Big]_{-\pi/4}^{\pi/4} = 8$, or $4\tan\left(\frac{1}{4}x - \frac{1}{4}\right)\Big]_{1-\pi}^{1+\pi} = 8$

15. $u = e^x + 4$, $du = e^x dx$, $u = e^{-\ln 3} + 4 = \dfrac{1}{3} + 4 = \dfrac{13}{3}$ when $x = -\ln 3$,

$u = e^{\ln 3} + 4 = 3 + 4 = 7$ when $x = \ln 3$, $\displaystyle\int_{13/3}^{7} \frac{1}{u} du = \ln u\Big]_{13/3}^{7} = \ln(7) - \ln(13/3) = \ln(21/13)$

16. $u = 3 - 4e^x$, $du = -4e^x dx$, $u = -1$ when $x = 0$, $u = -17$ when $x = \ln 5$

$$-\frac{1}{4}\int_{-1}^{-17} u\,du = -\frac{1}{8}u^2\Big]_{-1}^{-17} = -36$$

17. $u = \sqrt{x}$, $\displaystyle 2\int_1^{\sqrt{3}} \frac{1}{u^2+1} du = 2\tan^{-1} u\Big]_1^{\sqrt{3}} = 2(\tan^{-1}\sqrt{3} - \tan^{-1} 1) = 2(\pi/3 - \pi/4) = \pi/6$

18. $u = e^{-x}$, $\displaystyle -\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}} du = -\sin^{-1} u\Big]_{1/2}^{\sqrt{3}/2} = -\sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2} = -\frac{\pi}{3} + \frac{\pi}{6} = -\frac{\pi}{6}$

19. $\displaystyle \frac{1}{3}\int_0^5 \sqrt{25-u^2}\,du = \frac{1}{3}\left[\frac{1}{4}\pi(5)^2\right] = \frac{25}{12}\pi$

20. $\displaystyle \frac{1}{2}\int_0^4 \sqrt{16-u^2}\,du = \frac{1}{2}\left[\frac{1}{4}\pi(4)^2\right] = 2\pi$

21. $\displaystyle -\frac{1}{2}\int_1^0 \sqrt{1-u^2}\,du = \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du = \frac{1}{2}\cdot\frac{1}{4}[\pi(1)^2] = \pi/8$

22. $\displaystyle \int_{-6}^{6} \sqrt{36-u^2}du = \pi(6)^2/2 = 18\pi$

23. $\displaystyle \int_0^1 \sin \pi x dx = -\frac{1}{\pi}\cos \pi x\Big]_0^1 = -\frac{1}{\pi}(-1-1) = 2/\pi$

24. $\displaystyle A = \int_0^{\pi/8} 3\ \cos 2x\,dx = \frac{3}{2}\sin 2x\Big]_0^{\pi/8} = 3\sqrt{2}/4$

25. $\displaystyle \int_3^7 (x+5)^{-2}dx = -(x+5)^{-1}\Big]_3^7 = -\frac{1}{12} + \frac{1}{8} = \frac{1}{24}$

26. $\displaystyle A = \int_0^1 \frac{dx}{(3x+1)^2} = -\frac{1}{3(3x+1)}\Big]_0^1 = \frac{1}{4}$

27. $\displaystyle A = \int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}dx = \frac{1}{3}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}du = \frac{1}{3}\sin^{-1} u\Big]_0^{1/2} = \pi/18$

28. $x = \sin y$, $\displaystyle A = \int_0^{\pi/2} \sin y\,dy = -\cos y\Big]_0^{\pi/2} = 1$

29. $\displaystyle \frac{1}{2-0}\int_0^2 \frac{x}{(5x^2+1)^2}\,dx = -\frac{1}{2}\frac{1}{10}\frac{1}{5x^2+1}\Big|_0^2 = \frac{1}{21}$

30. $\displaystyle f_{\text{ave}} = \frac{1}{1/4-(-1/4)}\int_{-1/4}^{1/4} \sec^2 \pi x dx = \frac{2}{\pi}\tan \pi x\Big]_{-1/4}^{1/4} = \frac{4}{\pi}$

31. $f_{\text{ave}} = \dfrac{1}{4}\displaystyle\int_0^4 e^{-2x}\,dx = -\dfrac{1}{8}e^{-2x}\Big]_0^4 = \dfrac{1}{8}(1-e^{-8})$

32. $f_{\text{ave}} = \dfrac{2}{\ln 3}\displaystyle\int_{1/\sqrt{3}}^1 \dfrac{du}{1+u^2} = \dfrac{2}{\ln 3}\tan^{-1} u\Big]_{1/\sqrt{3}}^1 = \dfrac{2}{\ln 3}\left(\dfrac{\pi}{4}-\dfrac{\pi}{6}\right) = \dfrac{\pi}{6\ln 3}$

33. $\dfrac{2}{3}(3x+1)^{1/2}\Big]_0^1 = 2/3$

34. $\dfrac{2}{15}(5x-1)^{3/2}\Big]_1^2 = 38/15$

35. $\dfrac{2}{3}(x^3+9)^{1/2}\Big]_{-1}^1 = \dfrac{2}{3}(\sqrt{10}-2\sqrt{2})$

36. $\dfrac{1}{10}(t^3+1)^{20}\Big]_{-1}^0 = 1/10$

37. $u = x^2+4x+7,\ \dfrac{1}{2}\displaystyle\int_{12}^{28} u^{-1/2}du = u^{1/2}\Big]_{12}^{28} = \sqrt{28}-\sqrt{12} = 2(\sqrt{7}-\sqrt{3})$

38. $\displaystyle\int_1^2 \dfrac{1}{(x-3)^2}dx = -\dfrac{1}{x-3}\Big]_1^2 = 1/2$

39. $\dfrac{1}{2}\sin^2 x\Big]_{-3\pi/4}^{\pi/4} = 0$

40. $\dfrac{2}{3}(\tan x)^{3/2}\Big]_0^{\pi/4} = 2/3$

41. $\dfrac{5}{2}\sin(x^2)\Big]_0^{\sqrt{\pi}} = 0$

42. $u = \sqrt{x},\ 2\displaystyle\int_{\pi}^{2\pi} \sin u\,du = -2\cos u\Big]_{\pi}^{2\pi} = -4$

43. $u = 3\theta,\ \dfrac{1}{3}\displaystyle\int_{\pi/4}^{\pi/3} \sec^2 u\,du = \dfrac{1}{3}\tan u\Big]_{\pi/4}^{\pi/3} = (\sqrt{3}-1)/3$

44. $u = \sin 3\theta,\ \dfrac{1}{3}\displaystyle\int_0^{-1} u^2 du = \dfrac{1}{9}u^3\Big]_0^{-1} = -1/9$

