Calculus Chapter 1

Introduction to Calculus

This chapter, which replaces Chapter 4 in Physics 2000, is intended for students who have not had calculus, or as a calculus review for those whose calculus is not well remembered. If, after reading part way through this chapter, you feel your calculus background is not so bad after all, go back to Chapter 4 in Physics 2000, study the derivation of the constant acceleration formulas beginning on page 4-8, and work the projectile motion problems in the appendix to Chapter 4. Those who study all of this introduction to calculus should then proceed to the projectile motion problems in the appendix to Chapter 4 of the physics text.

In Chapter 3 of Physics 2000, we used strobe photographs to define velocity and acceleration vectors. The basic approach was to turn up the strobe flashing rate as we did in going from Figure (3-3) to (3-4) until all the kinks are clearly visible and the successive displacement vectors give a reasonable description of the motion. We did not turn the flashing rate too high, for the practical reason that the displacement vectors became too short for accurate work.

LIMITING PROCESS

In our discussion of instantaneous velocity we conceptually turned the strobe all the way up as illustrated in Figures (2-22a) through (2-22d), redrawn here in Figure (1). In these figures, we initially see a fairly large change in $\vec{v}_0$ as the strobe rate is increased and Δt reduced. But the change becomes smaller and it looks as if we are approaching some final value of $\vec{v}_0$ that does not depend on the size of Δt, provided Δt is small enough. It looks as if we have come close to the final value in Figure (1c).

The progression seen in Figure (1) is called a ***limiting process***. The idea is that there really is some true value of $\vec{v}_0$ which we have called the instantaneous velocity, and that we approach this true value for sufficiently small values of Δt. This is a calculus concept, and in the language of calculus, we are ***taking the limit as*** Δt ***goes to zero***.

The Uncertainty Principle

For over 200 years, from the invention of calculus by Newton and Leibnitz until 1924, the limiting process and the resulting concept of instantaneous velocity was one of the cornerstones of physics. Then in 1924 Werner Heisenberg discovered what he called the ***uncertainty principle*** which places a limit on the accuracy of experimental measurements.

Heisenberg discovered something very new and unexpected. He found that the act of making an experimental measurement unavoidably affects the results of an experiment. This had not been known previously because the effect on large objects like golf balls is undetectable. But on an atomic scale where we study small systems like electrons moving inside an atom, the effect is not only observable, it can dominate our study of the system.

One particular consequence of the uncertainly principle is that the more accurately we measure the position of an object, the more we disturb the motion of the object. This has an immediate impact on the concept of instantaneous velocity. If we turn the strobe all the way up, reduce Δt to zero, we are in effect trying to measure the position of the object with infinite precision. The consequence would be an infinitely big disturbance of the motion of the object we are studying. If we actually could turn the strobe all the way up, we would destroy the object we were trying to study.

It turns out that the uncertainty principle can have a significant impact on a larger scale of distance than the atomic scale. Suppose, for example, that we constructed a chamber 1 cm on a side, and wished to study the projectile motion of an electron inside. Using Galileo's idea that objects of different mass fall at the same rate, we would expect that the motion of the electron projectile should be the same as more massive objects. If we took a strobe photograph of the electron's motion, we would expect get results like those shown in Figure (2). This figure represents projectile motion with an acceleration $g = 980\,\text{cm/sec}^2$ and $\Delta t = .01\,\text{sec}$, as the reader can easily check.

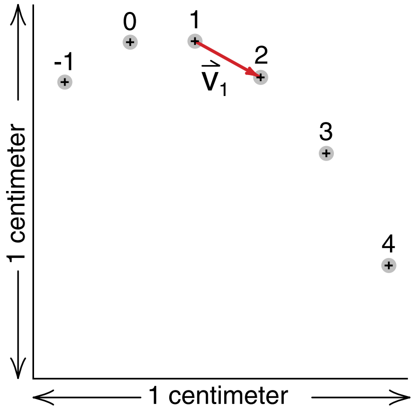


Figure 2
Hypothetical electron projectile motion experiment.

When we study the uncertainty principle in Chapter 40 of the physics text, we will see that a measurement which is accurate enough to show that Position (2) is below Position (1), could disturb the electron enough to reverse its direction of motion. The next position measurement could find the electron over where we drew Position (3), or back where we drew Position (0), or anywhere in the region in between. As a result we could not even determine what direction the electron is moving. This uncertainty would not be the result of a sloppy experiment, it is the best we can do with the most accurate and delicate measurements possible.

The uncertainty principle has had a significant impact on the way physicists think about motion. Because we now know that the measuring process affects the results of the measurement, we see that it is essential to provide experimental definitions to any physical quantity we wish to study. A conceptual definition, like turning the strobe all the way up to define instantaneous velocity, can lead to fundamental inconsistencies.

Even an experimental definition like our strobe definition of velocity can lead to inconsistent results when applied to something like the electron in Figure (2). But these inconsistencies are real. Their existence is telling us that the very concept of velocity is beginning to lose meaning for these small objects.

On the other hand, the idea of the limiting process and instantaneous velocity is very convenient when applied to larger objects where the effects of the uncertainty principle are not detectable. In this case we can apply all the mathematical tools of calculus developed over the past 250 years. The status of instantaneous velocity has changed from a basic concept to a useful mathematical tool. Those problems for which this mathematical tool works are called problems in ***classical physics***; and those problems for which the uncertainty principle is important, are in the realm of what we call ***quantum physics***.

CALCULUS DEFINITION OF VELOCITY

With the above perspective on the physical limitations on the limiting process, we can now return to the main topic of this chapter—the use of calculus in defining and working with velocity and acceleration.

In discussing the limiting process in calculus, one traditionally uses a special set of symbols which we can understand if we adopt the notation shown in Figure (3). In that figure we have drawn the coordinate vectors $\vec{R}_i$ and $\vec{R}_{i+1}$ for the i th and (i + 1) th positions of the object. We are now using the symbol $\overrightarrow{\Delta R}_i$ to represent the displacement of the ball during the i to i+1 interval. The vector equation for $\overrightarrow{\Delta R}_i$ is

$$\overrightarrow{\Delta R}_i = \vec{R}_{i+1} - \vec{R}_i \qquad (1)$$

In words, Equation (1) tells us that $\overrightarrow{\Delta R}_i$ is the change, during the time Δt, of the position vector $\vec{R}$ describing the location of the ball.

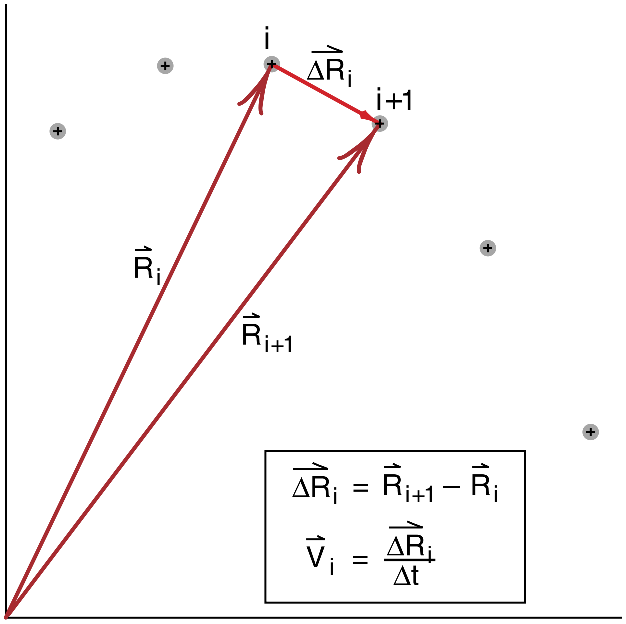


Figure 3
Definitions of $\overrightarrow{\Delta R}_i$ *and* $\vec{v}_i$.

The velocity vector $\vec{v}_i$ is now given by

$$\vec{v}_i = \frac{\overrightarrow{\Delta R}_i}{\Delta t} \qquad (2)$$

This is just our old strobe definition $\vec{v}_i = \vec{s}_i/\Delta t$, but using a notation which emphasizes that the displacement $\vec{s}_i = \overrightarrow{\Delta R}_i$ is the ***change in position*** that occurs during the time Δt. The Greek letter Δ (delta) is used both to represent the idea that the quantity $\overrightarrow{\Delta R}_i$ or Δt is small, and to emphasize that both of these quantities change as we change the strobe rate.

The limiting process in Figure (1) can be written in the form

$$\underline{\vec{v}}_i \equiv \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta R}_i}{\Delta t} \qquad (3)$$

where the word "limit" with $\Delta t \to 0$ underneath, is to be read as "limit as Δt goes to zero". For example we would read Equation (3) as "***the instantaneous velocity*** $\underline{\vec{v}}_i$ ***at position*** i ***is the limit, as*** Δt ***goes to zero, of the ratio*** $\overrightarrow{\Delta R}_i/\Delta t$."

For two reasons, Equation (3) is not quite yet in standard calculus notation. One is that in calculus, only the limiting value, in this case, the instantaneous velocity, is considered to be important. Our strobe definition $\vec{v}_i = \overrightarrow{\Delta R}_i/\Delta t$ is only a step in the limiting process. Therefore when we see the vector $\vec{v}_i$, we should assume that it is the limiting value, and no special symbol like the underline is used. For this reason we will drop the underline and write

$$\vec{v}_i = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta R}_i}{\Delta t} \qquad (3a)$$

The second change deals with the fact that when Δt goes to zero we need an infinite number to time steps to get through our strobe photograph, and thus it is not possible to locate a position by counting time steps. Instead we measure the time t that has elapsed since the beginning of the photograph, and use that time to tell us where we are, as illustrated in Figure (4). Thus instead of using $\vec{v}_i$ to represent the velocity at position i, we write $\vec{v}(t)$ to represent the velocity at time t. Equation (3) now becomes

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta R}(t)}{\Delta t} \tag{3b}$$

where we also replaced $\overrightarrow{\Delta R}_i$ by its value $\overrightarrow{\Delta R}(t)$ at time t.

Although Equation (3b) is in more or less standard calculus notation, the notation is clumsy. It is a pain to keep writing the word “limit” with a $\Delta t \to 0$ underneath. To streamline the notation, we replace the Greek letter Δ with the English letter d as follows

$$\boxed{\lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta R}(t)}{\Delta t} \equiv \frac{d\vec{R}(t)}{dt}} \tag{4}$$

(The symbol $\equiv$ means *defined equal to*.) To a mathematician, the symbol $d\vec{R}(t)/dt$ is just shorthand notation for the limiting process we have been describing. But to a physicist, there is a different, more practical meaning. Think of dt as a short Δt, short enough so that the limiting process has essentially occurred, but not too short to see what is going on. In Figure (1), a value of dt less than .025 seconds is probably good enough.

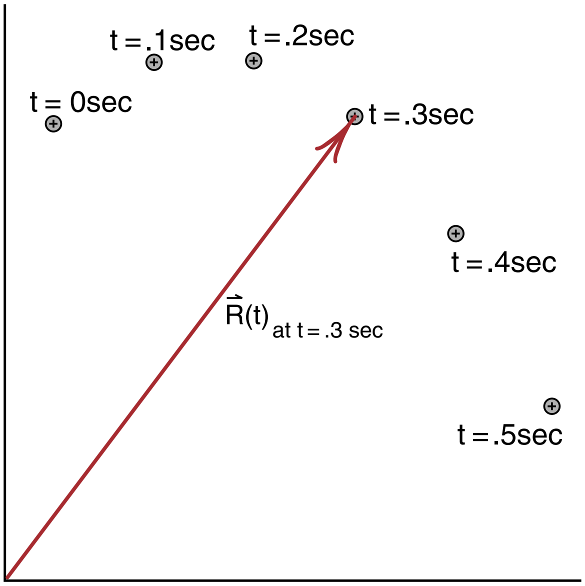


Figure 4
Rather than counting individual images, we can locate a position by measuring the elapsed time t. In this figure, we have drawn the displacement vector $\vec{R}(t)$ at time t = .3 sec.

If dt is small but finite, then we know exactly what the $d\vec{R}(t)$ is. It is the small but finite displacement vector at the time t. It is our old strobe definition of velocity, with the added condition that dt is such a short time interval that the limiting process has occurred. From this point of view, dt is a real time interval and $d\vec{R}(t)$ a real vector, which we can work with in a normal way. The only thing special about these quantities is that when we see the letter d instead of Δ, we must remember that a limiting process is involved. In this notation, the calculus definition of velocity is

$$\boxed{\vec{v}(t) = \frac{d\vec{R}(t)}{dt}} \tag{5}$$

where $\vec{R}(t)$ and $\vec{v}(t)$ are the particle’s coordinate vector and velocity vector respectively, as shown in Figure (5). Remember that this is just fancy shorthand notation for the limiting process we have been describing.

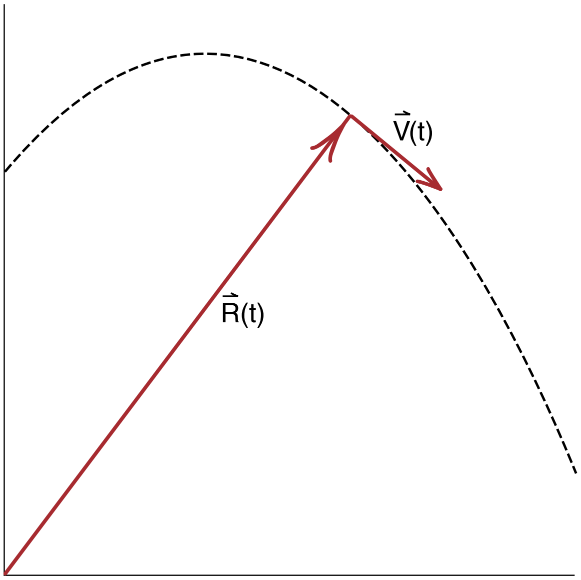


Figure 5
Instantaneous position and velocity at time t.

ACCELERATION

In the analysis of strobe photographs, we defined both a velocity vector $\vec{v}$ and an acceleration vector $\vec{a}$. The definition of $\vec{a}$, shown in Figure (2-12) reproduced here in Figure (6) was

$$\vec{a}_i \equiv \frac{\vec{v}_{i+1} - \vec{v}_i}{\Delta t} \tag{6}$$

In our graphical work we replaced $\vec{v}_i$ by $\vec{s}_i/\Delta t$ so that we could work directly with the displacement vectors $\vec{s}_i$ and experimentally determine the behavior of the acceleration vector for several kinds of motion.

Let us now change this graphical definition of acceleration over to a calculus definition, using the ideas just applied to the velocity vector. First, assume that the ball reached position i at time t as shown in Figure (6). Then we can write

$$\vec{v}_i = \vec{v}(t)$$

$$\vec{v}_{i+1} = \vec{v}(t+\Delta t)$$

to change the time dependence from a count of strobe flashes to the continuous variable t. Next, define the vector $\overrightarrow{\Delta v}(t)$ by

$$\overrightarrow{\Delta v}(t) \equiv \vec{v}(t+\Delta t) - \vec{v}(t) \qquad \left(= \vec{v}_{i+1} - \vec{v}_i \right) \tag{7}$$

We see that $\overrightarrow{\Delta v}(t)$ is the change in the velocity vector as the time advances from t to $t+\Delta t$. The strobe definition of $\vec{a}_i$ can now be written

$$\vec{a}(t)\begin{pmatrix}\textit{strobe}\\ \textit{definition}\end{pmatrix} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} \equiv \frac{\overrightarrow{\Delta v}(t)}{\Delta t} \tag{8}$$

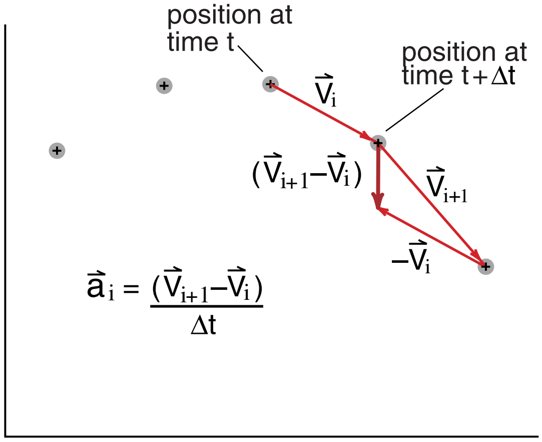


Figure 6
Experimental definition of the acceleration vector.

