Junior Problem Seminar

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Preface

From time to time I get to revise this problem seminar. Although my chances of addressing the type of students for which they were originally intended (middle-school, high-school) are now very remote, I have had very pleasant emails from people around the world finding this material useful.

I haven't compiled the solutions for the practice problems anywhere. This is a project that now, having more pressing things to do, I can't embark. But if you feel you can contribute to these notes, drop me a line, or even mail me your files!

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Chapter 1

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Essential Techniques

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Prove that $a \leq b$.

Solution: Assume contrariwise that $a > b$. Hence $\frac{a-b}{2} > 0$. Since the inequality $a < b + \varepsilon$ holds for every $\varepsilon > 0$ in particular it holds for $\varepsilon = \frac{a-b}{2}$. This implies that

$$a < b + \frac{a-b}{2} \quad \text{or} \quad a < b.$$

Thus starting with the assumption that $a > b$ we reach the incompatible conclusion that $a < b$. The original assumption must be wrong. We therefore conclude that $a \leq b$.

5 Example (Euclid) Shew that there are infinitely many prime numbers.

Solution: We need to assume for this proof that any integer greater than 1 is either a prime or a product of primes. The following beautiful proof goes back to Euclid. Assume that $\{p_1, p_2, \ldots, p_n\}$ is a list that exhausts all the primes. Consider the number

$$N = p_1 p_2 \cdots p_n + 1.$$

This is a positive integer, clearly greater than 1. Observe that none of the primes on the list $\{p_1, p_2, \ldots, p_n\}$ divides N, since division by any of these primes leaves a remainder of 1. Since N is larger than any of the primes on this list, it is either a prime or divisible by a prime outside this list. Thus we have shewn that the assumption that any finite list of primes leads to the existence of a prime outside this list. This implies that the number of primes is infinite.

6 Example Let $n > 1$ be a composite integer. Prove that n has a prime factor $p \leq \sqrt{n}$.

Solution: Since n is composite, n can be written as $n = ab$ where both $a > 1, b > 1$ are integers. Now, if both $a > \sqrt{n}$ and $b > \sqrt{n}$ then $n = ab > \sqrt{n}\sqrt{n} = n$, a contradiction. Thus one of these factors must be $\leq \sqrt{n}$ and *a fortiori* it must have a prime factor $\leq \sqrt{n}$.

The result in example 6 can be used to test for primality. For example, to shew that 101 is prime, we compute $\lfloor\!\lfloor \sqrt{101} \rfloor\!\rfloor = 10$. By the preceding problem, either 101 is prime or it is divisible by $2, 3, 5$, or 7 (the primes smaller than 10). Since neither of these primes divides 101, we conclude that 101 is prime.

7 Example Prove that a sum of two squares of integers leaves remainder 0, 1 or 2 when divided by 4.

Solution: An integer is either even (of the form $2k$) or odd (of the form $2k+1$). We have

$$\begin{aligned} (2k)^2 &= 4(k^2), \\ (2k+1)^2 &= 4(k^2+k)+1. \end{aligned}$$

Thus squares leave remainder 0 or 1 when divided by 4 and hence their sum leave remainder 0, 1, or 2.

8 Example Prove that 2003 is not the sum of two squares by proving that the sum of any two squares cannot leave remainder 3 upon division by 4.

Solution: 2003 leaves remainder 3 upon division by 4. But we know from example 7 that sums of squares do not leave remainder 3 upon division by 4, so it is impossible to write 2003 as the sum of squares.

9 Example If a, b, c are odd integers, prove that $ax^2 + bx + c = 0$ does not have a rational number solution.

Solution: Suppose $\frac{p}{q}$ is a rational solution to the equation. We may assume that p and q have no prime factors in common, so either p and q are both odd, or one is odd and the other even. Now

$$a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0 \implies ap^2 + bpq + cq^2 = 0.$$

If both p and p were odd, then $ap^2 + bpq + cq^2$ is also odd and hence $\neq 0$. Similarly if one of them is even and the other odd then either $ap^2 + bpq$ or $bpq + cq^2$ is even and $ap^2 + bpq + cq^2$ is odd. This contradiction proves that the equation cannot have a rational root.