45. $u = 4-3y,\ y = \dfrac{1}{3}(4-u),\ dy = -\dfrac{1}{3}du$

$$-\frac{1}{27}\int_4^1 \frac{16-8u+u^2}{u^{1/2}}du = \frac{1}{27}\int_1^4 (16u^{-1/2}-8u^{1/2}+u^{3/2})du$$
$$= \frac{1}{27}\left[32u^{1/2}-\frac{16}{3}u^{3/2}+\frac{2}{5}u^{5/2}\right]_1^4 = 106/405$$

46. $u = 5+x,\ \displaystyle\int_4^9 \dfrac{u-5}{\sqrt{u}}du = \int_4^9 (u^{1/2}-5u^{-1/2})du = \dfrac{2}{3}u^{3/2}-10u^{1/2}\Big]_4^9 = 8/3$

47. $\ln(x+e)\Big]_0^e = \ln(2e)-\ln e = \ln 2$

48. $-\dfrac{1}{2}e^{-x^2}\Big]_1^{\sqrt{2}} = (e^{-1}-e^{-2})/2$

49. $u = \sqrt{3}x^2,\ \dfrac{1}{2\sqrt{3}}\displaystyle\int_0^{\sqrt{3}} \dfrac{1}{\sqrt{4-u^2}}du = \dfrac{1}{2\sqrt{3}}\sin^{-1}\dfrac{u}{2}\Big]_0^{\sqrt{3}} = \dfrac{1}{2\sqrt{3}}\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{6\sqrt{3}}$

50. $u=\sqrt{x},\ 2\int_1^{\sqrt{2}}\frac{1}{\sqrt{4-u^2}}du=2\sin^{-1}\frac{u}{2}\Big]_1^{\sqrt{2}}=2(\pi/4-\pi/6)=\pi/6$

51. $u=3x,\ \frac{1}{3}\int_0^{2\sqrt{3}}\frac{1}{4+u^2}du=\frac{1}{6}\tan^{-1}\frac{u}{2}\Big]_0^{2\sqrt{3}}=\frac{1}{6}\frac{\pi}{3}=\frac{\pi}{18}$

52. $u=x^2,\ \frac{1}{2}\int_1^3\frac{1}{3+u^2}du=\frac{1}{2\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}\Big]_1^3=\frac{1}{2\sqrt{3}}(\pi/3-\pi/6)=\frac{\pi}{12\sqrt{3}}$

53. **(b)** $\int_0^{\pi/6}\sin^4 x(1-\sin^2 x)\cos x\,dx=\left(\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x\right)\Bigg|_0^{\pi/6}=\frac{1}{160}-\frac{1}{896}=\frac{23}{4480}$

54. **(b)** $\int_{-\pi/4}^{\pi/4}\tan^2 x(\sec^2 x-1)\,dx=\frac{1}{3}\tan^3 x\Bigg|_{-\pi/4}^{\pi/4}-\int_{-\pi/4}^{\pi/4}(\sec^2 x-1)\,dx$

$$=\frac{2}{3}+(-\tan x+x)\Bigg|_{-\pi/4}^{\pi/4}=\frac{2}{3}-2+\frac{\pi}{2}=-\frac{4}{3}+\frac{\pi}{2}$$

55. **(a)** $u=3x+1,\frac{1}{3}\int_1^4 f(u)du=5/3$ **(b)** $u=3x,\ \frac{1}{3}\int_0^9 f(u)du=5/3$

(c) $u=x^2,\ 1/2\int_4^0 f(u)du=-1/2\int_0^4 f(u)du=-1/2$

56. $u=1-x,\ \int_0^1 x^m(1-x)^n dx=-\int_1^0(1-u)^m u^n du=\int_0^1 u^n(1-u)^m du=\int_0^1 x^n(1-x)^m dx$

57. $\sin x=\cos(\pi/2-x)$,

$$\int_0^{\pi/2}\sin^n x\,dx=\int_0^{\pi/2}\cos^n(\pi/2-x)dx=-\int_{\pi/2}^0\cos^n u\,du\quad(u=\pi/2-x)$$

$$=\int_0^{\pi/2}\cos^n u\,du=\int_0^{\pi/2}\cos^n x\,dx\qquad\text{(by replacing } u \text{ by } x\text{)}$$

58. $u=1-x,\ -\int_1^0(1-u)u^n du=\int_0^1(1-u)u^n du=\int_0^1(u^n-u^{n+1})du=\frac{1}{n+1}-\frac{1}{n+2}$

$$=\frac{1}{(n+1)(n+2)}$$

59. $y(t)=(802.137)\int e^{1.528t}dt=524.959e^{1.528t}+C;\ y(0)=750=524.959+C,\ C=225.041,$

$y(t)=524.959e^{1.528t}+225.041,\ y(12)=48,233,500,000$

60. $V_{\text{ave}}=\frac{275000}{10-0}\int_0^{10}e^{-0.17t}dt=-161764.7059e^{-0.17t}\Big]_0^{10}=\$132,212.96$

61. $s(t)=\int(25+10e^{-0.05t})dt=25t-200e^{-0.05t}+C$

(a) $s(10)-s(0)=250-200(e^{-0.5}-1)=450-200/\sqrt{e}\approx 328.69$ ft

(b) yes; without it the distance would have been 250 ft

62. $\displaystyle\int_0^k e^{2x}dx = 3,\ \frac{1}{2}e^{2x}\Big]_0^k = 3,\ \frac{1}{2}(e^{2k}-1) = 3, e^{2k} = 7, k = \frac{1}{2}\ln 7$

63. The area is given by $\displaystyle\int_0^2 1/(1+kx^2)dx = (1/\sqrt{k})\tan^{-1}(2\sqrt{k}) = 0.6$; solve for k to get $k = 5.081435$.

64. **(a)** $\displaystyle\int_0^1 \sin \pi x dx = 2/\pi$

65. **(a)** $\displaystyle V_{\text{rms}}^2 = \frac{1}{1/f - 0}\int_0^{1/f} V_p^2 \sin^2(2\pi ft)dt = \frac{1}{2}fV_p^2 \int_0^{1/f}[1-\cos(4\pi ft)]dt$

$$= \frac{1}{2}fV_p^2\left[t - \frac{1}{4\pi f}\sin(4\pi ft)\right]\Bigg]_0^{1/f} = \frac{1}{2}V_p^2, \text{ so } V_{\text{rms}} = V_p/\sqrt{2}$$

(b) $V_p/\sqrt{2} = 120, V_p = 120\sqrt{2} \approx 169.7$ V

66. Let $u = t - x$, then $du = -dx$ and
$\displaystyle\int_0^t f(t-x)g(x)dx = -\int_t^0 f(u)g(t-u)du = \int_0^t f(u)g(t-u)du;$
the result follows by replacing u by x in the last integral.