Now go through the limiting process, turning the strobe up, reducing Δt until the value of $\vec{a}(t)$ settles down to its limiting value. We have

$$\begin{aligned}\vec{a}(t)\begin{pmatrix}\textit{calculus}\\ \textit{definition}\end{pmatrix} &= \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}\\ &= \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}(t)}{\Delta t}\end{aligned} \tag{9}$$

Finally use the shorthand notation d/dt for the limiting process:

$$\boxed{\vec{a}(t) = \frac{d\vec{v}(t)}{dt}} \tag{10}$$

Equation (10) does not make sense unless you remember that it is notation for all the ideas expressed above. Again, physicists think of dt as a short but finite time interval, and $d\vec{v}(t)$ as the small but finite change in the velocity vector during the time interval dt. It's our strobe definition of acceleration with the added requirement that Δt is short enough that the limiting process has already occurred.

Components

Even if you have studied calculus, you may not recall encountering formulas for the derivatives of vectors, like $d\vec{R}(t)/\Delta t$ and $d\vec{v}(t)/\Delta t$ which appear in Equations (5) and (10). To bring these equations into a more familiar form where you can apply standard calculus formulas, we will break the vector Equations (5) and (10) down into component equations.

In the chapter on vectors, we saw that any vector equation like

$$\vec{A} = \vec{B} + \vec{C} \tag{11}$$

is equivalent to the three component equations

$$\begin{aligned}A_x &= B_x + C_x\\ A_y &= B_y + C_y\\ A_z &= B_z + C_z\end{aligned} \tag{12}$$

The advantage of the component equations was that they are simply numerical equations and no graphical work or trigonometry is required.

The limiting process in calculus does not affect the decomposition of a vector into components, thus Equation (5) for $\vec{v}(t)$ and Equation (10) for $\vec{a}(t)$ become

$$\vec{v}(t) = d\vec{R}(t)/dt \qquad (5)$$

$$v_x(t) = dR_x(t)/dt \qquad (5a)$$

$$v_y(t) = dR_y(t)/dt \qquad (5b)$$

$$v_z(t) = dR_z(t)/dt \qquad (5c)$$

and

$$\vec{a}(t) = d\vec{v}(t)/dt \qquad (10)$$

$$a_x(t) = dv_x(t)/dt \qquad (10a)$$

$$a_y(t) = dv_y(t)/dt \qquad (10b)$$

$$a_z(t) = dv_z(t)/dt \qquad (10c)$$

Often we use the letter x for the x coordinate of the vector $\vec{R}$ and we use y for R_y and z for R_z. With this notation, Equation (5) assumes the shorter and perhaps more familiar form

$$v_x(t) = dx(t)/dt \qquad (5a')$$

$$v_y(t) = dy(t)/dt \qquad (5b')$$

$$v_z(t) = dz(t)/dt \qquad (5c')$$

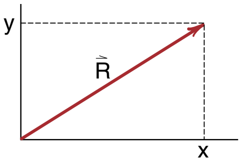


Figure 7

At this point the notation has become deceptively short. You now have to remember that x(t) stands for the x coordinate of the particle at a time t.

We have finally boiled the notation down to the point where it would be familiar from any calculus course. If we restrict our attention to one dimensional motion along the x axis. Then all we have to concern ourselves with are the x component equations

$$\boxed{\begin{aligned} v_x(t) &= \frac{dx(t)}{dt} \\ a_x(t) &= \frac{dv_x(t)}{dt} \end{aligned}} \qquad (10a)$$

INTEGRATION

When we worked with strobe photographs, the photograph told us the position $\vec{R}(t)$ of the ball as time passed. Knowing the position, we can then use Equation (5) to calculate the ball's velocity $\vec{v}(t)$ and then Equation (10) to determine the acceleration $\vec{a}(t)$. In general, however, we want to go the other way, and predict the motion from a knowledge of the acceleration. For example, imagine that you were in Galileo's position, hired by a prince to predict the motion of cannonballs. You know that a cannonball should not be much affected by air resistance, thus the acceleration throughout its trajectory should be the constant gravitational acceleration $\vec{g}$. You know that $\vec{a}(t) = \vec{g}$; how then do you use that knowledge in Equations (5) and (10) to predict the motion of the ball?

The answer is that you cannot with the equations in their present form. The equations tell you how to go from $\vec{R}(t)$ to $\vec{a}(t)$, while to predict motion you need to go the other way, from $\vec{a}(t)$ to $\vec{R}(t)$. The topic of this section is to see how to reverse the directions in which we use our calculus equations. Equations (5) and (10) involve the process called ***differentiation***. We will see that when we go the other way the reverse of differentiation is a process called ***integration***. We will see that integration is a simple concept, but a process that is sometimes hard to perform without the aid of a computer.

Prediction of Motion

In our earlier discussion, we have used strobe photographs to analyze motion. Let us see what we can learn from such a photograph for predicting motion. Figure (8) is our familiar projectile motion photograph showing the displacement $\vec{s}$ of a ball during the time the ball traveled from a position labeled (0) to the position labeled (4). If the ball is now at position (0) and each of the images is .1 seconds apart, then the vector $\vec{s}$ tells us where the ball will be at a time of .4 seconds from now. If we can predict $\vec{s}$, we can predict the motion of the ball. The general problem of predicting the motion of the ball is to be able to calculate $\vec{s}(t)$ for any time t.

From Figure (8) we see that $\vec{s}$ is the vector sum of the individual displacement vectors $\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$ and $\vec{s}_4$

$$\vec{s} = \vec{s}_1 + \vec{s}_2 + \vec{s}_3 + \vec{s}_4 \qquad (11)$$

We can then use the fact that $\vec{s}_1 = \vec{v}_1\Delta t$, $\vec{s}_2 = \vec{v}_2\Delta t$, etc. to get

$$\vec{s} = \vec{v}_1\Delta t + \vec{v}_2\Delta t + \vec{v}_3\Delta t + \vec{v}_4\Delta t \qquad (12)$$

Rather than writing out each term, we can use the ***summation sign*** Σ to write

$$\vec{s} = \sum_{i=1}^{4} \vec{v}_i\Delta t \qquad (12a)$$

Equation (12) is approximate in that the $\vec{v}_i$ are approximate (strobe) velocities, not the instantaneous velocities we want for a calculus discussion. In Figure (9) we improved the situation by cutting Δt to 1/4 of its previous value, giving us four times as many images and more accurate velocities $\vec{v}_i$.

We see that the displacement $\vec{s}$ is now the sum of 16 vectors

$$\vec{s} = \vec{s}_1 + \vec{s}_2 + \vec{s}_3 + \ldots + \vec{s}_{15} + \vec{s}_{16} \qquad (13)$$

Expressing this in terms of the velocity vectors $\vec{v}_1$ to $\vec{v}_{16}$ we have

$$\vec{s} = \vec{v}_1\Delta t + \vec{v}_2\Delta t + \vec{v}_3\Delta t + \ldots + \vec{v}_{15}\Delta t + \vec{v}_{16}\Delta t \quad (14)$$

or using our more compact notation

$$\vec{s} = \sum_{i=1}^{16} \vec{v}_i\Delta t \qquad (14a)$$

While Equation (14) for $\vec{s}$ looks quite different than Equation (12), the sum of sixteen vectors instead of four, the displacement vectors $\vec{s}$ in the two cases are exactly the same. Adding more intermediate images did not change where the ball was located at the time of t = .4 seconds. In going from Equation (12) to (14), what has changed in shortening the time step Δt, is that ***the individual velocity vectors $\vec{v}_i$ become more nearly equal to the instantaneous velocity of the ball at each image.***

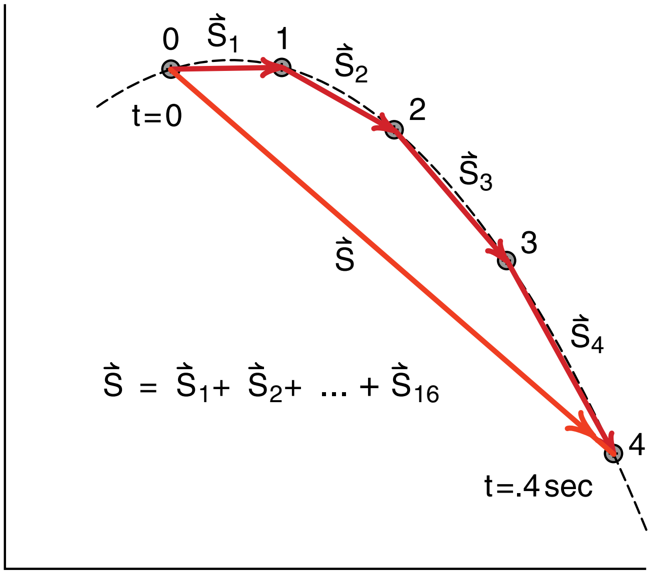


Figure 8
To predict the total displacement $\vec{s}$, we add up the individual displacements $\vec{s}_i$.

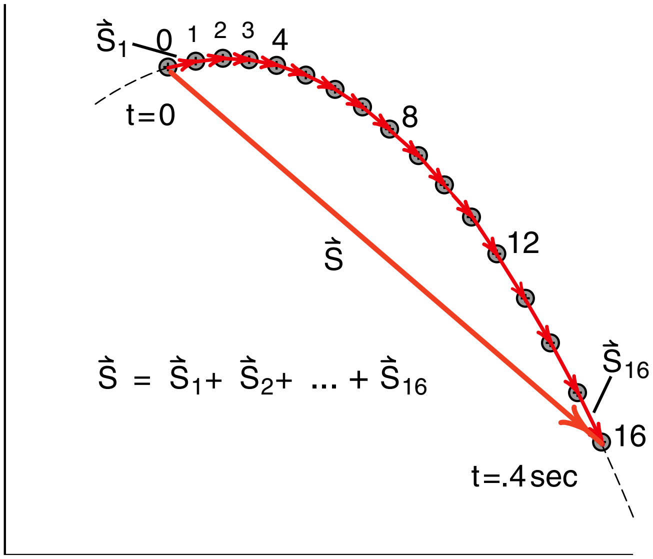


Figure 9
With a shorter time interval, we add up more displacement vectors to get the total displacement $\vec{s}$.

If we reduced Δt again by another factor of 1/4, so that we had 64 images in the interval t = 0 to t = .4 sec, the formula for $\vec{s}$ would become

$$\vec{s} = \sum_{i=1}^{64} \vec{v}_i \Delta t \tag{15a}$$

where now the $\vec{v}_i$ are still closer to representing the ball's instantaneous velocity. The more we reduce Δt, the more images we include, the closer each $\vec{v}_i$ comes to the instantaneous velocity $\vec{v}(t)$. While adding more images gives us more vectors we have to add up to get the total displacement $\vec{s}$, there is very little change in our formula for $\vec{s}$. If we had a million images, we would simply write

$$\vec{s} = \sum_{i=1}^{1000000} \vec{v}_i \Delta t \tag{16a}$$

In this case the $\vec{v}_i$ would be physically indistinguishable from the instantaneous velocity $\vec{v}(t)$. We have essentially reached a calculus limit, but we have problems with the notation. It is clearly inconvenient to label each $\vec{v}_i$ and then count the images. Instead we would like notation that involves the instantaneous velocity $\vec{v}(t)$ and expresses the beginning and end points in terms of the initial time t_1 and final time t_f, rather than the initial and final image numbers i.

In the calculus notation, we replace the summation sign Σ by something that looks almost like the summation sign, namely the ***integral sign*** $\int$. (The French word for integration is the same as their word for summation.) Next we replaced the individual $\vec{v}_i$ by the continuous variable $\vec{v}(t)$ and finally express the end points by the initial time t_i and the final time t_f. The result is

$$\boxed{\vec{s} = \sum_{i=1}^{n} \vec{v}_i \Delta t \rightarrow \left(\begin{array}{l}\textit{as the number}\\ \textit{n becomes}\\ \textit{infinitely}\\ \textit{large}\end{array}\right) \int_{t_i}^{t_f} \vec{v}(t)dt} \tag{17}$$

Calculus notation is more easily handled, or is at least more familiar, if we break vector equations up into component equations. Assume that the ball started at position i which has components $x_i = x(t_i)$ [read $x(t_i)$ as "x at time t_i"] and $y_i = y(t_i)$ as shown in Figure (10). The final position f is at $x_f = x(t_f)$ and $y_f = y(t_f)$.

Thus the displacement $\vec{s}$ has x and y components

$$s_x = x(t_f) - x(t_i)$$
$$s_y = y(t_f) - y(t_i)$$

Breaking Equation (17) into component equations gives

$$\boxed{s_x = x(t_f) - x(t_i) = \int_{t_i}^{t_f} v_x(t)dt} \tag{18a}$$

$$\boxed{s_y = y(t_f) - y(t_i) = \int_{t_i}^{t_f} v_y(t)dt} \tag{18b}$$

Here we will introduce one more piece of notation often used in calculus courses. On the left hand side of Equation (18a) we have $x(t_f) - x(t_i)$ which we can think of as the variable x(t) evaluated over the interval of time from t_i to t_f. We will often deal with variables evaluated over some interval and have a special notation for that. We will write

$$\boxed{x(t_f) - x(t_i) \equiv x(t)\Big|_{t_i}^{t_f}} \tag{19}$$

You are to read the symbol $x(t)\big|_{t_i}^{t_f}$ as "x of t evaluated from t_i to t_f". We write the initial time t_i at the bottom of the vertical bar, the final time t_f at the top.

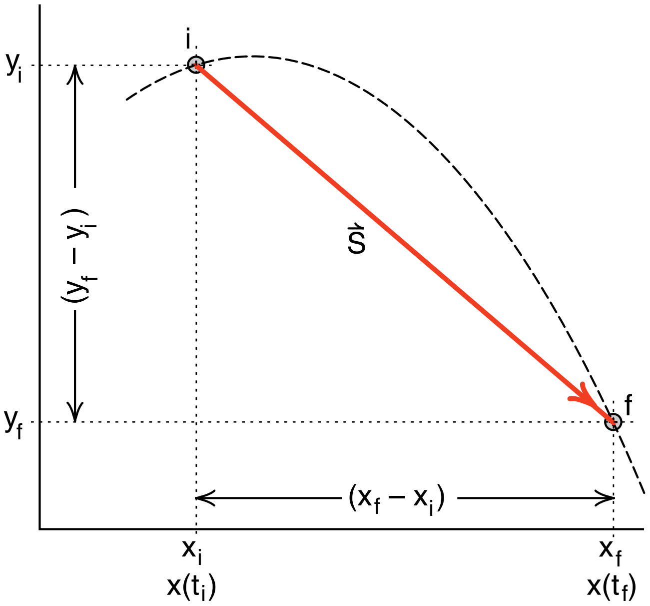


Figure 10
Breaking the vector $\vec{s}$ into components.

We use similar notation for any kind of variable, for example

$$f(x)\Big|_{x_1}^{x_2} \equiv f(x_2) - f(x_1) \qquad (19a)$$

Remember to subtract the variable when evaluated at the value at the bottom of the vertical bar.

With this notation, our Equation (18) can be written

$$s_x = x(t)\Big|_{t_i}^{t_f} = \int_{t_i}^{t_f} v_x(t)dt \qquad (18a')$$

$$s_y = y(t)\Big|_{t_i}^{t_f} = \int_{t_i}^{t_f} v_y(t)dt \qquad (18b')$$

Calculating Integrals

Equation (20) is nice and compact, but how do you use it? How do you calculate integrals? The key is to remember that an integral is just a fancy notation for a sum of terms, where we make the time step $\Delta(t)$ very small. Keeping this in mind, we will see that there is a very easy way to interpret an integral.

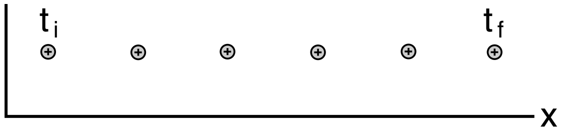


Figure 11a
Strobe photograph of ball moving at constant velocity in x direction.

Figure 11b
Graph of $v_x(t)$ versus t for the ball of Figure 11a.