Practice

10 Problem The product of 34 integers is equal to 1. Shew that their sum cannot be 0.

11 Problem Let $a_1, a_2, \ldots, a_{2000}$ be natural numbers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2000}} = 1.$$

Prove that at least one of the a_k's is even.

(Hint: Clear the denominators.)

12 Problem Prove that $\log_2 3$ is irrational.

13 Problem A *palindrome* is an integer whose decimal expansion is symmetric, e.g. $1, 2, 11, 121,\ 15677651$ (but not $010, 0110$) are palindromes. Prove that there is no positive palindrome which is divisible by 10.

14 Problem In $\triangle ABC$, $\angle A > \angle B$. Prove that $BC > AC$.

15 Problem Let $0 < \alpha < 1$. Prove that $\sqrt{\alpha} > \alpha$.

16 Problem Let $\alpha = 0.999\ldots$ where there are at least 2000 nines. Prove that the decimal expansion of $\sqrt{\alpha}$ also starts with at least 2000 nines.

17 Problem Prove that a quadratic equation

$$ax^2 + bx + c = 0,\ a \neq 0$$

has at most two solutions.

18 Problem Prove that if $ax^2 + bx + c = 0$ has real solutions and if $a > 0, b > 0, c > 0$ then both solutions must be negative.

1.2 Pigeonhole Principle

The Pigeonhole Principle states that if $n+1$ pigeons fly to n holes, there must be a pigeonhole containing at least two pigeons. This apparently trivial principle is very powerful. Thus in any group of 13 people, there are always two who have their birthday on the same month, and if the average human head has two million hairs, there are at least three people in NYC with the same number of hairs on their head.

The Pigeonhole Principle is useful in proving *existence* problems, that is, we shew that something exists without actually identifying it concretely.

Let us see some more examples.

19 Example (Putnam 1978) Let A be any set of twenty integers chosen from the arithmetic progression $1, 4, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104.

Solution: We partition the thirty four elements of this progression into nineteen groups

$$\{1\}, \{52\}, \{4, 100\}, \{7, 97\}, \{10, 94\}, \ldots, \{49, 55\}.$$

Since we are choosing twenty integers and we have nineteen sets, by the Pigeonhole Principle there must be two integers that belong to one of the pairs, which add to 104.

20 Example Shew that amongst any seven distinct positive integers not exceeding 126, one can find two of them, say a and b, which satisfy

$$b < a \leq 2b.$$

Solution: Split the numbers $\{1,2,3,\ldots,126\}$ into the six sets

$$\{1,2\},\{3,4,5,6\},\{7,8,\ldots,13,14\},\{15,16,\ldots,29,30\},$$

$$\{31,32,\ldots,61,62\} \text{ and } \{63,64,\ldots,126\}.$$

By the Pigeonhole Principle, two of the seven numbers must lie in one of the six sets, and obviously, any such two will satisfy the stated inequality.

21 Example No matter which fifty five integers may be selected from

$$\{1,2,\ldots,100\},$$

prove that one must select some two that differ by 10.

Solution: First observe that if we choose $n+1$ integers from any string of $2n$ consecutive integers, there will always be some two that differ by n. This is because we can pair the $2n$ consecutive integers

$$\{a+1,a+2,a+3,\ldots,a+2n\}$$

into the n pairs

$$\{a+1,a+n+1\},\{a+2,a+n+2\},\ldots,\{a+n,a+2n\},$$

and if $n+1$ integers are chosen from this, there must be two that belong to the same group.

So now group the one hundred integers as follows:

$$\{1,2,\ldots 20\},\{21,22,\ldots,40\},$$

$$\{41,42,\ldots,60\},\ \{61,62,\ldots,80\}$$

and

$$\{81,82,\ldots,100\}.$$

If we select fifty five integers, we must perforce choose eleven from some group. From that group, by the above observation (let $n=10$), there must be two that differ by 10.