67. **(a)** $\displaystyle I = -\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}du = \int_0^a \frac{f(a-u)+f(u)-f(u)}{f(a-u)+f(u)}du$

$$= \int_0^a du - \int_0^a \frac{f(u)}{f(a-u)+f(u)}du, I = a - I \text{ so } 2I = a, I = a/2$$

(b) $3/2$ **(c)** $\pi/4$

68. $\displaystyle x = \frac{1}{u},\ dx = -\frac{1}{u^2}du,\ I = \int_{-1}^1 \frac{1}{1+1/u^2}(-1/u^2)du = -\int_{-1}^1 \frac{1}{u^2+1}du = -I$ so $I = 0$ which is impossible because $\displaystyle\frac{1}{1+x^2}$ is positive on $[-1, 1]$. The substitution $u = 1/x$ is not valid because u is not continuous for all x in $[-1, 1]$.

69. **(a)** Let $u = -x$ then

$$\int_{-a}^a f(x)dx = -\int_a^{-a} f(-u)du = \int_{-a}^a f(-u)du = -\int_{-a}^a f(u)du$$

so, replacing u by x in the latter integral,

$$\int_{-a}^a f(x)dx = -\int_{-a}^a f(x)dx,\ 2\int_{-a}^a f(x)dx = 0,\ \int_{-a}^a f(x)dx = 0$$

The graph of f is symmetric about the origin so $\displaystyle\int_{-a}^0 f(x)dx$ is the negative of $\displaystyle\int_0^a f(x)dx$ thus $\displaystyle\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)dx = 0$

(b) $\displaystyle\int_{-a}^a f(x)dx = \int_{-a}^0 f(x)dx + \int_0^a f(x)dx$, let $u = -x$ in $\displaystyle\int_{-a}^0 f(x)dx$ to get

$$\int_{-a}^0 f(x)dx = -\int_a^0 f(-u)du = \int_0^a f(-u)du = \int_0^a f(u)du = \int_0^a f(x)dx$$

so $\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

The graph of $f(x)$ is symmetric about the y-axis so there is as much signed area to the left of the y-axis as there is to the right.

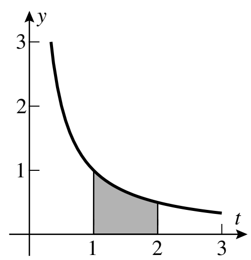
70. **(a)** By Exercise 69(a), $\int_{-1}^{1} x\sqrt{\cos(x^2)}\,dx = 0$

(b) $u = x - \pi/2, du = dx, \sin(u + \pi/2) = \sin u, \cos(u + \pi/2) = -\sin u$

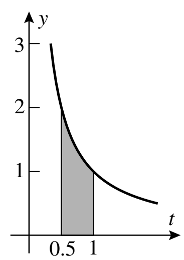
$\int_{0}^{\pi} \sin^8 x \cos^5 x\,dx = \int_{-\pi/2}^{\pi/2} \sin^8 u(-\sin^5 u)\,du = -\int_{-\pi/2}^{\pi/2} \sin^{13} u\,du = 0$ by Exercise 69(a).

EXERCISE SET 6.9

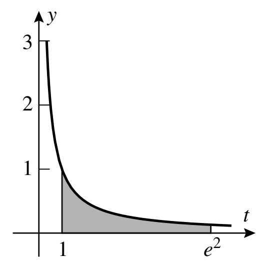
1. **(a)**



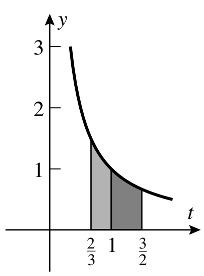
(b)



(c)



2.



3. **(a)** $\ln t\Big]_1^{ac} = \ln(ac) = \ln a + \ln c = 7$ **(b)** $\ln t\Big]_1^{1/c} = \ln(1/c) = -5$

(c) $\ln t\Big]_1^{a/c} = \ln(a/c) = 2 - 5 = -3$ **(d)** $\ln t\Big]_1^{a^3} = \ln a^3 = 3\ln a = 6$

4. **(a)** $\ln t\Big]_1^{\sqrt{a}} = \ln a^{1/2} = \dfrac{1}{2}\ln a = 9/2$ **(b)** $\ln t\Big]_1^{2a} = \ln 2 + 9$

(c) $\ln t\Big]_1^{2/a} = \ln 2 - 9$ **(d)** $\ln t\Big]_2^{a} = 9 - \ln 2$

5. $\ln 5 \approx 1.603210678$; $\ln 5 = 1.609437912$; magnitude of error is < 0.0063

6. $\ln 3 \approx 1.098242635$; $\ln 3 = 1.098612289$; magnitude of error is < 0.0004

7. **(a)** x^{-1}, $x > 0$ **(b)** x^2, $x \neq 0$

(c) $-x^2$, $-\infty < x < +\infty$ **(d)** $-x$, $-\infty < x < +\infty$

(e) x^3, $x > 0$ **(f)** $\ln x + x$, $x > 0$

(g) $x - \sqrt[3]{x}$, $-\infty < x < +\infty$ **(h)** $\dfrac{e^x}{x}$, $x > 0$

8. **(a)** $f(\ln 3) = e^{-2\ln 3} = e^{\ln(1/9)} = 1/9$

(b) $f(\ln 2) = e^{\ln 2} + 3e^{-\ln 2} = 2 + 3e^{\ln(1/2)} = 2 + 3/2 = 7/2$

9. **(a)** $3^{\pi} = e^{\pi \ln 3}$ **(b)** $2^{\sqrt{2}} = e^{\sqrt{2}\ln 2}$

10. **(a)** $\pi^{-x} = e^{-x\ln \pi}$ **(b)** $x^{2x} = e^{2x\ln x}$

11. **(a)** $\lim_{x\to+\infty}\left[\left(1+\frac{1}{x}\right)^x\right]^2 = \left[\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x\right]^2 = e^2$

(b) $y = 2x,\ \lim_{y\to 0}(1+y)^{2/y} = \lim_{y\to 0}\left[(1+y)^{1/y}\right]^2 = e^2$

12. **(a)** $y = 3x,\ \lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^{y/3} = \lim_{y\to+\infty}\left[\left(1+\frac{1}{y}\right)^y\right]^{1/3} = \left[\lim_{y\to+\infty}\left(1+\frac{1}{y}\right)^y\right]^{1/3} = e^{1/3}$

(b) $\lim_{x\to 0}(1+x)^{1/3x} = \lim_{x\to 0}\left[(1+x)^{1/x}\right]^{1/3} = e^{1/3}$

13. $g'(x) = x^2 - x$

14. $g'(x) = 1 - \cos x$

15. **(a)** $\frac{1}{x^3}(3x^2) = \frac{3}{x}$ **(b)** $e^{\ln x}\frac{1}{x} = 1$

16. **(a)** $2x\sqrt{x^2+1}$ **(b)** $-\left(\frac{1}{x^2}\right)\sin\left(\frac{1}{x}\right)$

17. $F'(x) = \frac{\cos x}{x^2+3},\ F''(x) = \frac{-(x^2+3)\sin\ x - 2x\cos x}{(x^2+3)^2}$