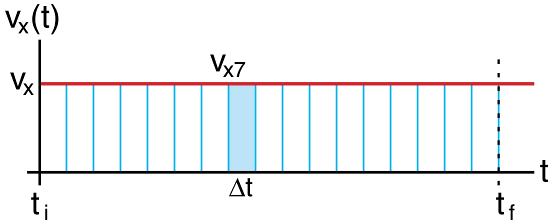


Figure 11c
Each $v_x\Delta t$ is the area of a rectangle.

To get this interpretation, let us start with the simple case of a ball moving in a straight line, for instance, the x direction, at a constant velocity v_x. A strobe picture of this motion would look like that shown in Figure (11a).

Figure (11b) is a graph of the ball's velocity $v_x(t)$ as a function of the time t. The vertical axis is the value of v_x, the horizontal axis is the time t. Since the ball is traveling at constant velocity, v_x has a constant value and is thus represented by a straight horizontal line. In order to calculate the distance that the ball has traveled during the time interval from t_i to t_f, we need to evaluate the integral

$$s_x = \int_{t_i}^{t_f} v_x(t)dt \quad \begin{matrix}\textit{distance ball}\\ \textit{travels in}\\ \textit{time interval}\\ t_i \textit{ to } t_f\end{matrix} \qquad (18a)$$

To actually evaluate the integral, we will go back to our summation notation

$$s_x = \sum_{i_{\text{initial}}}^{i_{\text{final}}} v_{xi}\Delta t \qquad (20)$$

and show individual time steps Δt in the graph of v_x versus t, as in Figure (11c).

We see that each term in Equation (20) is represented in Figure (11c) by a rectangle whose height is v_x and whose width is Δt. We have shaded in the rectangle representing the 7th term $v_{x7}\Delta t$. We see that $v_{x7}\Delta t$ is just the ***area*** of the shaded rectangle, and it is clear that the sum of all the areas of the individual rectangles is the total area under the curve, starting at time t_i and ending at time t_f. Here we are beginning to see that the process of integration is equivalent to finding the area under a curve.

With a simple curve like the constant velocity $v_x(t)$ in Figure (11c), we see by inspection that the total area from t_i to t_f is just the area of the complete rectangle of height v_x and width $(t_f - t_i)$. Thus

$$s_x = v_x \times (t_f - t_i) \qquad (21)$$

This is the expected result for constant velocity, namely

$$\begin{matrix}\text{distance}\\ \text{traveled}\end{matrix} = \text{velocity} \times \text{time} \quad \begin{matrix}\textit{for}\\ \textit{constant}\\ \textit{velocity}\end{matrix} \qquad (21a)$$

To see that you are not restricted to the case of constant velocity, suppose you drove on a freeway due east (the x direction) starting at 9:00 AM and stopping for lunch at 12 noon. Every minute during your trip you wrote down the speedometer reading so that you had an accurate plot of $v_x(t)$ for the entire morning, a plot like that shown in Figure (12). From such a plot, could you determine the distance s_x that you had travelled?

Your best answer is to multiply each value v_i of your velocity by the time Δt to calculate the average distance traveled each minute. Summing these up from the initial time $t_i = 9{:}00\text{AM}$ to the final time $t_f = \text{noon}$, you have as your estimate

$$s_x \approx \sum_i v_{xi}\Delta t$$

(The symbol ≈ means ***approximately equal***.)

To get a more accurate value for the distance traveled, you should measure your velocity at shorter time intervals Δt and add up the larger number of smaller rectangles. The precise answer should be obtained in the limit as Δt goes to zero

$$s_x = \underset{\Delta t \to 0}{\text{limit}} \sum_i v_{xi}\Delta t = \int_{t_i}^{t_f} v_x(t)dt \qquad (22)$$

This limit is just the area under the curve that is supposed to represent the instantaneous velocity $v_x(t)$.

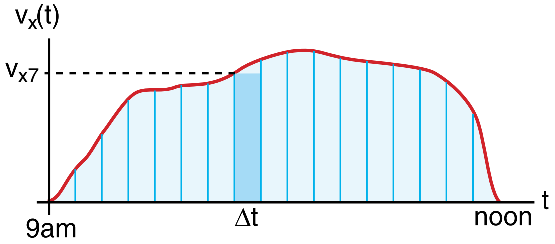


Figure 12
Plot of $v_x(t)$ for a trip starting at 9:00 AM and finishing at noon. The distance traveled is the area under the curve.

Thus we can interpret the integral of a curve as the area under the curve even when the curve is not constant or flat. Mathematicians concern themselves with curves that are so wild that it is difficult or impossible to determine the area under them. Such curves seldom appear in physics problems.

While the basic idea of integration is simple—just finding the area under a curve–in practice it can be quite difficult to calculate the area. Much of an introductory calculus course is devoted to finding the formulas for the areas under various curves. There are also books called ***tables of integrals*** where you look up the formula for a curve and the table tells you the formula for the area under that curve.

In Chapter 16 of the physics text, we will discuss a mathematical technique called ***Fourier analysis***. This is a technique in which we can describe the shape of any continuous curve in terms of a sum of sin waves. (Why we want to do that will become clear then.) The process of Fourier analysis involves finding the area under some very complex curves, curves often involving experimental data for which we have no formula, only graphs. Such curves cannot be integrated by using a table of integrals, with the result that Fourier analysis was not widely used until the advent of the modern digital computer.

The computer made a difference, because we can find the area under almost any curve by breaking the curve into short pieces of length Δt, calculating the area $v_i\Delta t$ of each narrow rectangle, and adding up the area of the rectangles to get the total area. If the curve is so wild that we have to break it into a million segments to get an accurate answer, that might be too hard to do by hand, but it usually a very simple and rapid job for a computer. Computers can be much more efficient than people at integration.

The Process of Integrating

There is a language for the process of integration which we will now take you through. In each case we will check that the results are what we would expect from our summation definition, or the idea that an integral is the area under a curve.

The simplest integral we will encounter in the calculation of the area under a curve of unit height as shown in Figure (13). We have the area of a rectangle of height 1 and length $(t_f - t_i)$

$$\int_{t_i}^{t_f} 1\,dt = \int_{t_i}^{t_f} dt = (t_f - t_i) \tag{22}$$

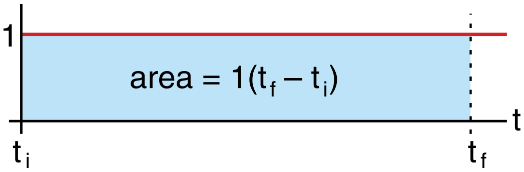


Figure 13
Area under a curve of unit height.

We will use some special language to describe this integration. We will say that the integral of dt is simply the time t, and that the integral of dt from t_i to t_f is equal to t evaluated from t_i to t_f. In symbols this is written as

$$\int_{t_i}^{t_f} dt = t\Big|_{t_i}^{t_f} = (t_f - t_i) \tag{23}$$

Recall that the vertical line after a variable means to evaluate that variable at the final position t_f (upper value), minus that variable evaluated at the initial position t_i (lower value). Notice that this prescription gives the correct answer.

The next simplest integral is the integral of a constant, like a constant velocity v_x over the interval t_i to t_f

$$\int_{t_i}^{t_f} v_x\,dt = v_x(t_f - t_i) \tag{24}$$

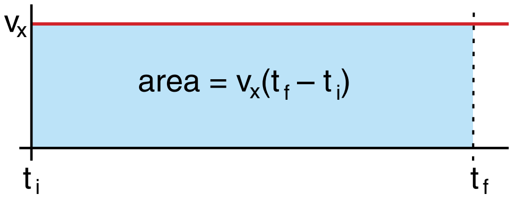


Figure 14
Area under the constant v_x curve.

Since $(t_f - t_i) = \int_{t_i}^{t_f} dt$, we can replace $(t_f - t_i)$ in Equation (24) by the integral to get

$$\int_{t_i}^{t_f} v_x\,dt = v_x\int_{t_i}^{t_f} dt \qquad v_x \text{ a constant} \tag{25}$$

and we see that a constant like v_x can be taken outside the integral sign.

Let us try the simplest case we can think of where v_x is not constant. Suppose v_x starts at zero at time $t_i = 0$ and increases linearly according to the formula

$$v_x = at \tag{26}$$

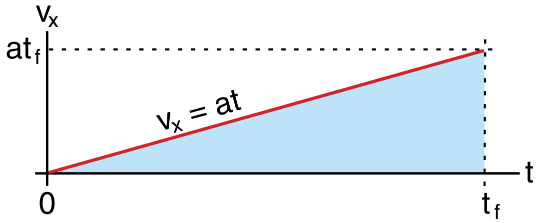


Figure 15

When we get up to the time t_f the velocity will be (at_f) as shown in Figure (15). The area under the curve $v_x = at$ is a triangle whose base is of length t_f and height is at_f. The area of this triangle is one half the base times the height, thus we get for the distance s_x traveled by an object moving with this velocity

$$\begin{aligned} s_x &= \int_0^{t_f} v_x\,dt = \frac{1}{2}(\text{base}) \times (\text{altitude}) \\ &= \frac{1}{2}(t_f)(at_f) = \frac{1}{2}at_f^{\,2} \end{aligned} \tag{27}$$

Now let us repeat the same calculation using the language one would find in a calculus book. We have

$$s_x = \int_0^{t_f} v_x\,dt = \int_0^{t_f} (at)\,dt \tag{28}$$

The constant (a) can come outside, and we know that the answer is $1/2\,at_f^{\,2}$, thus we can write

$$s_x = a\int_0^{t_f} t\,dt = \frac{1}{2}at_f^{\,2} \tag{29}$$

In Equation (29) we can cancel the a's to get the result

$$\int_0^{t_x} t\,dt = \frac{1}{2}t_f^{\,2} \tag{30}$$

In a calculus text, you would find the statement that the integral $\int t\,dt$ is equal to $t^2/2$ and that the integral should be evaluated as follows

$$\int_0^{t_f} t\,dt = \left.\frac{t^2}{2}\right|_0^{t_f} = \frac{{t_f}^2}{2} - \frac{0}{2} = \frac{{t_f}^2}{2} \qquad (31)$$

Indefinite Integrals

When we want to measure an actual area under a curve, we have to know where to start and stop. When we put these limits on the integral sign, like t_i and t_f, we have what is called a ***definite integral***. However there are times where we just want to know what the form of the integral is, with the idea that we will put in the limits later. In this case we have what is called an ***indefinite integral***, such as

$$\int t\,dt = \frac{t^2}{2} \qquad \textit{indefinite integral} \qquad (32)$$

The difference between our definite integral in Equation (31) and the indefinite one in Equation (32) is that we have not chosen the limits yet in Equation (32). If possible, a table of integrals will give you a formula for the indefinite integral and let you put in whatever limits you want.

Integration Formulas

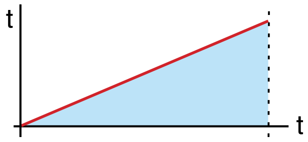
For some sets of curves, there are simple formulas for the area under them. One example is the set of curves of the form t^n. We have already considered the cases where n = 0 and n = 1.

n = 0

$$\int t^0 dt = \int dt = t$$

n = 1

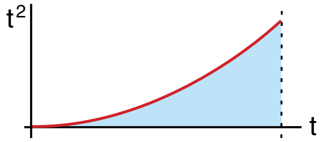
$$\int t^1 dt = \int t\,dt = \frac{t^2}{2}$$



Some results we will prove later are

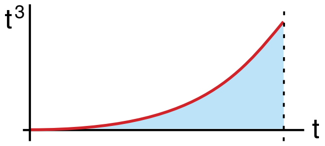
n = 2

$$\int t^2 dt = \frac{t^3}{3}$$



n = 3

$$\int t^3 dt = \frac{t^4}{4}$$



(33a,b,c,d)

Looking at the way these integrals are turning out, we suspect that the general rule is

$$\boxed{\int t^n dt = \frac{t^{n+1}}{n+1}} \qquad (34)$$

It turns out that Equation (34) is a general result for any value of n except n = –1. If n = –1, then you would have division by zero, which cannot be the answer. (We will shortly discuss the special case where n = –1.)

As long as we stay away from the n = –1 case, the formula works for negative numbers. For example

$$\int t^{-2} dt = \int \frac{dt}{t^2} = \frac{t^{(-2+1)}}{-2+1} = \frac{t^{-1}}{(-1)}$$

$$\boxed{\int \frac{dt}{t^2} = -\frac{1}{t}} \qquad (35)$$

In our discussion of gravitational and electrical potential energy, we will encounter integrals of the form seen in Equation (35).

Exercise 1

Using Equation (34) and the fact that constants can come outside the integral, evaluate the following integrals:

(a) $\int x\,dx$ *it does not matter whether we call the variable t or x*

(b) $\int_{x=1}^{x=2} x^5 dx$ *also sketch the area being evaluated*

(c) $\int_{t=1}^{t=2} \frac{dt}{t^2}$ *Show that you get a positive area.*

(d) $\int \frac{GmM}{r^2} dr$ *where G, m, and M are constants*

(e) $\int \frac{a}{y^{3/2}} dy$ *(a) is a constant*

NEW FUNCTIONS

Logarithms

We have seen that when we integrate a curve or function like t^2, we get a new function $t^3/3$. The functions t^2 and t^3 appear to be fairly similar; the integration did not create something radically different. However, the process of integration can lead to some curves with entirely different behavior. This happens, for example, in that special case $n = -1$ when we try to do the integral of t^{-1}.

It is certainly not hard to plot t^{-1}, the result is shown in Figure (16). Also there is nothing fundamentally difficult or peculiar about measuring the area under the t^{-1} curve from some t_i to t_f, as long as we stay away from the origin $t = 0$ where t^{-1} blows up. The formula for this area turns out, however, to be the new function called the ***natural logarithm***, abbreviated by the symbol *ln*. The area in Figure (16) is given by the formula

$$\int_{t_i}^{t_f} \frac{1}{t}\,dt = \ln(t_f) - \ln(t_i) \tag{36}$$

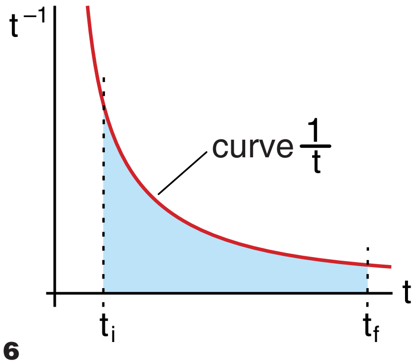


Figure 16
Plot of t^{-1}. The area under this curve is the natural logarithm ln.

Two of the important but peculiar features of the natural logarithm are

$$\ln(ab) = \ln(a) + \ln(b) \tag{37}$$

$$\ln\left(\frac{1}{a}\right) = -\ln(a) \tag{38}$$

Thus we get, for example

$$\begin{aligned} \ln(t_f) - \ln(t_i) &= \ln(t_f) + \ln\left(\frac{1}{t_i}\right) \\ &= \ln\left(\frac{t_f}{t_i}\right) \end{aligned} \tag{39}$$

Thus the area under the curve in Figure (16) is

$$\int_{t_i}^{t_f} \frac{dt}{t} = \ln\left(\frac{t_f}{t_i}\right) \tag{40}$$

While the natural logarithm has some rather peculiar properties it is easy to evaluate because it is available on all scientific calculators. For example, if $t_i = .5$ seconds and $t_f = 4$ seconds, then we have

$$\ln\left(\frac{t_f}{t_i}\right) = \ln\left(\frac{4}{.5}\right) = \ln(8) \tag{41}$$

Entering the number 8 on a scientific calculator and pressing the button labeled *ln*, gives

$$\ln(8) = 2.079 \tag{42}$$

which is the answer.

Exercise 2

Evaluate the integrals

$$\int_{.001}^{1000} \frac{dx}{x} \qquad \int_{.000001}^{1} \frac{dx}{x}$$

Why are the answers the same?

The Exponential Function

We have just seen that, while the logarithm function may have some peculiar properties, it is easy to evaluate using a scientific calculator. The question we now want to consider is whether there is some function that undoes the logarithm. When we enter the number 8 into the calculator and press *ln*, we get the number 2.079. Now we are asking if, when we enter the number 2.079, can we press some key and get back the number 8? The answer is, you press the key labeled e^x. The e^x key performs the ***exponential function*** which undoes the logarithm function. We say that the exponential function e^x is the ***inverse*** of the logarithm function *ln*.