(a) 0 **(b)** 1/3 **(c)** 0

18. $F'(x) = \sqrt{3x^2+1},\ F''(x) = \frac{3x}{\sqrt{3x^2+1}}$

(a) 0 **(b)** $\sqrt{13}$ **(c)** $6/\sqrt{13}$

19. **(a)** $\frac{d}{dx}\int_1^{x^2} t\sqrt{1+t}\,dt = x^2\sqrt{1+x^2}(2x) = 2x^3\sqrt{1+x^2}$

(b) $\int_1^{x^2} t\sqrt{1+t}\,dt = -\frac{2}{3}(x^2+1)^{3/2} + \frac{2}{5}(x^2+1)^{5/2} - \frac{4\sqrt{2}}{15}$

20. **(a)** $\frac{d}{dx}\int_x^a f(t)dt = -\frac{d}{dx}\int_a^x f(t)dt = -f(x)$

(b) $\frac{d}{dx}\int_{g(x)}^a f(t)dt = -\frac{d}{dx}\int_a^{g(x)} f(t)dt = -f(g(x))g'(x)$

21. **(a)** $-\sin x^2$ **(b)** $-\frac{\tan^2 x}{1+\tan^2 x}\sec^2 x = -\tan^2 x$

22. **(a)** $-(x^2+1)^{40}$ **(b)** $-\cos^3\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = \frac{\cos^3(1/x)}{x^2}$

23. $-3\dfrac{3x-1}{9x^2+1}+2x\dfrac{x^2-1}{x^4+1}$

24. If f is continuous on an open interval I and $g(x)$, $h(x)$, and a are in I then

$$\int_{h(x)}^{g(x)} f(t)dt = \int_{h(x)}^{a} f(t)dt + \int_{a}^{g(x)} f(t)dt = -\int_{a}^{h(x)} f(t)dt + \int_{a}^{g(x)} f(t)dt$$

so $\dfrac{d}{dx}\displaystyle\int_{h(x)}^{g(x)} f(t)dt = -f(h(x))h'(x) + f(g(x))g'(x)$

25. **(a)** $\sin^2(x^3)(3x^2) - \sin^2(x^2)(2x) = 3x^2\sin^2(x^3) - 2x\sin^2(x^2)$

(b) $\dfrac{1}{1+x}(1) - \dfrac{1}{1-x}(-1) = \dfrac{2}{1-x^2}$

26. $F'(x) = \dfrac{1}{3x}(3) - \dfrac{1}{x}(1) = 0$ so $F(x)$ is constant on $(0, +\infty)$. $F(1) = \ln 3$ so $F(x) = \ln 3$ for all $x > 0$.

27. from geometry, $\displaystyle\int_0^3 f(t)dt = 0$, $\displaystyle\int_3^5 f(t)dt = 6$, $\displaystyle\int_5^7 f(t)dt = 0$; and $\displaystyle\int_7^{10} f(t)dt$

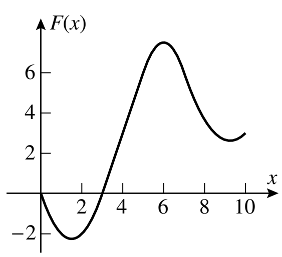
$$= \int_7^{10} (4t-37)/3dt = -3$$

(a) $F(0) = 0$, $F(3) = 0$, $F(5) = 6$, $F(7) = 6$, $F(10) = 3$

(b) F is increasing where $F' = f$ is positive, so on $[3/2, 6]$ and $[37/4, 10]$, decreasing on $[0, 3/2]$ and $[6, 37/4]$

(c) critical points when $F'(x) = f(x) = 0$, so $x = 3/2, 6, 37/4$; maximum $15/2$ at $x = 6$, minimum $-9/4$ at $x = 3/2$

(d)



28. $f_{\text{ave}} = \dfrac{1}{10-0}\displaystyle\int_0^{10} f(t)dt = \dfrac{1}{10}F(10) = 0.3$

29. $x < 0 : F(x) = \displaystyle\int_{-1}^{x} (-t)dt = -\frac{1}{2}t^2\Big]_{-1}^{x} = \frac{1}{2}(1-x^2)$,

$x \geq 0 : F(x) = \displaystyle\int_{-1}^{0} (-t)dt + \int_0^x t\,dt = \frac{1}{2} + \frac{1}{2}x^2$; $F(x) = \begin{cases} (1-x^2)/2, & x < 0 \\ (1+x^2)/2, & x \geq 0 \end{cases}$

30. $0 \leq x \leq 2 : F(x) = \displaystyle\int_0^x t\,dt = \frac{1}{2}x^2$,

$x > 2 : F(x) = \displaystyle\int_0^2 t\,dt + \int_2^x 2\,dt = 2 + 2(x-2) = 2x - 2$; $F(x) = \begin{cases} x^2/2, & 0 \leq x \leq 2 \\ 2x-2, & x > 2 \end{cases}$

31. $y(x) = 2 + \int_1^x t^{1/3} dt = 2 + \frac{3}{4} t^{4/3} \Big]_1^x = \frac{5}{4} + \frac{3}{4} x^{4/3}$

32. $y(x) = \int_1^x (t^{1/2} + t^{-1/2}) dt = \frac{2}{3} x^{3/2} - \frac{2}{3} + 2x^{1/2} - 2 = \frac{2}{3} x^{3/2} + 2x^{1/2} - \frac{8}{3}$

33. $y(x) = 1 + \int_{\pi/4}^x (\sec^2 t - \sin t) dt = \tan x + \cos x - \sqrt{2}/2$

34. $y(x) = \int_0^x t e^{t^2} dt = \frac{1}{2} e^{-x^2} - \frac{1}{2}$

35. $P(x) = P_0 + \int_0^x r(t) dt$ individuals

36. $s(T) = s_1 + \int_1^T v(t) dt$

37. II has a minimum at $x = 12$, and I has a zero there, so I could be the derivative of II; on the other hand I has a minimum near $x = 1/3$, but II is not zero there, so II could not be the derivative of I, so I is the graph of $f(x)$ and II is the graph of $\int_0^x f(t)\, dt$.

38. **(b)** $\lim_{k \to 0} \frac{1}{k}(x^k - 1) = \frac{d}{dt} x^t \Big]_{t=0} = \ln x$

39. **(a)** where $f(t) = 0$; by the First Derivative Test, at $t = 3$

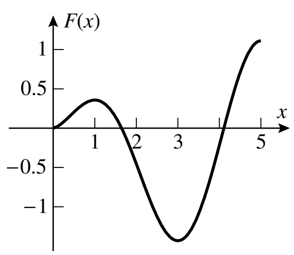
(b) where $f(t) = 0$; by the First Derivative Test, at $t = 1, 5$

(c) at $t = 0, 1$ or 5; from the graph it is evident that it is at $t = 5$

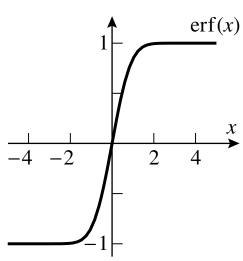
(d) at $t = 0, 3$ or 5; from the graph it is evident that it is at $t = 3$

(e) F is concave up when $F'' = f'$ is positive, i.e. where f is increasing, so on $(0, 1/2)$ and $(2, 4)$; it is concave down on $(1/2, 2)$ and $(4, 5)$

(f)



40. **(a)**



(c) $\text{erf}'(x) > 0$ for all x, so there are no relative extrema

(e) $\text{erf}''(x) = -4xe^{-x^2}/\sqrt{\pi}$ changes sign only at $x = 0$ so that is the only point of inflection

(g) $\lim_{x \to +\infty} \text{erf}(x) = +1$, $\lim_{x \to -\infty} \text{erf}(x) = -1$

41. $C'(x) = \cos(\pi x^2/2)$, $C''(x) = -\pi x \sin(\pi x^2/2)$

(a) $\cos t$ goes from negative to positive at $2k\pi - \pi/2$, and from positive to negative at $t = 2k\pi + \pi/2$, so $C(x)$ has relative minima when $\pi x^2/2 = 2k\pi - \pi/2$, $x = \pm\sqrt{4k-1}$, $k = 1, 2, \ldots$, and $C(x)$ has relative maxima when $\pi x^2/2 = (4k+1)\pi/2$, $x = \pm\sqrt{4k+1}$, $k = 0, 1, \ldots$.

(b) $\sin t$ changes sign at $t = k\pi$, so $C(x)$ has inflection points at $\pi x^2/2 = k\pi$, $x = \pm\sqrt{2k}$, $k = 1, 2, \ldots$; the case $k = 0$ is distinct due to the factor of x in $C''(x)$, but x changes sign at $x = 0$ and $\sin(\pi x^2/2)$ does not, so there is also a point of inflection at $x = 0$

42. Let $F(x) = \displaystyle\int_1^x \ln t\,dt$, $F'(x) = \displaystyle\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \ln t\,dt$; but $F'(x) = \ln x$ so $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} \ln t\,dt = \ln x$

43. Differentiate: $f(x) = 3e^{3x}$, so $2 + \displaystyle\int_a^x f(t)dt = 2 + \int_a^x 3e^{3t}dt = 2 + e^{3t}\Big]_a^x = 2 + e^{3x} - e^{3a} = e^{3x}$ provided $e^{3a} = 2$, $a = (\ln 2)/3$.

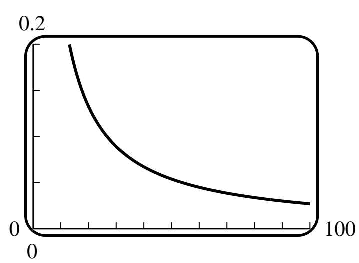
44. **(a)** The area under $1/t$ for $x \le t \le x+1$ is less than the area of the rectangle with altitude $1/x$ and base 1, but greater than the area of the rectangle with altitude $1/(x+1)$ and base 1.

(b) $\displaystyle\int_x^{x+1} \frac{1}{t}dt = \ln t\Big]_x^{x+1} = \ln(x+1) - \ln x = \ln(1 + 1/x)$, so $1/(x+1) < \ln(1 + 1/x) < 1/x$ for $x > 0$.

(c) from Part (b), $e^{1/(x+1)} < e^{\ln(1+1/x)} < e^{1/x}$, $e^{1/(x+1)} < 1 + 1/x < e^{1/x}$, $e^{x/(x+1)} < (1 + 1/x)^x < e$; by the Squeezing Theorem, $\displaystyle\lim_{x\to+\infty}(1 + 1/x)^x = e$.

(d) Use the inequality $e^{x/(x+1)} < (1 + 1/x)^x$ to get $e < (1 + 1/x)^{x+1}$ so $(1 + 1/x)^x < e < (1 + 1/x)^{x+1}$.

45. From Exercise 44(d) $\left| e - \left(1 + \dfrac{1}{50}\right)^{50} \right| < y(50)$, and from the graph $y(50) < 0.06$



46. $F'(x) = f(x)$, thus $F'(x)$ has a value at each x in I because f is continuous on I so F is continuous on I because a function that is differentiable at a point is also continuous at that point

CHAPTER 6 SUPPLEMENTARY EXERCISES

5. If the acceleration $a = \text{const}$, then $v(t) = at + v_0$, $s(t) = \frac{1}{2}at^2 + v_0t + s_0$.

6. **(a)** Divide the base into n equal subintervals. Above each subinterval choose the lowest and highest points on the curved top. Draw a rectangle above the subinterval going through the lowest point, and another through the highest point. Add the rectangles that go through the lowest points to obtain a lower estimate of the area; add the rectangles through the highest points to obtain an upper estimate of the area.

(b) $n = 10$: 25.0 cm, 22.4 cm

(c) $n = 20$: 24.4 cm, 23.1 cm

7. **(a)** $\dfrac{1}{2} + \dfrac{1}{4} = \dfrac{3}{4}$ **(b)** $-1 - \dfrac{1}{2} = -\dfrac{3}{2}$

(c) $5\left(-1 - \dfrac{3}{4}\right) = -\dfrac{35}{4}$ **(d)** -2

(e) not enough information **(f)** not enough information

8. **(a)** $\dfrac{1}{2} + 2 = \dfrac{5}{2}$ **(b)** not enough information

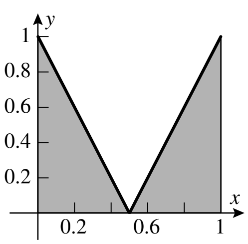
(c) not enough information **(d)** $4(2) - 3\dfrac{1}{2} = \dfrac{13}{2}$

9. **(a)** $\displaystyle\int_{-1}^{1} dx + \int_{-1}^{1} \sqrt{1 - x^2}\, dx = 2(1) + \pi(1)^2/2 = 2 + \pi/2$

(b) $\displaystyle\frac{1}{3}(x^2 + 1)^{3/2}\Big]_0^3 - \pi(3)^2/4 = \frac{1}{3}(10^{3/2} - 1) - 9\pi/4$

(c) $u = x^2$, $du = 2xdx$; $\displaystyle\frac{1}{2}\int_0^1 \sqrt{1 - u^2}du = \frac{1}{2}\pi(1)^2/4 = \pi/8$

10. $\dfrac{1}{2}$



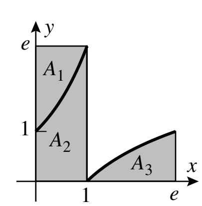
11. The rectangle with vertices $(0, 0)$, $(\pi, 0)$, $(\pi, 1)$ and $(0, 1)$ has area π and is much too large; so is the triangle with vertices $(0, 0)$, $(\pi, 0)$ and $(\pi, 1)$ which has area $\pi/2$; $1 - \pi$ is negative; so the answer is $35\pi/128$.