Exponents to the Base 10

You are already familiar with exponents to the base 10, as in the following examples

$$\begin{array}{ll} 10^0 = 1 & \\ 10^1 = 10 & 10^{-1} = 1/10 = .1 \\ 10^2 = 100 & 10^{-2} = 1/100 = .01 \\ \cdots\cdots & \cdots\cdots \\ 10^6 = 1{,}000{,}000 & 10^{-6} = .000001 \end{array} \qquad (43)$$

The exponent, the number written above the 10, tells us how many factors of 10 are involved. A minus sign means how many factors of 10 we divide by. From this alone we deduce the following rules for the exponent to the base 10.

$$\boxed{10^{-a} = \frac{1}{10^a}} \qquad (44)$$

$$\boxed{10^a \times 10^b = 10^{a+b}} \qquad (45)$$

(Example $10^2 \times 10^3 = 100 \times 1000 = 100{,}000$.)

The inverse of the *exponent to the base 10* is the function called *logarithm to the base 10* which is denoted by the key labeled *log* on a scientific calculator. Formally this means that

$$\boxed{\log\left(10^y\right) = y} \qquad (46)$$

Check this out on your scientific calculator. For example, enter the number 1,000,000 and press the *log* button and see if you get the number 6. Try several examples so that you are confident of the result.

The Exponential Function y^x

Another key on your scientific calculator is labeled y^x. This allows you to determine the value of any number y raised to the power (or exponent) x. For example, enter the number $y = 10$, and press the y^x key. Then enter the number $x = 6$ and press the = key. You should see the answer

$$y^x = 10^6 = 1000000$$

It is quite clear that all exponents obey the same rules we saw for powers of 10, namely

$$\boxed{y^a \times y^b = y^{a+b}} \qquad (47)$$

(Example $y^2 \times y^3 = \left(y \times y\right)\left(y \times y \times y\right) = y^5$.)

And as before

$$\boxed{y^{-a} \equiv \frac{1}{y^a}} \qquad (48)$$

Exercise 3

Use your scientific calculator to evaluate the following quantities. (You should get the answers shown.)

(a) 10^6	(1000000)
(b) 2^3	(8)
(c) 23^0	(1)
(d) 10^{-1}	(.1)

(To do this calculation, enter 10, then press y^x . Then enter 1, then press the +/– key to change it to –1, then press = to get the answer .1)

(e) $2^{-.5}$	($1/\sqrt{2} = .707$)
(f) log(10)	(1)
(g) ln(2.7183)	(1) (very close to 1)

Try some other examples on your own to become completely familiar with the y^x key. (You should note that any positive number raised to the 0 power is 1. Also, some calculators, in particular the one I am using, cannot handle any negative values of y, not even $(-2)^2$ which is +4)

Euler's Number e = 2.7183. . .

We have seen that the function *log* on the scientific calculator undoes, is the inverse of, powers of 10. For example, we saw that

$$\log\left(10^x\right) = x \tag{46 repeated}$$

$$\text{Example: } \log\left(10^6\right) = 6$$

Earlier we saw that the exponential function e^x was the inverse of the natural logarithm *ln*. This means that

$$\ln\left(e^x\right) = x \tag{49}$$

The difference between the logarithm *log* and the natural logarithm *ln*, is that *log* undoes exponents of the number 10, while *ln* undoes exponents of the number *e*. This special number *e*, one of the fundamental mathematical constants like π , is known as ***Euler's number***, and is always denoted by the letter *e*.

You can find the numerical value of Euler's number *e* on your calculator by evaluating

$$e^1 = e \tag{50}$$

To do this, enter 1 into your calculator, press the e^x key, and you should see the result

$$e^1 = e = 2.718281828 \tag{51}$$

We will run into this number throughout the course. You should remember that *e* is about 2.7, or you might even remember 2.718. (Only remembering *e* as 2.7 is as klutzy as remembering π as 3.1)

The terminology in math courses is that the function *log*, which undoes exponents of the number 10, is the *logarithm to the base 10*. The function *ln*, what we have called the *natural logarithm*, which undoes exponents of the number *e*, is the *logarithm to the base e*. You can have logarithms to any base you want, but in practice we only use base 10 (because we have 10 fingers) and the base *e*. The base *e* is special, in part because that is the logarithm that naturally arises when we integrate the function 1/x. We will see shortly that the functions *ln* and e^x have several more, very special features.

DIFFERENTIATION AND INTEGRATION

The scientific calculator is a good tool for seeing how the functions like *ln* and e^x are inverse of each other. Another example of inverse operations is integration and differentiation. We have seen that integration allows us to go the other way from differentiation [finding x(t) from v(t), rather than v(t) from x(t)]. However it is not so obvious that integration and differentiation are inverse operations when you think of integration as finding the area under a curve, and differentiation as finding limits of $\Delta x/\Delta t$ as Δt goes to zero. It is time now to make this relationship clear.

First, let us review our concept of a derivative. Going back to our strobe photograph of Figure (3), replacing $\vec{R}_i$ by $\vec{R}(t)$ and $\vec{R}_{i+1}$ by $\vec{R}(t+\Delta t)$, as shown in Figure (3a), our strobe velocity was then given by

$$\vec{v}(t) = \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t} \tag{52}$$

The calculus definition of the velocity is obtained by reducing the strobe time interval Δt until we obtain the instantaneous velocity $\vec{v}$.

$$\boxed{\vec{v}_{\text{calculus}} = \lim_{\Delta t \to 0} \frac{\vec{R}(t+\Delta t) - \vec{R}(t)}{\Delta t}} \tag{53}$$

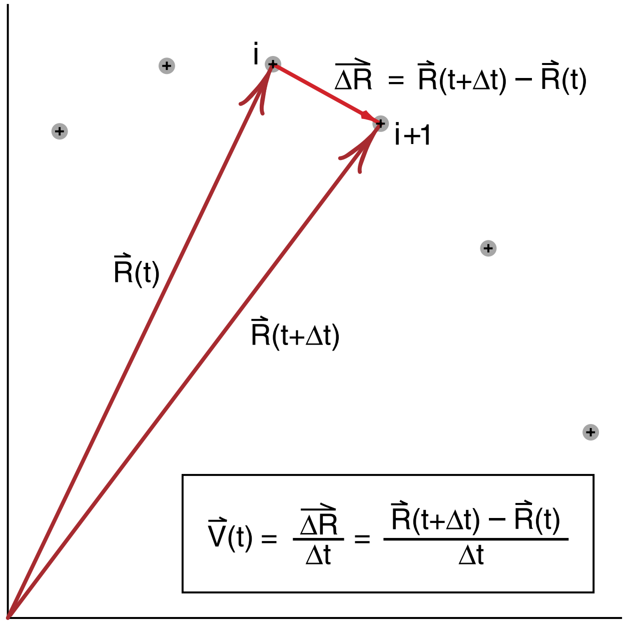


Figure 3a
Defining the strobe velocity.

While Equation (53) looks like it is applied to the explicit case of the strobe photograph of projectile motion, it is easily extended to cover any process of differentiation. Whatever function we have [we had $\vec{R}(t)$, suppose it is now f(t)], evaluate it at two closely spaced times, subtract the older value from the newer one, and divide by the time difference Δt. Taking the limit as Δt becomes very small gives us the derivative

$$\frac{d\,f(t)}{dt} \equiv \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{54}$$

The variable with which we are differentiating does not have to be time t. It can be any variable that we can divide into small segments, such as x;

$$\boxed{\frac{d}{dx} f(x) \equiv \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}} \tag{55}$$

Let us see how the operation defined in Equation (55) is the inverse of finding the area under a curve.

Suppose we have a curve, like our old $v_x(t)$ graphed as a function of time, as shown in Figure (17). To find out how far we traveled in a time interval from t_i to some later time T, we would do the integral

$$x(T) = \int_{t_i}^{T} v_x(t)\,dt \tag{56}$$

The integral in Equation (56) tells us how far we have gone at any time T during the trip. The quantity x(T) is a function of this time T.

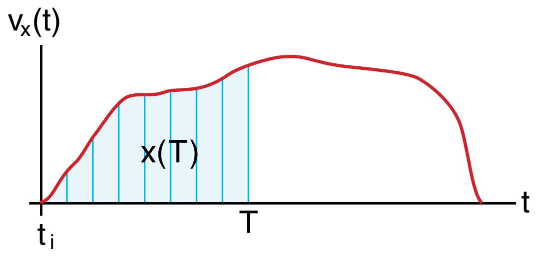


Figure 17
The distance traveled by the time T is the area under the velocity curve up to the time T.

Now let us differentiate the function x(T) with respect to the variable T. By our definition of differentiation we have

$$\frac{d}{dT}x(T) = \lim_{\Delta T\to 0}\frac{x(T+\Delta T)-x(T)}{\Delta T} \tag{57}$$

Figure (17) shows us the function x(T). It is the area under the curve v(t) starting at t_i and going up to time t = T. Figure (18) shows us the function $x(T + \Delta T)$. It is the area under the same curve, starting at t_i but going up to $t = T + \Delta t$. When we subtract these two areas, all we have left is the area of the slender rectangle shown in Figure (19).

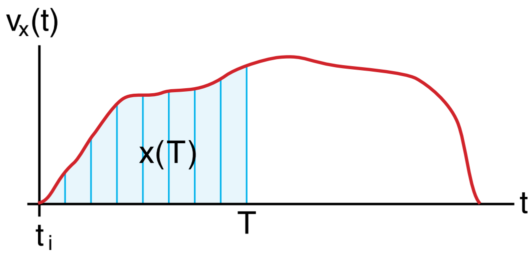


Figure 17 repeated
The distance x(T) traveled by the time T

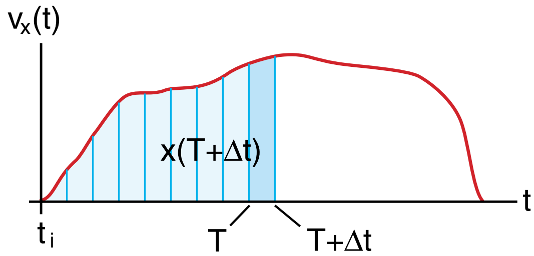


Figure 18
The distance $x(T+\Delta t)$ traveled by the time $T+\Delta t$.

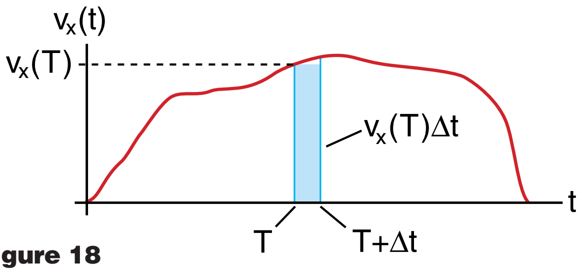


Figure 18
The distance $x(T+\Delta t) - x(T)$ traveled during the time Δt.

The rectangle has a height approximately v(T) and a width ΔT for an area

$$x(T+\Delta T) - x(T) = v_x(T)\Delta t \tag{58}$$

Dividing through by ΔT gives

$$v_x(T) = \frac{x(T+\Delta T)-x(T)}{\Delta T} \tag{59}$$

The only approximation in Equation (59) is at the top of the rectangle. If the curve is not flat, $v_x(T+\Delta T)$ will be different from $v_x(T)$ and the area of the sliver will have a value somewhere between $v_x(T)\Delta t$ and $v_x(T+\Delta t)\Delta t$. But if we take the limit as ΔT goes to zero, the value of $v_x(T+\Delta T)$ must approach $v_x(T)$, and we end up with the exact result

$$v_x(T) = \lim_{\Delta t\to 0}\frac{x(T+\Delta t)-x(T)}{\Delta t} \tag{60}$$

This is just the derivative dx(t)/dt evaluated at t = T.

$$\boxed{v_x(T) = \left.\frac{dx(t)}{dt}\right|_{t=T}} \tag{61a}$$

where we started from

$$\boxed{x(T) = \int_{t_i}^{T} v_x(t)\,dt} \tag{61b}$$

Equations (61a) and (61b) demonstrate explicitly how differentiation and integration are inverse operations. The derivative allowed us to go from x(t) to $v_x(t)$ while the integral took us from $v_x(t)$ to x(t). This inverse is not as simple as pushing a button on a calculator to go from *ln* to e^x. Here we have to deal with limits on the integration and a shift of variables from t to T. But these two processes do allow us to go back and forth.

A Fast Way to go Back and Forth

We introduced our discussion of integration by pointing out that equations

$$v_x(t) = \frac{dx(t)}{dt}; \qquad a_x(t) = \frac{dv_x(t)}{dt} \qquad (62a,b)$$

went the wrong way in that we were more likely to know the acceleration $a_x(t)$ and from that want to calculate the velocity $v_x(t)$ and distance traveled x(t). After many steps, we found that integration was what we needed.

We do not want to repeat all those steps. Instead we would like a quick and simple way to go the other way around. Here is how you do it. Think of the dt in (62a) as a small but finite time interval. That means we can treat it like any other number and multiply both sides of Equation (62a) through by it.

$$v_x(t) = \frac{dx(t)}{dt}$$

$$dx(t) = v_x(t)dt \qquad (63)$$

Now integrate both sides of Equation (63) from some initial time t_i to a final time T. (If you do the same thing to both sides of an equation, both sides should still be equal to each other.)

$$\int_{t_i}^{T} dx(t) = \int_{t_i}^{T} v_x(t)\,dt \qquad (64)$$

If dt is to be thought of as a small but finite time step, then dx(t) is the small but finite distance we moved in the time dt. The integral on the left side of Equation (64) is just the sum of all these short distances moved, which is just the total distance moved during the time from t_i to T.

$$\int_{t_i}^{T} dx(t) = x(t)\Big|_{t_i}^{T} = x(T) - x\,(t_i) \qquad (65)$$

Thus we end up with the result

$$x(t)\Big|_{t_i}^{T} = \int_{t_i}^{T} v_x(t)dt \qquad (66)$$

Equation (66) is a little more general than (62b) for it allows for the fact that $x(t_i)$ might not be zero. If, however, we say that we started our trip at $x(t_i) = 0$, then we get the result

$$x(T) = \int_{t_i}^{T} v_x(t)dt \qquad (67)$$

representing the distance traveled since the start of the trip.

Constant Acceleration Formulas

The constant acceleration formulas, so well known from high school physics courses, are an excellent application of the procedures we have just described.

We will begin with motion in one dimension. Suppose a car is traveling due east, in the x direction, and for a while has a constant acceleration a_x. The car passes us at a time $t_i = 0$, traveling at a speed v_{x0}. At some later time T, if the acceleration a_x remains constant, how far away from us will the car be?

We start with the equation

$$a_x(t) = \frac{dv_x(t)}{dt} \qquad (68)$$

Multiplying through by dt to get

$$dv_x(t) = a_x(t)dt$$

then integrating from time $t_i = 0$ to time $t_f = T$, we get

$$\int_{0}^{T} dv_x(t) = \int_{0}^{T} a_x(t)dt \qquad (69)$$

Since the integral $\int dv_x(t) = v_x(t)$, we have

$$\int_{0}^{T} dv_x(t) = v_x(t)\Big|_{0}^{T} = v_x(T) - v_x(0) \qquad (70)$$

where $v_x(0)$ is the velocity v_{x0} of the car when it passed us at time t = 0.

While we can always do the left hand integral in Equation (69), we cannot do the right hand integral until we know $a_x(t)$. For the constant acceleration problem, however, we know that $a_x(t) = a_x$ is constant, and we have

$$\int_{0}^{T} a_x(t)dt = \int_{0}^{T} a_x dt \qquad (71)$$

Since constants can come outside the integral sign, we get

$$\int_0^T a_x dt = a_x \int_0^T dt = a_x t \Big|_0^T = a_x T \qquad (72)$$

where we used $\int dt = t$. Substituting Equations (70) and (72) in (69) gives

$$v_x T - v_{x0} = a_x T \qquad (73)$$

Since Equation (73) applies for any time T, we can replace T by t to get the well known result

$$\boxed{v_x(t) = v_{x0} + a_x t} \qquad (a_x \textit{ constant}) \qquad (74)$$

Equation (74) tells us the speed of the car at any time t after it passed us, as long as the acceleration remains constant.