12. Divide $e^x + 3$ into e^{2x} to get $\dfrac{e^{2x}}{e^x + 3} = e^x - \dfrac{3e^x}{e^x + 3}$ so

$$\int \frac{e^{2x}}{e^x + 3}dx = \int e^x dx - 3\int \frac{e^x}{e^x + 3}dx = e^x - 3\ln(e^x + 3) + C$$

13. Since $y = e^x$ and $y = \ln x$ are inverse functions, their graphs are symmetric with respect to the line $y = x$; consequently the areas A_1 and A_3 are equal (see figure). But $A_1 + A_2 = e$, so

$$\int_1^e \ln x dx + \int_0^1 e^x dx = A_2 + A_3 = A_2 + A_1 = e$$



14. **(a)** $\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}} = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = \sqrt{x}$, $x_k^* = k/n$, and $\Delta x = 1/n$ for $0 \le x \le 1$. Thus

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}} = \int_0^1 x^{1/2}dx = \frac{2}{3}$$

(b) $\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4 = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = x^4$, $x_k^* = k/n$, and $\Delta x = 1/n$ for $0 \le x \le 1$. Thus

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4 = \int_0^1 x^4 dx = \frac{1}{5}$$

(c) $\sum_{k=1}^{n}\frac{e^{k/n}}{n} = \sum_{k=1}^{n} f(x_k^*)\Delta x$ where $f(x) = e^x, x_k^* = k/n$, and $\Delta x = 1/n$ for $0 \le x \le 1$. Thus

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{e^{k/n}}{n} = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x = \int_0^1 e^x dx = e - 1.$$

15. Since $f(x) = \frac{1}{x}$ is positive and increasing on the interval $[1, 2]$, the left endpoint approximation overestimates the integral of $\frac{1}{x}$ and the right endpoint approximation underestimates it.

(a) For $n = 5$ this becomes

$$0.2\left[\frac{1}{1.2}+\frac{1}{1.4}+\frac{1}{1.6}+\frac{1}{1.8}+\frac{1}{2.0}\right] < \int_1^2 \frac{1}{x}dx < 0.2\left[\frac{1}{1.0}+\frac{1}{1.2}+\frac{1}{1.4}+\frac{1}{1.6}+\frac{1}{1.8}\right]$$

(b) For general n the left endpoint approximation to $\int_1^2 \frac{1}{x}dx = \ln 2$ is

$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(k-1)/n} = \sum_{k=1}^{n}\frac{1}{n+k-1} = \sum_{k=0}^{n-1}\frac{1}{n+k}$ and the right endpoint approximation is $\sum_{k=1}^{n}\frac{1}{n+k}$. This yields $\sum_{k=1}^{n}\frac{1}{n+k} < \int_1^2 \frac{1}{x}dx < \sum_{k=0}^{n-1}\frac{1}{n+k}$ which is the desired inequality.

(c) By telescoping, the difference is $\frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$ so $\frac{1}{2n} \le 0.1$, $n \ge 5$

(d) $n \ge 1{,}000$

16. The direction field is clearly an even function, which means that the solution is even, its derivative is odd. Since $\sin x$ is periodic and the direction field is not, that eliminates all but x, the solution of which is the family $y = x^2/2 + C$.

17. **(a)**

$$1\cdot 2 + 2\cdot 3 + \cdots + n(n+1) = \sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$
$$= \frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1) = \frac{1}{3}n(n+1)(n+2)$$

(b) $\sum_{k=1}^{n-1}\left(\frac{9}{n} - \frac{k}{n^2}\right) = \frac{9}{n}\sum_{k=1}^{n-1} 1 - \frac{1}{n^2}\sum_{k=1}^{n-1} k = \frac{9}{n}(n-1) - \frac{1}{n^2}\cdot\frac{1}{2}(n-1)(n) = \frac{17}{2}\left(\frac{n-1}{n}\right)$;

$$\lim_{n\to+\infty}\frac{17}{2}\left(\frac{n-1}{n}\right) = \frac{17}{2}$$

(c) $\sum_{i=1}^{3}\left[\sum_{j=1}^{2} i + \sum_{j=1}^{2} j\right] = \sum_{i=1}^{3}\left[2i + \frac{1}{2}(2)(3)\right] = 2\sum_{i=1}^{3} i + \sum_{i=1}^{3} 3 = 2\cdot\frac{1}{2}(3)(4) + (3)(3) = 21$

18. **(a)** $\sum_{k=0}^{14}(k+4)(k+1)$ **(b)** $\sum_{k=5}^{19}(k-1)(k-4)$

19. For $1 \le k \le n$ the k-th L-shaped strip consists of the corner square, a strip above and a strip to the right for a combined area of $1+(k-1)+(k-1) = 2k-1$, so the total area is $\sum_{k=1}^{n}(2k-1) = n^2$.

20. $1+3+5+\cdots+(2n-1) = \sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n}k - \sum_{k=1}^{n}1 = 2\cdot\frac{1}{2}n(n+1) - n = n^2$

21. $(3^5-3^4)+(3^6-3^5)+\cdots+(3^{17}-3^{16}) = 3^{17}-3^4$

22. $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{50}-\frac{1}{51}\right) = \frac{50}{51}$

23. $\left(\frac{1}{2^2}-\frac{1}{1^2}\right)+\left(\frac{1}{3^2}-\frac{1}{2^2}\right)+\cdots+\left(\frac{1}{20^2}-\frac{1}{19^2}\right) = \frac{1}{20^2}-1 = -\frac{399}{400}$

24. $(2^2-2)+(2^3-2^2)+\cdots+(2^{101}-2^{100}) = 2^{101}-2$

25. **(a)**
$$\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)$$
$$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right]$$
$$= \frac{1}{2}\left[1-\frac{1}{2n+1}\right] = \frac{n}{2n+1}$$

(b) $\lim_{n\to+\infty}\frac{n}{2n+1} = \frac{1}{2}$

26. **(a)**
$$\sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$$
$$= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$= 1-\frac{1}{n+1} = \frac{n}{n+1}$$

(b) $\lim_{n\to+\infty}\frac{n}{n+1} = 1$

27. $\sum_{i=1}^{n}(x_i-\bar{x}) = \sum_{i=1}^{n}x_i - \sum_{i=1}^{n}\bar{x} = \sum_{i=1}^{n}x_i - n\bar{x}$ but $\bar{x} = \frac{1}{n}\sum_{i=1}^{n}x_i$ thus

$\sum_{i=1}^{n}x_i = n\bar{x}$ so $\sum_{i=1}^{n}(x_i-\bar{x}) = n\bar{x}-n\bar{x} = 0$

28.
$$S - rS = \sum_{k=0}^{n}ar^k - \sum_{k=0}^{n}ar^{k+1}$$
$$= (a+ar+ar^2+\cdots+ar^n)-(ar+ar^2+ar^3+\cdots+ar^{n+1})$$
$$= a - ar^{n+1} = a(1-r^{n+1})$$

so $(1-r)S = a(1-r^{n+1})$, hence $S = a(1-r^{n+1})/(1-r)$

29. **(a)** $\displaystyle\sum_{k=0}^{19} 3^{k+1} = \sum_{k=0}^{19} 3(3^k) = \frac{3(1-3^{20})}{1-3} = \frac{3}{2}(3^{20}-1)$

(b) $\displaystyle\sum_{k=0}^{25} 2^{k+5} = \sum_{k=0}^{25} 2^5 2^k = \frac{2^5(1-2^{26})}{1-2} = 2^{31} - 2^5$

(c) $\displaystyle\sum_{k=0}^{100}(-1)\left(\frac{-1}{2}\right)^k = \frac{(-1)(1-(-1/2)^{101})}{1-(-1/2)} = -\frac{2}{3}(1+1/2^{101})$

30. **(a)** $1.999023438, 1.999999046, 2.000000000;\ 2$ **(b)** $2.831059456, 2.990486364, 2.999998301;\ 3$

31. **(a)** If $u = \sec x$, $du = \sec x \tan x dx$, $\displaystyle\int \sec^2 x \tan x dx = \int u du = u^2/2 + C_1 = (\sec^2 x)/2 + C_1$;

if $u = \tan x$, $du = \sec^2 x dx$, $\displaystyle\int \sec^2 x \tan x dx = \int u du = u^2/2 + C_2 = (\tan^2 x)/2 + C_2$.

(b) They are equal only if $\sec^2 x$ and $\tan^2 x$ differ by a constant, which is true.

32. $\displaystyle\frac{1}{2}\sec^2 x\Big]_0^{\pi/4} = \frac{1}{2}(2-1) = 1/2$ and $\displaystyle\tfrac{1}{2}\tan^2 x\Big]_0^{\pi/4} = \frac{1}{2}(1-0) = 1/2$

33. $\displaystyle\int \sqrt{1+x^{-2/3}}dx = \int x^{-1/3}\sqrt{x^{2/3}+1}dx$; $u = x^{2/3}+1$, $du = \frac{2}{3}x^{-1/3}dx$

$\displaystyle\frac{3}{2}\int u^{1/2}du = u^{3/2} + C = (x^{2/3}+1)^{3/2} + C$

34. **(a)** $\displaystyle\int_a^b \sum_{k=1}^{n} f_k(x)dx = \sum_{k=1}^{n}\int_a^b f_k(x)dx$

(b) yes; substitute $c_k f_k(x)$ for $f_k(x)$ in part (a), and then use $\displaystyle\int_a^b c_k f_k(x)dx = c_k\int_a^b f_k(x)dx$ from Theorem 6.5.4

35. left endpoints: $x_k^* = 1, 2, 3, 4$; $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = (2+3+2+1)(1) = 8$

right endpoints: $x_k^* = 2, 3, 4, 5$; $\displaystyle\sum_{k=1}^{4} f(x_k^*)\Delta x = (3+2+1+2)(1) = 8$

36. **(a)** $x_k^* = 0, 1, 2, 3, 4$

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(e^0+e^1+e^2+e^3+e^4\right)(1) = (1-e^5)/(1-e) = 85.791$$

(b) $x_k^* = 1, 2, 3, 4, 5$

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(e^1+e^2+e^3+e^4+e^5\right)(1) = e(1-e^5)/(1-e) = 233.204$$

(c) $x_k^* = 1/2, 3/2, 5/2, 7/2, 9/2$

$$\sum_{k=1}^{4} f(x_k^*)\Delta x = \left(e^{1/2}+e^{3/2}+e^{5/2}+e^{7/2}+e^{9/2}\right)(1) = e^{1/2}(1-e^5)/(1-e) = 141.446$$

37. $\displaystyle f\text{ave} = \frac{1}{e-1}\int_1^e \frac{1}{x}dx = \frac{1}{e-1}\ln x\Big]_1^e = \frac{1}{e-1};\ \frac{1}{x^*} = \frac{1}{e-1},\ x^* = e-1$

38. $\lim_{n\to+\infty}\sum_{k=1}^{n}\left[\frac{25(k-1)}{n}-\frac{25(k-1)^2}{n^2}\right]\frac{5}{n}=\frac{125}{6}$

39. $0.351220577, 0.420535296, 0.386502483$

40. $1.63379940, 1.805627583, 1.717566087$

41. $f(x)=e^x, [a,b]=[0,1], \Delta x=\frac{1}{n}$; $\lim_{n\to+\infty}\sum_{k=1}^{n}f(x_k^*)\frac{1}{n}=\int_0^1 e^x\,dx=e-1$

42. **(a)** e^{x^2} **(b)** $\ln x$

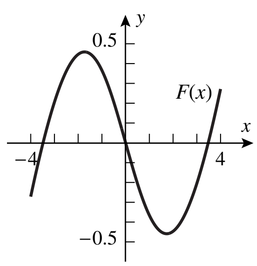
43. **(a)** $\int_1^x \frac{1}{1+e^t}dt$ **(b)** $\int_{-\ln(e^2+e-1)}^{x}\frac{1}{1+e^t}dt$

44. **(a)** $F'(x)=\frac{x^2-3}{x^2+7}$; increasing on $(-\infty,-\sqrt{3}], [\sqrt{3},+\infty)$, decreasing on $[-\sqrt{3},\sqrt{3}]$

(b) $F''(x)=\frac{20x}{(x^2+7)^2}$; concave down on $(-\infty,0)$, concave up on $(0,+\infty)$

(c) $\lim_{x\to\pm\infty}F(x)=\mp\infty$, so F has no absolute extrema.

(d)



45. $F'(x)=\frac{1}{1+x^2}+\frac{1}{1+(1/x)^2}(-1/x^2)=0$ so F is constant on $(0,+\infty)$.

46. $(-3,3)$ because f is continuous there and 1 is in $(-3,3)$

47. **(a)** The domain is $(-\infty,+\infty)$; $F(x)$ is 0 if $x=1$, positive if $x>1$, and negative if $x<1$, because the integrand is positive, so the sign of the integral depends on the orientation (forwards or backwards).

(b) The domain is $[-2,2]$; $F(x)$ is 0 if $x=-1$, positive if $-1<x\le 2$, and negative if $-2\le x<-1$; same reasons as in Part (a).