To find out how far away the car is, we start with the equation

$$v_x(t) = \frac{dx(t)}{dt} \qquad (62a)$$

Multiplying through by dt to get

$$dx(t) = v_x(t)\,dt$$

then integrating from time $t = 0$ to time $t = T$ gives (as we saw earlier)

$$\int_0^T dx(t) = \int_0^T v_x(t)\,dt \qquad (75)$$

The left hand side is

$$\int_0^T dx(t) = x(t)\Big|_0^T = x(T) - x(0) \qquad (76)$$

If we measure along the x axis, starting from where we are (where the car was at $t = 0$) then $x(0) = 0$.

In order to do the right hand integral in Equation (75), we have to know what the function $v_x(t)$ is. But for constant acceleration, we have from Equation (74) $v_x(t) = v_{x0} + a_x t$, thus

$$\int_0^T v_x(t)\,dt = \int_0^T (v_{x0} + a_x t)\,dt \qquad (77)$$

One of the results of integration that you should prove for yourself (just sketch the areas) is the rule

$$\int_i^f \left[a(x) + b(x)\right]dx = \int_i^f a(x)dx + \int_i^f b(x)dx \qquad (78)$$

thus we get

$$\int_0^T (v_{x0} + a_x t)dt = \int_0^T v_{x0}dt + \int_0^T a_x t\,dt \qquad (79)$$

Since constants can come outside the integrals, this is equal to

$$\int_0^T (v_{x0} + a_x t)dt = v_{x0}\int_0^T dt + a_x \int_0^T t\,dt \qquad (80)$$

Earlier we saw that

$$\int_0^T dt = t\Big|_0^T = T - 0 = T \qquad (23)$$

$$\int_0^T t dt = \frac{t^2}{2}\Big|_0^T = \frac{T^2}{2} - 0 = \frac{T^2}{2} \qquad (30)$$

Thus we get

$$\int_0^T (v_{x0} + a_x t)dt = v_{x0}T + \frac{1}{2}a_x T^2 \qquad (81)$$

Using Equations (76) and (81) in (75) gives

$$x(T) - x_0 = v_{x0}T + \frac{1}{2}a_x T^2$$

Taking $x_0 = 0$ and replacing T by t gives the other constant acceleration formula

$$\boxed{x(t) = v_{x0}t + \frac{1}{2}a_x t^2} \qquad (a_x \textit{ constant}) \qquad (82)$$

You can now see that the factor of $t^2/2$ in the constant acceleration formulas comes from the integral $\int t\,dt$.

Exercise 4

Find the formula for the velocity v(t) and position x(t) for a car moving with constant acceleration a_x, that was located at position x_i at some initial time t_i.

Start your calculation from the equations

$$v_x(t) = \frac{dx(t)}{dt}$$

$$a_x(t) = \frac{dv_x(t)}{dt}$$

and go through all the steps that we did to get Equations (74) and (82). See if you can do this without looking at the text.

If you have to look back to see what some steps are, then finish the derivation looking at the text. Then a day or so later, clean off your desk, get out a blank sheet of paper, write down this problem, put the book away and do the derivation. Keep doing this until you can do the derivation of the constant acceleration formulas without looking at the text.

Constant Acceleration Formulas in Three Dimensions

To handle the case of motion with constant acceleration in three dimensions, you start with the separate equations

$$\begin{aligned} v_x(t) &= \frac{dx(t)}{dt} & a_x(t) &= \frac{dv_x(t)}{dt} \\ v_y(t) &= \frac{dy(t)}{dt} & a_y(t) &= \frac{dv_y(t)}{dt} \\ v_z(t) &= \frac{dz(t)}{dt} & a_z(t) &= \frac{dv_z(t)}{dt} \end{aligned} \qquad (83)$$

Then repeat, for each pair of equations, the steps that led to the constant acceleration formulas for motion in the x direction. The results will be

$$\begin{aligned} x(t) &= v_{x0}t + \frac{1}{2}a_x t^2 & v_x(t) &= v_{x0} + a_x t \\ y(t) &= v_{y0}t + \frac{1}{2}a_y t^2 & v_y(t) &= v_{y0} + a_y t \\ z(t) &= v_{z0}t + \frac{1}{2}a_z t^2 & v_z(t) &= v_{z0} + a_z t \end{aligned} \qquad (84)$$

The final step is to combine these six equations into the two vector equations

$$\boxed{\vec{x}(t) = \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 \ ; \quad \vec{v}(t) = \vec{v}_0 + \vec{a}t} \qquad (85)$$

These are the equations we analyzed graphically in Chapter 3 of the physics text, in Figure (3-34) and Exercise (3-9). (There we wrote $\vec{s}$ instead of $\vec{x}(t)$, and $\vec{v}_i$ rather than v_0.)

In many introductory physics courses, considerable emphasis is placed on solving constant acceleration problems. You can spend weeks practicing on solving these problems, and become very good at it. However, when you have done this, you have not learned very much physics because most forms of motion are not with constant acceleration, and thus the formulas do not apply. The formulas were important historically, for they were the first to allow the accurate prediction of motion (of cannonballs). But if too much emphasis is placed on these problems, students tend to use them where they do not apply. For this reason we have placed the exercises using the constant acceleration equations in an appendix at the end of chapter 4 of the physics text. There are plenty of problems there for all the practice you will need with these equations. Doing these exercises requires only algebra, there is no practice with calculus. To get some experience with calculus, be sure that you can confidently do Exercise 4.

MORE ON DIFFERENTIATION

In our discussion of integration, we saw that the basic idea was that the integral of some curve or function f(t) was equal to the area under that curve. That is an easy enough concept. The problems arose when we actually tried to find the formulas for the areas under various curves. The only areas we actually calculated were the rectangular area under f(t) = constant and the triangular area under f(t) = at. It was perhaps a surprise that the area under the simple curve 1/t should turn out to be a logarithm.

For differentiation, the basic idea of the process is given by the formula

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \qquad \text{(54 repeated)}$$

Equation (54) is short hand notation for a whole series of steps which we introduced through the use of strobe photographs. The basic idea of differentiation is more complex than integration, but, as we will now see, it is often a lot easier to find the derivative of a curve than its integral.

Series Expansions

An easy way to find the formula for the derivative of a curve is to use a series expansion. We will illustrate the process by using the binomial expansion to calculate the derivative of the function x^n where n is any constant.

We used the binomial expansion, or at least the first two terms, in Chapter 1 of the physics text. That was during our discussion of the approximation formulas that are useful in relativistic calculations. As we mentioned in Exercise (1-5), the binomial expansion is

$$(x + \alpha)^n = x^n + n\alpha x^{n-1} + \frac{n(n-1)}{2!}\alpha^2 x^{n-2} \ \cdots \qquad (86)$$

When α is a number much smaller than 1 $(\alpha << 1)$, we can neglect α^2 compared to α (if $\alpha = .01$, $\alpha^2 = .0001$), with the result that we can accurately approximate $(x + \alpha)^n$ by

$$\boxed{(x + \alpha)^n \approx x^n + n\alpha x^{n-1}} \qquad \alpha << 1 \qquad (87)$$

Equation (87) gives us all the approximation formulas found in Equations (1-20) through (1-25) on page 1-28 of the physics text.

As an example of Equation (87), just to see that it works, let us take x = 5, n = 7 and $\alpha = .01$ to calculate $(5.01)^7$. From the calculator we get

$$(5.01)^7 = 79225.3344 \qquad (88)$$

(To do this enter 5.01, press the y^x button, then enter 7 and press the = button.) Let us now see how this result compares with

$$\begin{aligned} (x + \alpha)^n &\approx x^n + n\alpha x^{n-1} \\ (5 + .01)^7 &\approx 5^7 + 7(.01)5^6 \end{aligned} \qquad (89)$$

We have

$$5^7 = 78125 \qquad (90)$$

$$7 \times .01 \times 5^6 = 7 \times .01 \times 15625 = 1093.75 \qquad (91)$$

Adding the numbers in (90) and (91) together gives

$$5^7 + 7(.01)5^6 = 79218.75 \qquad (92)$$

Thus we end up with 79218 instead of 79225, which is not too bad a result. The smaller α is compared to one, the better the approximation.

Derivative of the Function x^n

We are now ready to use our approximation formula (87) to calculate the derivative of the function x^n. From the definition of the derivative we have

$$\frac{d(x^n)}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \tag{93}$$

Since Δx is to become infinitesimally small, we can use our approximation formula for $(x+\alpha)^n$. We get

$$(x+\alpha)^n \approx x^n + n(\alpha)x^{n-1} \qquad (\alpha << 1)$$

$$(x+\Delta x)^n \approx x^n + n(\Delta x)x^{n-1} \qquad (\Delta x << 1) \tag{94}$$

Using this in Equation (93) gives

$$\frac{d(x^n)}{dx} = \lim_{\Delta x \to 0}\left[\frac{\left(x^n + n(\Delta x)x^{n-1}\right) - x^n}{\Delta x}\right] \tag{95}$$

We used an equal sign rather than an approximately equal sign in Equation (95) because our approximation formula (94) becomes exact when Δx becomes infinitesimally small.

In Equation (95), the terms x^n cancel and we are left with

$$\frac{d(x^n)}{dx} = \lim_{\Delta x \to 0}\left[\frac{n(\Delta x)x^{n-1}}{\Delta x}\right] \tag{96}$$

At this point, the factors Δx cancel and we have

$$\frac{d(x^n)}{dx} = \lim_{\Delta x \to 0}\left[nx^{n-1}\right] \tag{97}$$

Since no Δx's remain in our formula, we end up with the exact result

$$\boxed{\frac{d(x^n)}{dx} = nx^{n-1}} \tag{98}$$

Equation (98) is the general formula for the derivative of the function x^n.

In our discussion of integration, we saw that a constant could come outside the integral. The same thing happens with a derivative. Consider, for example,

$$\frac{d}{dx}\left[a f(x)\right] = \lim_{\Delta x \to 0}\left[\frac{a f(x+\Delta x) - a f(x)}{\Delta x}\right]$$

Since the constant a has nothing to do with the limiting process, this can be written

$$\frac{d}{dx}\left[af(x)\right] = a \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] = a\frac{df(x)}{dx} \tag{99}$$

Exercise 5

Calculate the derivative with respect to x (i.e., d/dx) of the following functions. (When negative powers of x are involved, assume x is not equal to zero.)

(a) x

(b) x^2

(c) x^3

(d) $5x^2 - 3x$

(Before you do part (d), use the definition of the derivative to prove that $\frac{d}{dx}\left[f(x) + g(x)\right] = \frac{df(x)}{dx} + \frac{dg(x)}{dx}$)

(e) x^{-1}

(f) x^{-2}

(g) $\sqrt{x}$

(h) $1/\sqrt{x}$

(i) $3x^{.73}$

(j) $7x^{-.2}$

(k) 1

(In part (k) first show that this should be zero from the definition of the derivative. Then write $1 = x^0$ and show that Equation (98) also works, as long as x is not zero.)

(l) 5

The Chain Rule

There is a simple trick called the ***chain rule*** that makes it easy to differentiate a wide variety of functions. The rule is

$$\boxed{\frac{df\left[y(x)\right]}{dx} = \frac{df(y)}{dy}\,\frac{dy}{dx}} \qquad \textit{chain rule} \qquad (100)$$

To see how this rule works, consider the function

$$f(x) = \left(x^2\right)^n \qquad (101)$$

We know that this is just $f(x) = x^{2n}$, and the derivative is

$$\frac{df(x)}{dx} = \frac{d}{dx}\left(x^{2n}\right) = 2nx^{2n-1} \qquad (102)$$

But suppose that we did not know this trick, and therefore did not know how to differentiate $(x^2)^n$. We do, however, know how to differentiate powers like x^2 and y^n. The chain rule allows us to use this knowledge in order to figure out how to differentiate the more complex function $(x^2)^n$.

We begin by defining y(x) as

$$y(x) = x^2 \qquad (103)$$

Then our function $f(x) = (x^2)^n$ can be written in terms of y as follows

$$f(x) = (x^2)^n = \left(y(x)\right)^n = (y)^n = f(y)$$

$$f(y) = (y)^n \qquad (104)$$

Differentiating (103) and (104) gives

$$\frac{dy(x)}{dx} = \frac{d}{dx}\left(x^2\right) = 2x \qquad (105)$$

$$\frac{df(y)}{dy} = \frac{d}{dy}\left(y^n\right) = ny^{n-1} \qquad (106)$$

Using (104) and (105) in the chain rule (100) gives

$$\begin{aligned}
\frac{df(y)}{dx} &= \frac{df}{dy} \times \frac{dy}{dx} = \left(ny^{n-1}\right) \times \left(2x\right) \\
&= 2ny^{n-1}x \\
&= 2n\left(x^2\right)^{n-1}x \\
&= 2n\left(x^{2(n-1)}\right)x \\
&= 2n\left(x^{(2n-2)}\right)x \qquad (107) \\
&= 2n\left(x^{(2n-2)+1}\right) \\
&= 2nx^{2n-1}
\end{aligned}$$

which is the answer we expect.

In our example, using the chain rule was more difficult than differentiating directly because we already knew how to differentiate x^{2n} . But we will shortly encounter examples of new functions that we do not know how to differentiate directly, but which can be written in the form f[y(x)]; and where we know df/dy and dy/dx. We can then use the chain rule to evaluate the derivative df/dx. We will give you practice with the chain rule when we encounter these functions.

Remembering The Chain Rule

The chain rule can be remembered by thinking of the dy's as cancelling as shown.

$$\boxed{\frac{df(y)}{\cancel{dy}}\,\frac{\cancel{dy}}{dx} = \frac{df(y)}{dx}} \qquad \begin{matrix}\textit{remembering}\\ \textit{the chain rule}\end{matrix} \qquad (108)$$

Partial Proof of the Chain Rule (optional)

The proof of the chain rule is closely related to cancellation we showed in Equation (108). A partial proof of the rule proceeds as follows.

Suppose we have some function f(y) where y is a function of the variable x. As a result f[y(x)] is itself a function of x and can be differentiated with respect to x.

$$\frac{d}{dx} f\left[y(x)\right] = \underset{\Delta x \to 0}{\text{limit}} \frac{f\left[y(x+\Delta x)\right] - f\left[y(x)\right]}{\Delta x} \qquad (123)$$

Now define the quantity Δy by

$$\Delta y \equiv y(x+\Delta x) - y(x) \qquad (124)$$

so that

$$y(x+\Delta x) = y(x) + \Delta y$$

$$f[y(x+\Delta x)] = f(y+\Delta y)$$

and Equation (123) becomes

$$\frac{d}{dx} f\left[y(x)\right] = \underset{\Delta x \to 0}{\text{limit}} \frac{f(y+\Delta y) - f(y)}{\Delta x} \qquad (125)$$

Now multiply (125) through by

$$1 = \frac{\Delta y}{\Delta y} = \frac{y(x+\Delta x) - y(x)}{\Delta y} \qquad (126)$$

to get

$$\frac{d}{dx} f\left[y(x)\right]$$

$$= \underset{\Delta x \to 0}{\text{limit}} \left[\frac{f(y+\Delta y) - f(y)}{\Delta x} \times \frac{y(x+\Delta x) - y(x)}{\Delta y}\right]$$

$$= \underset{\Delta x \to 0}{\text{limit}} \left[\frac{f(y+\Delta y) - f(y)}{\Delta y} \times \frac{y(x+\Delta x) - y(x)}{\Delta x}\right] \qquad (127)$$

where we interchanged Δx and Δy in the denominator.

(We call this a partial proof for the following reason. For some functions y(x), the quantity $\Delta y = y\left(x + \Delta x\right) - y(x)$ may be identically zero for a small range of Δx. In that case we would be dividing by zero (the $1/\Delta y$) even before we took the limit as Δx goes to zero. A more complete proof handles the special cases separately. The resulting chain rule still works however, even for these special cases.)

Since $\Delta y = y(x+\Delta x) - y(x)$ goes to zero as Δx goes to zero, we can write Equation (127) as

$$\frac{d}{dx} f\left[y(x)\right]$$

$$= \left[\underset{\Delta y \to 0}{\text{limit}} \frac{f(y+\Delta y) - f(y)}{\Delta y}\right]$$

$$\times \left[\underset{\Delta x \to 0}{\text{limit}} \frac{y(x+\Delta x) - y(x)}{\Delta x}\right]$$

$$= \left[\frac{df(y)}{dy}\right]\left[\frac{dy}{dx}\right] \qquad (100\ \text{repeated})$$

This rule works as long as the derivatives df/dy and dy/dx are meaningful, i.e., we stay away from kinks or discontinuities in f and y.