48. The left endpoint of the top boundary is $((b-a)/2,h)$ and the right endpoint of the top boundary is $((b+a)/2,h)$ so

$$f(x)=\begin{cases}2hx/(b-a), & x<(b-a)/2\\ h, & (b-a)/2<x<(b+a)/2\\ 2h(x-b)/(a-b), & x>(a+b)/2\end{cases}$$

The area of the trapezoid is given by

$$\int_0^{(b-a)/2}\frac{2hx}{b-a}dx+\int_{(b-a)/2}^{(b+a)/2}h\,dx+\int_{(b+a)/2}^{b}\frac{2h(x-b)}{a-b}dx=(b-a)h/4+ah+(b-a)h/4=h(a+b)/2.$$

49. **(a)** no, since the velocity curve is not a straight line
(b) $25 < t < 40$ **(c)** 3.54 ft/s **(d)** 141.5 ft
(e) no since the velocity is positive and the acceleration is never negative
(f) need the position at any one given time (e.g. s_0)

50. $w_(t) = \int_0^t \tau/7\, d\tau = t^2/14$, assuming $w_0 = w(0) = 0$; $w_{\text{ave}} = \frac{1}{26}\int_{26}^{52} t^2/7\, dt = \frac{1}{26}\frac{t^3}{21}\Big]_{26}^{52} = 676/3$
Set $676/3 = t^2/14, t = \pm\frac{26}{3}\sqrt{21}$, so $t \approx 39.716$, so during the 40th week.

51. $u = 5 + 2\sin 3x$, $du = 6\cos 3x dx$; $\int \frac{1}{6\sqrt{u}}du = \frac{1}{3}u^{1/2} + C = \frac{1}{3}\sqrt{5 + 2\sin 3x} + C$

52. $u = 3 + \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}dx$; $\int 2\sqrt{u}du = \frac{4}{3}u^{3/2} + C = \frac{4}{3}(3 + \sqrt{x})^{3/2} + C$

53. $u = ax^3 + b$, $du = 3ax^2 dx$; $\int \frac{1}{3au^2}du = -\frac{1}{3au} + C = -\frac{1}{3a^2x^3 + 3ab} + C$

54. $u = ax^2$, $du = 2ax dx$; $\frac{1}{2a}\int \sec^2 u du = \frac{1}{2a}\tan u + C = \frac{1}{2a}\tan(ax^2) + C$

55. $\left(-\frac{1}{3u^3} - \frac{3}{u} + \frac{1}{4u^4}\right)\Big]_{-2}^{-1} = 389/192$

56. $\frac{1}{3\pi}\sin^3 \pi x\Big]_0^1 = 0$

57. $u = \ln x$, $du = (1/x)dx$; $\int_1^2 \frac{1}{u}du = \ln u\Big]_1^2 = \ln 2$

58. $\int_0^1 e^{-x/2}dx = 2(1 - 1/\sqrt{e})$

59. $u = e^{-2x}$, $du = -2e^{-2x}dx$; $-\frac{1}{2}\int_1^{1/4}(1 + \cos u)du = \frac{3}{8} + \frac{1}{2}\left(\sin 1 - \sin\frac{1}{4}\right)$

60. $100{,}000/(\ln 100{,}000) \approx 8686$; $\int_2^{100{,}000}\frac{1}{\ln t}dt \approx 9629$, so the integral is better

61. With $b = 1.618034$, area $= \int_0^b (x + x^2 - x^3)dx = 1.007514$.

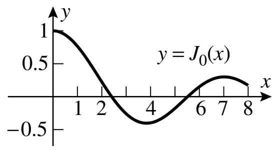
62. **(a)** $f(x) = \frac{1}{3}x^2\sin 3x - \frac{2}{27}\sin 3x + \frac{2}{9}x\cos 3x - 0.251607$

(b) $f(x) = \sqrt{4 + x^2} + \frac{4}{\sqrt{4 + x^2}} - 6$

63. **(a)** Solve $\frac{1}{4}k^4 - k - k^2 + \frac{7}{4} = 0$ to get $k = 2.073948$.

(b) Solve $-\frac{1}{2}\cos 2k + \frac{1}{3}k^3 + \frac{1}{2} = 3$ to get $k = 1.837992$.

64. $F(x) = \int_{-1}^{x}\frac{t}{\sqrt{2 + t^3}}dt$, $F'(x) = \frac{x}{\sqrt{2 + x^3}}$, so F is increasing on $[1, 3]$; $F_{\text{max}} = F(3) \approx 1.152082854$ and $F_{\text{min}} = F(1) \approx -0.07649493141$

65. **(a)** 

(b) 0.7651976866 **(c)** $J_0(x) = 0$ if $x = 2.404826$

CHAPTER 6 HORIZON MODULE

1. $v_x(0) = 35\cos\alpha$, so from Equation (1), $x(t) = (35\cos\alpha)t$; $v_y(0) = 35\sin\alpha$, so from Equation (2), $y(t) = (35\sin\alpha)t - 4.9t^2$.

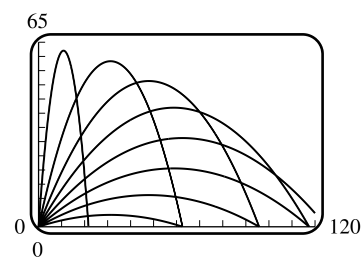
2. **(a)** $v_x(t) = \dfrac{dx(t)}{dt} = 35\cos\alpha$, $v_y(t) = \dfrac{dy(t)}{dt} = 35\sin\alpha - 9.8t$

(b) $v_y(t) = 35\sin\alpha - 9.8t$, $v_y(t) = 0$ when $t = 35\sin\alpha/9.8$;
$y = v_y(0)t - 4.9t^2 = (35\sin\alpha)(35\sin\alpha)/9.8 - 4.9((35\sin\alpha)/9.8)^2 = 62.5\sin^2\alpha$, so
$y_{\max} = 62.5\sin^2\alpha$.

3. $t = x/(35\cos\alpha)$ so $y = (35\sin\alpha)(x/(35\cos\alpha)) - 4.9(x/(35\cos\alpha))^2 = (\tan\alpha)x - \dfrac{0.004}{\cos^2\alpha}x^2$;
the trajectory is a parabola because y is a quadratic function of x.

4.

15°	25°	35°	45°	55°	65°	75°	85°
no	yes	no	no	no	yes	no	no



5. $y(t) = (35\sin\alpha\text{ s})t - 4.9t^2 = 0$ when $t = 35\sin\alpha/4.9$, at which time
$x = (35\cos\alpha)(35\sin\alpha/4.9) = 125\sin 2\alpha$; this is the maximum value of x, so $R = 125\sin 2\alpha$ m.

6. **(a)** $R = 95$ when $\sin 2\alpha = 95/125 = 0.76$, $\alpha = 0.4316565575, 1.139139769$ rad $\approx 24.73°, 65.27°$.

(b) $y(t) < 50$ is required; but $y(1.139) \approx 51.56$ m, so his height would be 56.56 m.

7. $0.4019 < \alpha < 0.4636$ (radians), or $23.03° < \alpha < 26.57°$