INTEGRATION FORMULAS

Knowing the formula for the derivative of the function x^n, and knowing that integration undoes differentiation, we can now use Equation (98)

$$\frac{dx^n}{dx} = nx^{n-1} \qquad (98\ repeated)$$

to find the integral of the function x^n. We will see that this trick works for all cases except the special case where $n = -1$, i.e., the special case where the integral is a natural logarithm.

To integrate x^n, let us go back to our calculation of the distance s_x or x(t) traveled by an object moving in the x direction at a velocity v_x. This was given by Equations (19) or (56) as

$$x(t)\Big|_{t_i}^{T} = \int_{t_i}^{T} v_x(t)\,dt \qquad (128)$$

where the instantaneous velocity $v_x(t)$ is defined as

$$v_x(t) = \frac{dx(t)}{dt} \qquad (129)$$

Suppose x(t) had the special form

$$x(t) = t^{n+1} \qquad \textit{(a special case)} \qquad (130)$$

then we know from our derivative formulas that

$$v(t) = \frac{dx(t)}{dt} = \frac{dt^{(n+1)}}{dt} = (n+1)t^n \qquad (131)$$

Substituting $x(t) = t^{n+1}$ and $v(t) = (n+1)t^n$ into Equation (128) gives

$$x(t)\Big|_{t_i}^{T} = \int_{t_i}^{T} v_x(t)\,dt \qquad (128)$$

$$\begin{aligned} t^{n+1}\Big|_{t_i}^{T} &= \int_{t_i}^{T} (n+1)t^n dt \\ &= (n+1)\int_{t_i}^{T} t^n dt \end{aligned} \qquad (132)$$

Dividing through by (n+1) gives

$$\boxed{\int_{t_i}^{T} t^n dt = \frac{1}{n+1}\, t^{n+1}\Big|_{t_i}^{T}} \qquad (133)$$

If we choose $t_i = 0$, we get the simpler result

$$\int_{0}^{T} t^n dt = \frac{T^{n+1}}{n+1} \qquad (134)$$

and the indefinite integral can be written

$$\boxed{\int t^n\,dt = \frac{t^{n+1}}{n+1}} \qquad (135)(also\ 34)$$

This is the general rule we stated without proof back in Equation (34). Note that this formula says nothing about the case $n = -1$, i.e., when we integrate $t^{-1} = 1/t$, because $n + 1 = -1 + 1 = 0$ and we end up with division by zero. But for all other values of n, we now have derived a general formula for finding the area under any curve of the form x^n (or t^n). This is a rather powerful result considering the problems one encounters actually finding areas under curves. (If you did not do Exercise 1, the integration exercises on page 14, or had difficulty with them, go back and do them now.)

Derivative of the Exponential Function

The previous work shows us that if we have a series expansion for a function, it is easy to obtain a formula for the derivative of the function. We will now apply this technique to calculate the derivative and integral of the exponential function e^x.

There is a series expansion for the function e^x that works for any value of α in the range –1 to +1.

$$e^{\alpha} \approx 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots \tag{136}$$

where $2! = 2\times 1$, $3! = 3\times 2\times 1 = 6$, etc. (The quantities 2!, 3! are called ***factorials***. For example 3! is called *three factorial*.)

To see how well the series (136) works, consider the case $\alpha = .01$. From the series we have, up to the α^3 term

$$\alpha = .01$$

$$\alpha^2 = .0001\ ; \qquad \alpha^2/2 = .00005$$

$$\alpha^3 = .000001\ ; \qquad \alpha^3/6 = .000000167$$

Giving us the approximate value

$$1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} = 1.010050167 \tag{137}$$

When we enter .01 into a scientific calculator and press the e^x button, we get exactly the same result. Thus the calculator is no more accurate than including the α^3 term in the series, for values of α equal to .01 or less.

Let us now see how to use the series 136 for calculating the derivative of e^x. We have, from the definition of a derivative,

$$\frac{d}{dx}f(x) \equiv \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{56repeat}$$

If $f(x) = e^x$, we get

$$\frac{d(e^x)}{dx} = \lim_{\Delta x\to 0}\left[\frac{e^{x+\Delta x} - e^x}{\Delta x}\right] \tag{138}$$

To do this calculation, we have to evaluate the quantity $e^{x+\Delta x}$. First, we use the fact that for exponentials

$$e^{a+b} = e^a e^b$$

(Remember that $10^{2+3} = 10^2 \times 10^3 = 10^5$.) Thus

$$e^{x+\Delta x} = e^x e^{\Delta x} \tag{139}$$

Now use the approximation formula (136), setting $\alpha = \Delta x$ and throwing out the α^2 and α^3 and higher terms because we are going to let Δx go to zero

$$e^{\Delta x} \approx 1 + \Delta x \tag{140}$$

Substituting (140) in (139) gives

$$e^{x+\Delta x} \approx e^x(1+\Delta x)$$

$$= e^x + e^x\Delta x \tag{141}$$

Next use (141) in (138) to get

$$\frac{d(e^x)}{dx} = \lim_{\Delta x\to 0}\left[\frac{\left(e^x + e^x\Delta x\right) - e^x}{\Delta x}\right] \tag{142}$$

The e^x terms cancel and we are left with

$$\frac{d(e^x)}{dx} = \lim_{\Delta x\to 0}\left[\frac{e^x\Delta x}{\Delta x}\right] = \lim_{\Delta x\to 0} e^x \tag{143}$$

Since the Δx's cancelled, we are left with the exact result

$$\boxed{\frac{d(e^x)}{dx} = e^x} \tag{144}$$

We see that the exponential function e^x has the special property that it is its own derivative.

We will often want to know the derivative, not just of the function e^x but of the slightly more general result e^{ax} where a is a constant. That is, we want to find

$$\frac{d}{dx}e^{ax} \qquad (a = constant) \tag{145}$$

Solving this problem provides us with our first meaningful application of the chain rule

$$\frac{df(y)}{dx} = \frac{df(y)}{dy}\frac{dy}{dx} \tag{100 repeated}$$

If we set

$$y=ax \tag{146}$$

then we have

$$\frac{de^{ax}}{dx} = \frac{de^y}{dy}\frac{dy}{dx} \tag{147}$$

Now

$$\frac{de^y}{dy} = e^y \tag{148}$$

$$\frac{dy}{dx} = \frac{d}{dx}(ax) = a\frac{dx}{dx} = a \times 1 = a \tag{149}$$

Using (148) and (149) in (147) gives

$$\frac{de^{ax}}{dx} = \left(e^y\right)(a) = \left(e^{ax}\right)(a) = ae^{ax}$$

Thus we have

$$\boxed{\frac{d}{dx}e^{ax} = ae^{ax}} \tag{150}$$

This result will be used so often it is worth memorizing.

Exercise 6

For further practice with the chain rule, show that

$$\frac{de^{ax^2}}{dx} = 2axe^{ax^2}$$

Do this by choosing $y = ax^2$, and then do it again by choosing $y = x^2$.

Integral of the Exponential Function

To calculate the integral of e^{ax}, we will use the same trick as we used for the integral of x^n, but we will be a bit more formal this time. Let us start with Equation (128) relating position x(t) and velocity v(t) = dx(t)/dt go get

$$x(t)\Big|_{t_i}^{t_f} = \int_{t_i}^{t_f} v_x(t)\,dt = \int_{t_i}^{t_f} \frac{dx(t)}{dt}\,dt \tag{128}$$

Since Equation (128) holds for any function x(t) [we did not put any restrictions on x(t)], we can write Equation (128) in a more abstract way relating any function f(x) to its derivative df(x)/dx;

$$\boxed{f(x)\Big|_{x_i}^{x_f} = \int_{x_i}^{x_f} \frac{df(x)}{dx}\,dx} \tag{151}$$

To calculate the integral of e^{ax}, we set $f(x) = e^{ax}$ and $df(x)/dx = ae^{ax}$ to get

$$e^{ax}\Big|_{x_i}^{x_f} = \int_{x_i}^{x_f} a\,e^{ax}\,dx \tag{152}$$

Dividing (157) through by (a) gives us the definite integral

$$\boxed{\int_{x_i}^{x_f} e^{ax}dx = \frac{1}{a}e^{ax}\Big|_{x_i}^{x_f}} \qquad (a = constant) \tag{153}$$

The corresponding indefinite integral is

$$\boxed{\int e^{ax}dx = \frac{e^{ax}}{a}} \qquad (a = constant) \tag{154}$$

Exercise 7

The natural logarithm is defined by the equation

$$\ln(x) = \int\left(\frac{1}{x}\right)dx \qquad \text{(see Equations 33-40)}$$

Use Equation (151) to show that

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \tag{155}$$

(Hint—integrate both sides of Equation (155) with respect to x.)

DERIVATIVE AS THE SLOPE OF A CURVE

Up to now, we have emphasized the idea that the derivative of a function f(x) is given by the limiting process

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \quad (55 \text{ repeated})$$

We saw that this form was convenient when we had an explicit way of calculating $f(x+\Delta x)$, as we did by using a series expansion. However, a lot of words are required to explain the steps involved in doing the limiting process indicated in Equation (55). In contrast, the idea of an integral as being the area under a curve is much easier to state and visualize. Now we will provide an easy way to state and interpret the derivative of a curve.

Consider the function f(x) graphed in Figure (20). At a distance x down the x axis, the curve had a height f(x) as shown. Slightly farther down the x axis, at $x+\Delta x$, the curve has risen to a height $f(x+\Delta x)$.

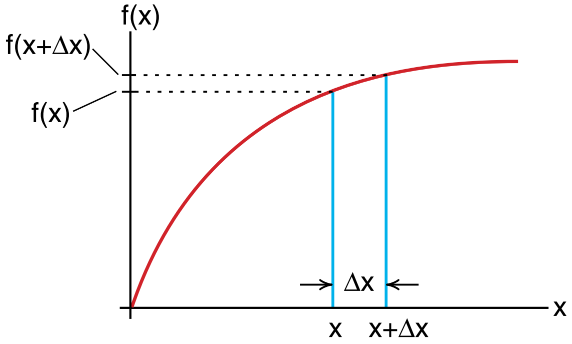


Figure 20

Two points on a curve, a distance Δx apart.

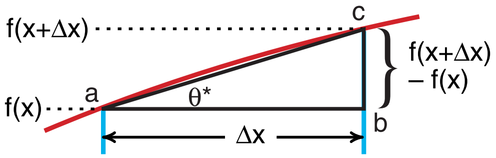


Figure 20a

At this point, the curve is tilted by approximately an angle θ.*

Figure (20a) is a blowup of the curve in the region between x and $x+\Delta x$. If the distance Δx is sufficiently small, the curve between x and $x+\Delta x$ should be approximately a straight line and that part of the curve should be approximately the hypotenuse of the right triangle abc seen in Figure (20a). Since the side opposite to the angle θ^* is $f(x+\Delta x) - f(x)$, and the adjacent side is Δx, we have the result that the tangent of the angle θ^* is

$$\tan\left(\theta^*\right) = \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (156)$$

When we make Δx smaller and smaller, take the limit as $\Delta x \to 0$, we see that the angle θ^* becomes more nearly equal to the angle θ shown in Figure (21), the angle of the curve when it passes through the point x. Thus

$$\tan\theta = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (157)$$

The tangent of the angle at which the curve passes through the point x is called the ***slope of the curve at the point x.*** Thus from Equation (157) we see that the slope of the curve is equal to the derivative of the curve at that point. We now have the interpretation that the derivative of a curve at some point is equal to the slope of the curve at that point, while the integral of a curve is equal to the area under the curve up to that point.

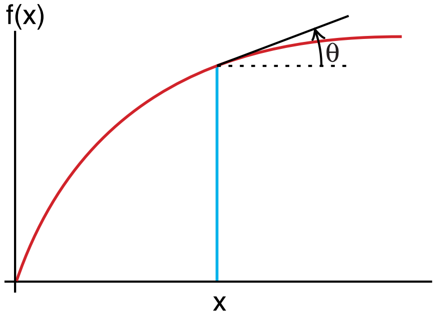


Figure 21

The tangent of the angle θ at which the curve passes through the point x is called the slope of the curve at that point.

Negative Slope

In Figure (22) we compare the slopes of a rising and a falling curve. In (22a), where the curve is rising, the quantity $f(x + \Delta x)$ is greater than $f(x)$ and the derivative or slope

$$\frac{df(x)}{dx} = \underset{\Delta x \to 0}{\text{limit}} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right]$$

is a positive number.

In contrast, for the downward curve of Figure (22b), $f(x + \Delta x)$ is less than $f(x)$ and the slope is negative. For a curve headed downward, we have

$$\frac{df(x)}{dx} = -\tan(\theta) \qquad \textit{downward heading curve} \tag{158}$$

(For this case you can think of θ as a negative angle, so that $\tan(\theta)$ would automatically come out negative. However it is easier simply to remember that the slope of an upward directed curve is positive and that of a downward directed cure is negative.)

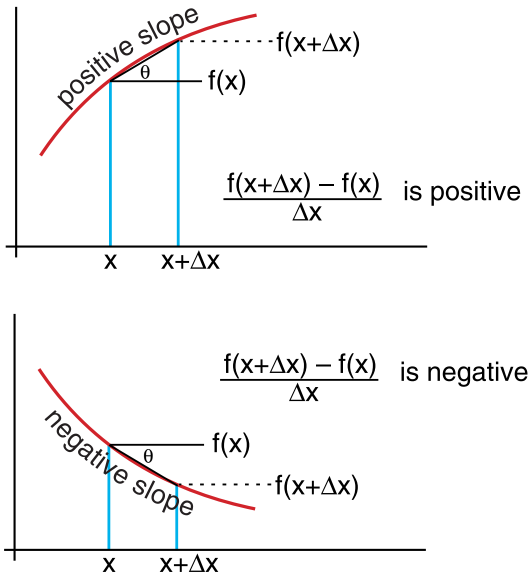


Figure 22
Going uphill is a positive slope, downhill is a negative slope.

Exercise 8

Estimate the numerical value of the slope of the curve shown in Figure (23) at points (a), (b), (c), (d) and (e). In each case do a sketch of $\left[f(x+\Delta x)-f(x)\right]$ for a small Δx, and let the slope be the ratio of $\left[f(x+\Delta x)-f(x)\right]$ to Δx. Your answers should be roughly 1, 0, –1, $+\infty$, $-\infty$.

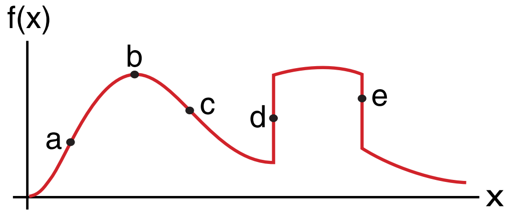


Figure 23
Estimate the slope at the various points indicated.

THE EXPONENTIAL DECAY

A curve that we will encounter several times during the course is the function e^{-ax} shown in Figure (24), which we call an exponential decay. Since exponents always have to be dimensionless numbers, we are writing the constant (a) in the form $1/x_0$ so that the exponent x/x_0 is more obviously dimensionless.

The function e^{-x/x_0} has several very special properties. At x = 0, it has the numerical value 1 $(e^0 = 1)$. When we get up to $x = x_0$, the curve has dropped to a value

$$\begin{aligned} e^{-x/x_0} &= e^{-1} = \frac{1}{e} \qquad (at\ x = x_0) \\ &\approx \frac{1}{2.7} \end{aligned} \tag{159}$$

When we go out to $x = 2x_0$, the curve has dropped to

$$e^{-2x_0/x_0} = e^{-2} = \frac{1}{e^2} \tag{160}$$

Out at $x = 3x_0$, the curve has dropped by another factor of e to (1/e)(1/e)(1/e). This decrease continues indefinitely. It is the characteristic feature of an exponential decay.

Muon Lifetime

In the muon lifetime experiment, we saw that the number of muons surviving decreased with time. At the end of two microseconds, more than half of the original 648 muons were still present. By 6 microseconds, only 27 remained. The decay of these muons is an example of an exponential decay of the form

$$\begin{matrix}\text{number of}\\ \text{surviving}\\ \text{muons}\end{matrix} = \begin{pmatrix}\text{number of}\\ \text{muons at}\\ \text{time } t = 0\end{pmatrix} \times e^{-t/t_0} \tag{161}$$

where t_0 is the time it takes for the number of muons remaining to drop by a factor of 1/e = 1/2.7. That time is called the muon ***lifetime***.

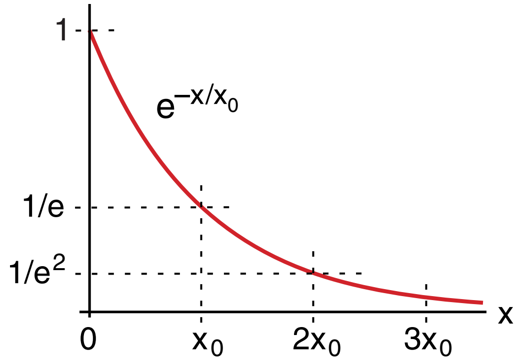


Figure 24

As we go out an additional distance x_0, the exponential curve drops by another factor of 1/e.

We can use Equation (161) to estimate the muon lifetime t_0. In the movie, the number of mesons at the top of the graph, reproduced in Figure (25), is 648. That is at time t = 0. Down at time t = 6 microseconds, the number surviving is 27. Putting these numbers into Equation (161) gives

$$27 \begin{matrix}\text{surviving}\\ \text{muons}\end{matrix} = 648 \begin{matrix}\text{initial}\\ \text{muons}\end{matrix} \times e^{-6/t_0}$$

$$e^{-6/t_0} = \frac{27}{648} = .042 \tag{162}$$

Take the natural logarithm *ln* of both sides of Equation (162), [remembering that $ln\left(e^x\right) = x$] gives

$$ln\left(e^{-6/t_0}\right) = \frac{-6}{t_0} = ln\left(.042\right) = -3.17$$

where we entered .042 on a scientific calculator and pressed the *ln* key. Solving for t_0 we get

$$t_0 = \frac{6}{3.17} = 1.9 \text{ microseconds} \tag{163}$$

This is close to the accepted value of t_0 = 2.2 0 microseconds which has been determined from the study of many thousands of muon decays.

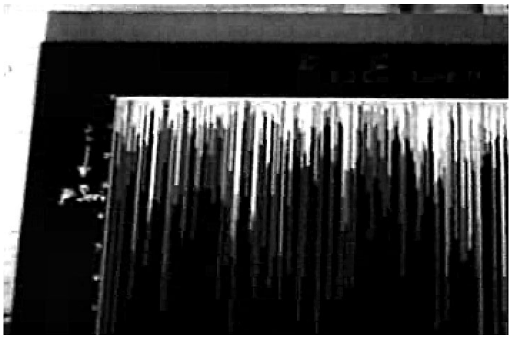


Figure 25

The lifetime of each detected muon is represented by the length of a vertical line. We can see that many muons live as long as 2 microseconds (2µs), but few live as long as 6 microseconds.

Half Life

The exponential decay curve e^{-t/t_0} decays to $1/e = 1/2.7$ of its value at time t_0. While $1/e$ is a very convenient number from a mathematical point of view, it is easier to think of the time $t_{1/2}$ it takes for half of the muons to decay. This time $t_{1/2}$ is called the ***half life*** of the particle.

From Figure (26) we can see that the half life $t_{1/2}$ is slightly shorter than the lifetime t_0. To calculate the half life from t_0, we have

$$e^{-t/t_0}\Big|_{t=t_{1/2}} = e^{-t_{1/2}/t_0} = \frac{1}{2} \qquad (164)$$

Again taking the natural logarithm of both sides of Equation (164) gives

$$ln\left(e^{-t_{1/2}/t_0}\right) = \frac{-t_{1/2}}{t_0} = ln\left(\frac{1}{2}\right) = -.693$$

$$\boxed{t_{1/2} = .693\, t_0} \qquad (165)$$

From Equation (165) you can see that a half life $t_{1/2}$ is about .7 of the lifetime t_0. If the muon lifetime is $2.2\mu sec$ (we will abbreviate microseconds as μsec), and you start with a large number of muons, you would expect about half to decay in a time of

$$\left(t_{1/2}\right)_{muon} = .693 \times 2.2\mu sec = 1.5\ \mu sec$$

The basic feature of the exponential decay curve e^{-t/t_0} is that for every time t_0 that passes, the curve decreases by another factor of 1/e. The same applies to the half life $t_{1/2}$. After one half life, e^{-t/t_0} has decreased to half its value. After a second half life, the curve is down to 1/4 = 1/2 x 1/2. After 3 half lives it is down to 1/8 = 1/2 x 1/2 x 1/2 as shown in Figure (27).

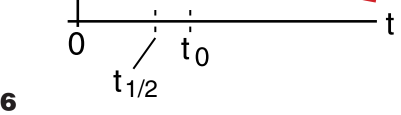


Figure 26
Comparison of the lifetime t_0 and the half-life $t_{1/2}$.

To help illustrate the nature of exponential decays, suppose that you started with a million muons. How long would you expect to wait before there was, on the average, only one left?

To solve this problem, you would want the number e^{-t/t_0} to be down by a factor of 1 million

$$e^{-t/t_0} = 1 \times 10^{-6}$$

Taking the natural logarithm of both sides gives

$$ln\left(e^{-t/t_0}\right) = \frac{-t}{t_0} = ln\left(1\times 10^{-6}\right) = -13.8 \qquad (166)$$

(To calculate $ln\left(1\times 10^{-6}\right)$, enter 1, then press the *exp* key and enter 6, then press the +/– key to change it to –6. Finally press = to get the answer –13.8.)

Solving Equation (166) for t gives

$$t = 13.8\, t_0 = 13.8 \times 2.2\ \mu sec$$

$$\boxed{t = 30 \text{ microseconds}} \qquad (167)$$

That is the nature of an exponential decay. While you have nearly half a million left after around 2 microseconds, they are essentially all gone by 30 microseconds.

Exercise 9

How many factors of 1/2 do you have to multiply together to get approximately 1/1,000,000? Multiply this number by the muon half-life to see if you get about 30 microseconds.

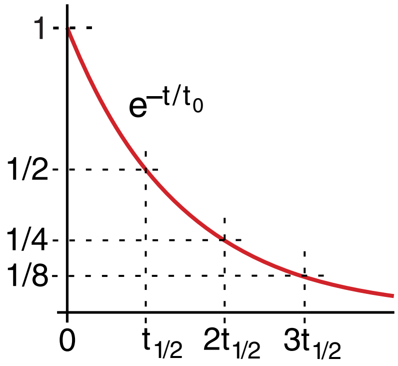


Figure 27
After each half-life, the curve decreases by another factor of 1/2.

Measuring the Time Constant from a Graph

The idea that the derivative of a curve is the slope of the curve, leads to an easy way to estimate a lifetime t_0 from an exponential decay curve e^{-t/t_0}.

The formula for the derivative of an exponential curve is

$$\frac{de^{at}}{dt} = ae^{at} \qquad \text{(150 repeated)}$$

Setting $a = -1/t_0$ gives

$$\frac{d}{dt}\left(e^{-t/t_0}\right) = -\frac{1}{t_0}e^{-t/t_0} \qquad (168)$$

Since the derivative of a curve is the slope of the curve, we set the derivative equal to the tangent of the angle the curve makes with the horizontal axis.

$$\frac{d}{dt}\left(e^{-t/t_0}\right) = -\frac{1}{t_0}e^{-t/t_0} = \tan\theta \qquad (168a)$$

The minus sign tells us that the curve is headed down.

In Figure (28), we have drawn a line tangent to the curve at the point $t = T$. This line intersects the (t) axis (the axis where e^{-t/t_0} goes to zero) at a distance (x) down the *t* axis.

The height (y) of the point where we drew the tangent curve is just the value of the function e^{-T/t_0}. The tangent of the angle θ is the opposite side (y) divided by the adjacent side (x)

$$\tan\theta = \frac{y}{x} = \frac{e^{-T/t_0}}{x} \qquad (169)$$

Equating the two magnitudes of $\tan\theta$ in Equations (169) in (168a) gives us

$$\frac{1}{t_0}e^{-T/t_0} = \frac{1}{x}e^{-T/t_0}$$

which requires that

$$x = t_0 \qquad (170)$$

Equation (170) tells us that the distance (x), the distance down the axis where the tangent lines intersect the axis, is simply the time constant t_0.

The result gives us a very quick way of determining the time constant t_0 of an exponential decay curve. As illustrated in Figure (29), choose any point on the curve, draw a tangent to the curve at that point and measure the distance down the axis where the tangent line intersects the axis. That distance will be the time constant t_0. We will use this technique in several laboratory exercises later in the course.

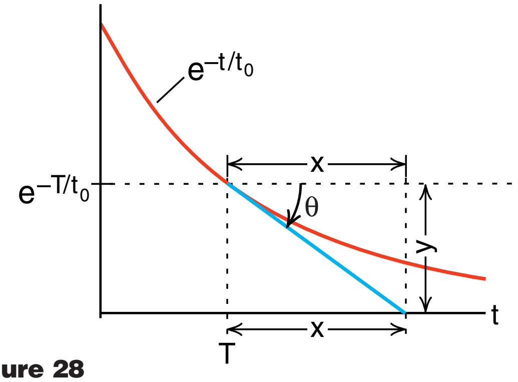


Figure 28

A line, drawn tangent to the exponential decay curve at some point T, intersects the axis a distance x down the axis. We show that this distance x is equal to the time constant t_0. This is true no matter what point T we start with.

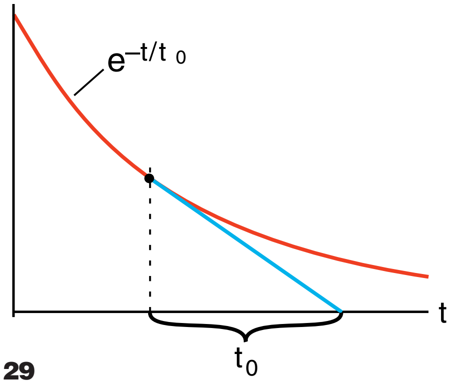


Figure 29

A quick way to estimate the time constant t_0 for an exponential decay curve is to draw the tangent line as shown.

THE SINE AND COSINE FUNCTIONS

The final topic in our introduction to calculus will be the functions $\sin\theta$ and $\cos\theta$ and their derivatives and integrals. We will need these functions when we come to rotational motion and wave motion.

The definition of $\sin\theta$ and $\cos\theta$, which should be familiar from trigonometry, are

$$\sin\theta = \frac{a}{c} \qquad \left(\frac{\textit{opposite}}{\textit{hypotenuse}}\right) \tag{171a}$$

$$\cos\theta = \frac{b}{c} \qquad \left(\frac{\textit{adjacent}}{\textit{hypotenuse}}\right) \tag{171b}$$

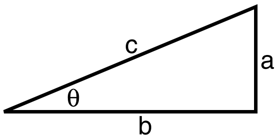


Figure 30

where θ is an angle of a right triangle as sown in Figure (30), (a) is the length of the side opposite to θ, (b) the side adjacent to θ and (c) the hypotenuse.

The formulas are simplified if we consider a right triangle whose hypotenuse is of length $c = 1$ as in Figure (31). Then we have

$$\sin\theta = a \tag{172a}$$

$$\cos\theta = b \tag{172b}$$

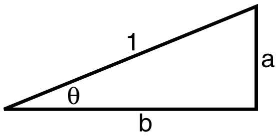


Figure 31

We can then fit our right triangle inside a circle of radius 1 as shown in Figure (32).

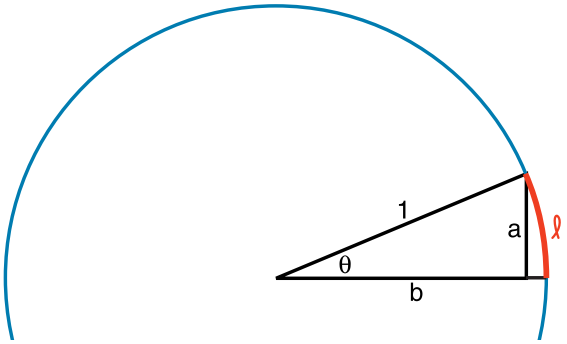


Figure 32
Fitting our right triangle inside a unit radius circle.

Radian Measure

We are brought up to measure angles in degrees, but physicists and mathematicians usually measure angles in ***radians***. The angle θ measured in radians is defined as the ***arc length*** ℓ subtended by the angle θ on a circle of unit radius, as shown in Figure (32).

$$\theta_{\text{radians}} = \ell \qquad \begin{matrix}\textit{arc length subtended}\\ \textit{by } \theta \textit{ on a unit circle}\end{matrix} \tag{173}$$

(If we had a circle of radius c, then we would define $\theta_{\text{radians}} = \ell/c$, a dimensionless ratio. In the special case c = 1, this reduces to $\theta_{\text{radians}} = \ell$.)

Since the circumference of a unit circle is 2π, we see that θ for a complete circle is 2π radians, which is the same as 360 degrees. This tells us how to convert from degrees to radians. We have the conversion factor

$$\frac{360\text{ degrees}}{2\pi\text{ radians}} = 57.3\,\frac{\text{degrees}}{\text{radian}} \tag{174}$$

As an example of using this conversion factor, suppose we want to convert 30 degrees to radians. We would have

$$\frac{30\text{ degrees}}{57.3\text{ degrees/radian}} = .52\text{ radians} \tag{175}$$

To decide whether to divide by or multiply by a conversion factor, use the dimensions of the conversion factor. For example, if we had multiplied 30 degrees by our conversion factor, we would have gotten

$$30\text{ degrees} \times 57.3\,\frac{\text{degrees}}{\text{radian}} = 1719\,\frac{\text{degrees}^2}{\text{radian}}$$

This answer may be correct, but it is useless.

The numbers to remember in using radians are the following:

$$\begin{aligned} 90^\circ &= \pi/2\text{ radians}\\ 180^\circ &= \pi\text{ radians}\\ 270^\circ &= 3\pi/2\text{ radians}\\ 360^\circ &= 2\pi\text{ radians}\end{aligned} \tag{176}$$

The other values you can work out as you need them.

The Sine Function

In Figure (33) we have started with a circle of radius 1 and, in a somewhat random way, labeled 10 points around the circle. The arc length up to each of these points is equal to the angle, in radian measure, subtended by that point. The special values are:

$\theta_0 = 0$ radians
$\theta_4 = \pi/2$ radians (90°)
$\theta_6 = \pi$ radians (180°)
$\theta_8 = 3\pi/2$ radians (270°)
$\theta_{10} = 2\pi$ radians (360°)

In each case the $\sin\theta$ is equal to the height (a) at that point. For example

$$\sin\theta_1 = a_1$$
$$\sin\theta_2 = a_2$$
$$\cdots\cdots$$
$$\sin\theta_{10} = a_{10}$$

We see that the height (a) starts out at $a_0 = 0$ for θ_0, increases up to $a_4 = 1$ at the top of the circle, drops back down to $a_6 = 0$ at $\theta_6 = \pi$, goes negative, down to $a_8 = -1$ at $\theta_8 = 3\pi/2$, and returns to $a_{10} = 0$ at $\theta_{10} = 2\pi$.

Our next step is to construct a graph in which θ is shown along the horizontal axis, and we plot the value of $\sin\theta = (a)$ on the vertical axis. The result is shown in Figure (34). The eleven points, representing the heights a_0 to a_{10} at θ_0 to θ_{10} are shown as large dots in Figure (34). We have also sketched in a smooth curve through these points, it is the curve we would get if we had plotted the value of (a) for every value of θ from $\theta = 0$ to $\theta = 2\pi$. The smooth curve is a graph of the function ***sin θ***.

Exercise 10

Using the fact that the cosine function is defined as

$\cos\theta = b$ (b is defined in Figures 31, 32)

plot the values of $b_0, b_1, \cdots, b_{10}$ on a graph similar to Figure (34), and show that the cosine function $\cos\theta$ looks like the curve shown in Figure (35).

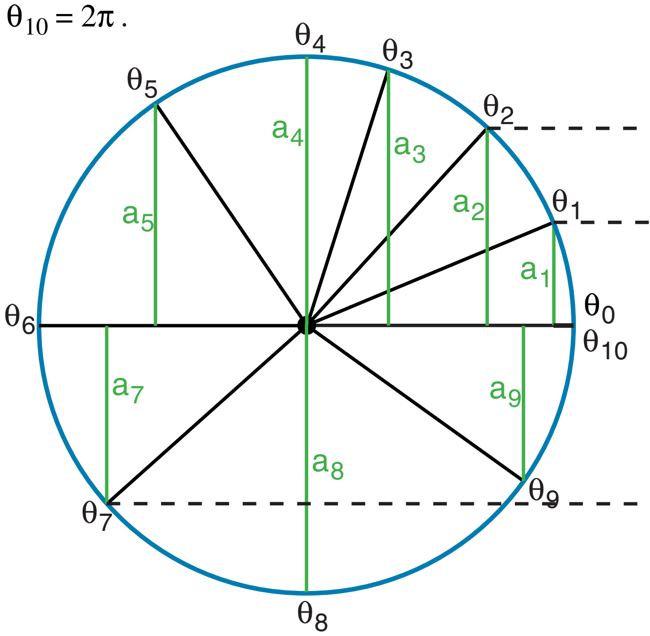


Figure 33
The heights a_i at various points around a unit circle.

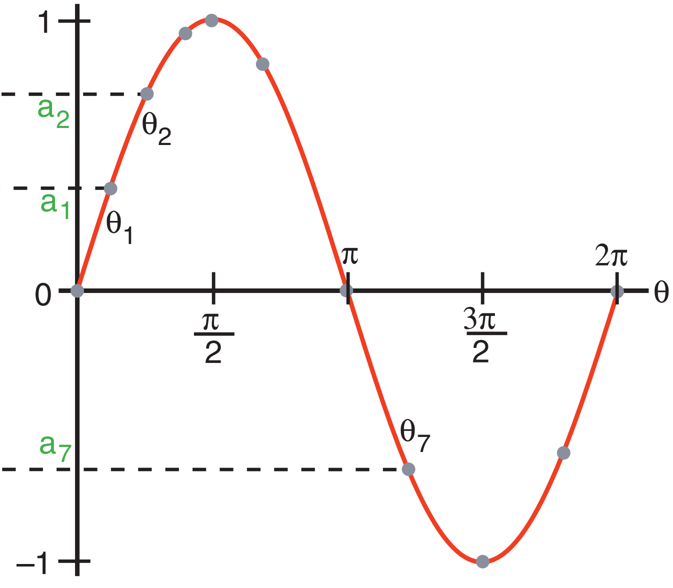


Figure 34
Graph of the function $\sin(\theta)$.

There is nothing that says we have to stop measuring the angle θ after we have gone around once. On the second trip around, θ increases from 2π up to 4π, and the curve $\sin\theta$ repeats itself. If we go around several times, we get a result like that shown in Figure (36).

Several cycles of the curve $\cos\theta$ are shown in Figure (37). You can see that the only difference between a sine and a cosine curve is where you set $\theta = 0$. If you move the origin of the cosine axis back (to the left) 90° $(\pi/2)$, you get a sine wave.

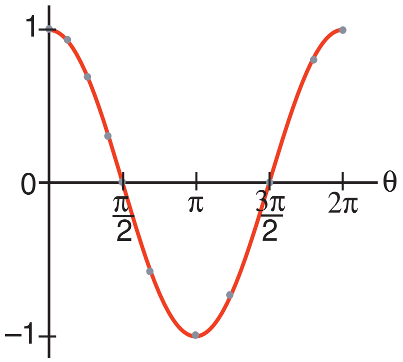


Figure 35
The cosine function.

Amplitude of a Sine Wave

A graph of the function $y(\theta) = c\sin\theta$ looks just like the curve in Figure (36), except the curve goes up to a height c and down to –c as shown in Figure (38). We would get the curve of Figure (38) by plotting points around a circle as in Figure (33), but using a circle of radius c. We call this factor c the *amplitude* of the sine wave. The function $\sin\theta$ has an amplitude 1, while the sine wave in Figure (38) has an ***amplitude c*** (its values range from +c to –c).

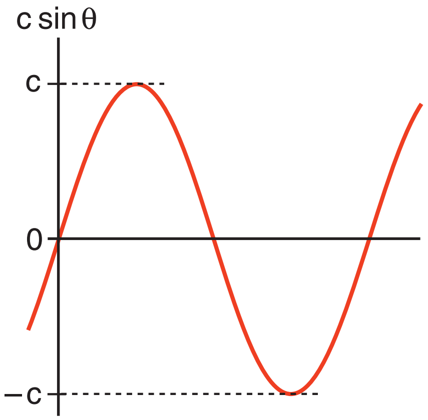


Figure 38
A sine wave of amplitude c.

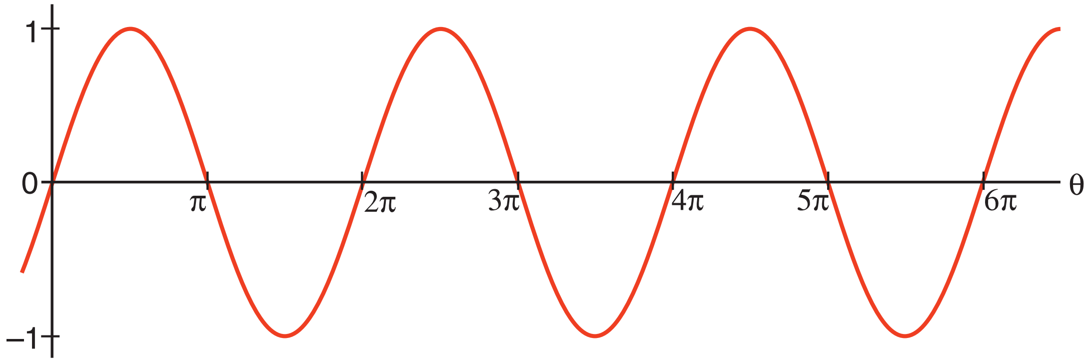


Figure 36
Several cycles of the curve $\sin(\theta)$.

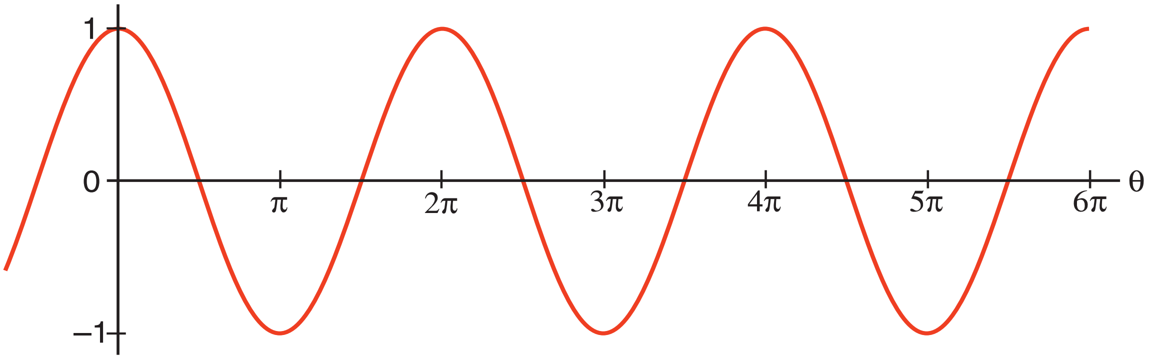


Figure 37
Several cycles of the curve $\cos(\theta)$.

Derivative of the Sine Function

Since the sine and cosine functions are smooth curves, we should be able to calculate the derivatives and integrals of them. We will do this by first calculating the derivative, and then turning the process around to find the integral, just as we did for the functions x^n and e^x.

The derivative of the function $\sin\theta$ is defined as usual by

$$\frac{d\left(\sin\theta\right)}{d\theta} = \operatorname*{limit}_{\Delta\theta\to0}\left[\frac{\sin\left(\theta+\Delta\theta\right)-\sin\theta}{\Delta\theta}\right] \tag{177}$$

where $\Delta\theta$ is a small change in the angle θ.

The easiest way to evaluate this limit is to go back to the unit circle of Figure (25) and construct both $\sin\theta$ and $\sin\left(\theta+\Delta\theta\right)$ as shown in Figure (39). We see that $\sin\theta$ is the height of the triangle with an angle θ, while $\sin\left(\theta+\Delta\theta\right)$ is the height of the triangle whose center angle is $\left(\theta+\Delta\theta\right)$. What we have to do is calculate the difference in heights of these two triangles.

In Figure (40) we start by focusing our attention on the slender triangle abc with an angle $\Delta\theta$ at (a) and long sides of length 1 (since we have a unit circle). Since the angle $\Delta\theta$ is small, the short side of this triangle is essentially equal to the arc length along the circle from point (b) to point (c). And since we are using radian measure, this arc length is equal to the angle $\Delta\theta$.

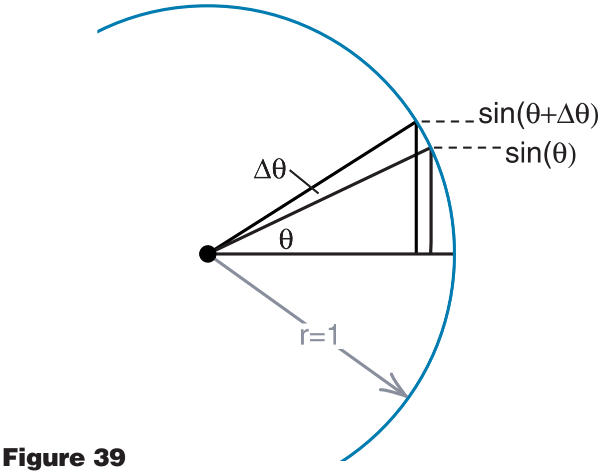


Figure 39
Triangles for the $\sin\boldsymbol{\theta}$ *and the* $\sin\left(\boldsymbol{\theta}+\boldsymbol{\Delta\theta}\right)$.

Now draw a line vertically down from point (c) and horizontally over from point (b) to form the triangle bcd shown in Figure (40). The important point is that the angle at point (c) in this tiny triangle is the same as the angle θ at point (a). To prove this, consider the sketch in Figure (41). A line bf is drawn tangent to the circle at point (b), so that the angle abf is a right angle. That means the other two angles in the triangle add up to 90°, the total angle in any triangle being 180°

$$\theta+\varphi = 90° \tag{178}$$

Since the angle at (e) in triangle bef is also a right angle, the other two angles in the triangle bef, must also add up to 90°.

$$\alpha+\varphi = 90° \tag{179}$$

For both Equations (178) and (179) to be true, we must have $\alpha=\theta$.

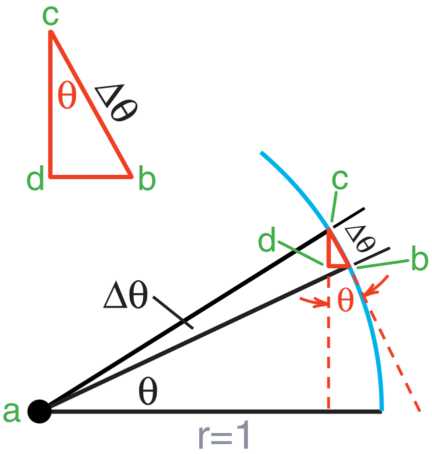


Figure 40
The difference between $\sin\boldsymbol{\theta}$ *and* $\sin\left(\boldsymbol{\theta}+\boldsymbol{\Delta\theta}\right)$ *is equal to the height of the side cd of the triangle cdb.*

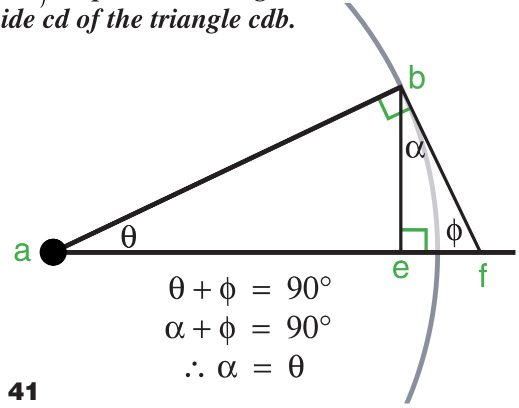


Figure 41
Demonstration that the angle $\boldsymbol{\alpha}$ *equals the angle* $\boldsymbol{\theta}$.

The final step is to note that when $\Delta\theta$ in Figure (40) is very small, the side cb of the very small triangle is essentially tangent to the circle, and thus parallel to the side bf in Figure (41). As a result the angle between cb and the vertical is also the same angle θ.

Because the tiny triangle, shown again in Figure (42) has a hypotenuse $\Delta\theta$ and a top angle θ, the vertical side, which is equal to the difference between $\sin\theta$ and $\sin(\theta+\Delta\theta)$ has a height $(\cos\theta)\Delta\theta$. Thus we have

$$\sin(\theta+\Delta\theta)-\sin\theta = (\cos\theta)\Delta\theta \qquad (180)$$

Equation (180) becomes exact when $\Delta\theta$ becomes an infinitesimal angle.

We can now evaluate the derivative

$$\frac{d(\sin\theta)}{d\theta} = \underset{\Delta\theta\to 0}{\text{limit}}\left[\frac{\sin(\theta+\Delta\theta)-\sin\theta}{\Delta\theta}\right]$$

$$= \underset{\Delta\theta\to 0}{\text{limit}}\left[\frac{(\cos\theta)\Delta\theta}{\Delta\theta}\right]$$

$$= \underset{\Delta\theta\to 0}{\text{limit}}\ \cos\theta$$

Thus we get the exact result

$$\boxed{\frac{d}{d\theta}(\sin\theta) = \cos\theta} \qquad (181)$$

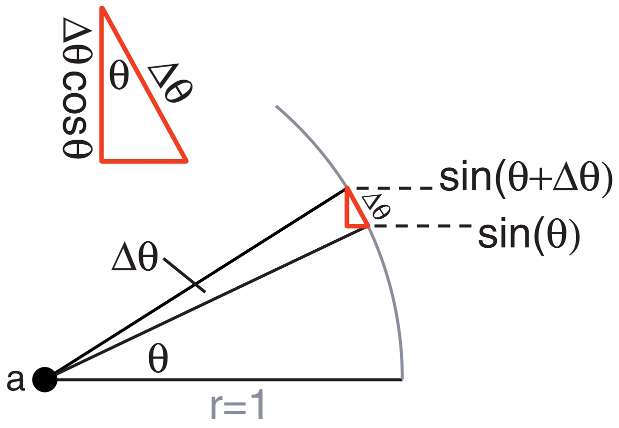


Figure 42
The difference between $\sin\theta$ and $\sin(\theta+\Delta\theta)$ is equal to $\Delta\theta\cos\theta$.

Exercise 11
Using a similar derivation, show that

$$\boxed{\frac{d}{d\theta}(\cos\theta) = -\sin\theta} \qquad (182)$$

Exercise 12
Using the chain rule for differentiation, show that

$$\boxed{\begin{aligned}\frac{d}{d\theta}(\sin a\theta) &= a\cos a\theta\\ \frac{d}{d\theta}(\cos a\theta) &= -a\sin a\theta\end{aligned}} \quad (a = constant) \qquad (183)$$

(Hint—if you need to, look at Equation (145) through (150).

Exercise 13
Using the fact that integration reverses differentiation, as we did in integrating the function e^x (Equations (151) through (154), show that

$$\boxed{\begin{aligned}\int_{\theta_i}^{\theta_f}(\cos a\theta)d\theta &= \frac{1}{a}\sin a\theta\Big|_{\theta_i}^{\theta_f}\\ \int_{\theta_i}^{\theta_f}(\sin a\theta)d\theta &= -\frac{1}{a}\cos a\theta\Big|_{\theta_i}^{\theta_f}\end{aligned}} \quad \begin{aligned}&(184a)\\ &(a = constant)\\ &(184b)\end{aligned}$$

Use sketches of the integrals from $\theta_i = 0$ to $\theta_f = \pi/2$ to show that Equations (184a) and (184b) have the correct numerical sign. (Explicitly explain the minus sign in (184b).