22 Example (AHSME 1994) Label one disc "**1**", two discs "**2**", three discs "**3**", ..., fifty discs "**50**". Put these $1+2+3+\cdots+50=1275$ labeled discs in a box. Discs are then drawn from the box at random without replacement. What is the minimum number of discs that must me drawn in order to guarantee drawing at least ten discs with the same label?

Solution: If we draw all the $1+2+\cdots+9=45$ labelled "**1**", ..., "**9**" and any nine from each of the discs "**10**", ..., "**50**", we have drawn $45+9\cdot 41=414$ discs. The 415-th disc drawn will assure at least ten discs from a label.

23 Example (IMO 1964) Seventeen people correspond by mail with one another—each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there at least three people who write to each other about the same topic.

Solution: Choose a particular person of the group, say Charlie. He corresponds with sixteen others. By the Pigeonhole Principle, Charlie must write to at least six of the people of one topic, say topic I. If any pair of these six people corresponds on topic I, then Charlie and this pair do the trick, and we are done. Otherwise, these six correspond amongst themselves only on topics II or III. Choose a particular person from this group of six, say Eric. By the Pigeonhole Principle, there must be three of the five remaining that correspond with Eric in one of the topics, say topic II. If amongst these three there is a pair that corresponds with each other on topic II, then Eric and this pair correspond on topic II, and we are done. Otherwise, these three people only correspond with one another on topic III, and we are done again.

24 Example Given any set of ten natural numbers between 1 and 99 inclusive, prove that there are two disjoint nonempty subsets of the set with equal sums of their elements.

Solution: There are $2^{10}-1=1023$ non-empty subsets that one can form with a given 10-element set. To each of these subsets we associate the sum of its elements. The maximum value that any such sum can achieve is $90+91+\cdots+99=945<1023$. Therefore, there must be at least two different subsets S,T that have the same element sum. Then $S\setminus(S\cap T)$ and $T\setminus(S\cap T)$ also have the same element sum.

25 Example Given any 9 integers whose prime factors lie in the set $\{3,7,11\}$ prove that there must be two whose product is a square.

Solution: For an integer to be a square, all the exponents of its prime factorisation must be even. Any integer in the given set has a prime factorisation of the form $3^a7^b11^c$. Now each triplet (a,b,c) has one of the following 8 parity patterns: (even, even, even), (even, even, odd), (even, odd, even), (even, odd, odd), (odd, even, even), (odd, even, odd), (odd, odd, even), (odd, odd, odd). In a group of 9 such integers, there must be two with the same parity patterns in the exponents. Take these two. Their product is a square, since the sum of each corresponding exponent will be even.

Practice

26 Problem Prove that among $n+1$ integers, there are always two whose difference is always divisible by n.

27 Problem (AHSME 1991) A circular table has exactly sixty chairs around it. There are N people seated at this table in such a way that the next person to be seated must sit next to someone. What is the smallest possible value of N?

28 Problem Shew that if any five points are all in, or on, a square of side 1, then some pair of them will be at most at distance $\sqrt{2}/2$.

29 Problem (Hungarian Math Olympiad, 1947) Prove that amongst six people in a room there are at least three who know one another, or at least three who do not know one another.

30 Problem Shew that in any sum of nonnegative real numbers there is always one number which is at least the average of the numbers and that there is always one member that it is at most the average of the numbers.

31 Problem We call a set "sum free" if no two elements of the set add up to a third element of the set. What is the maximum size of a sum free subset of $\{1,2,\ldots,2n-1\}$.

Hint: Observe that the set $\{n+1,n+2,\ldots,2n-1\}$ of $n+1$ elements is sum free. Shew that any subset with $n+2$ elements is not sum free.

32 Problem (MMPC 1992) Suppose that the letters of the English alphabet are listed in an arbitrary order.

1. Prove that there must be four consecutive consonants.
2. Give a list to shew that there need not be five consecutive consonants.
3. Suppose that all the letters are arranged in a circle. Prove that there must be five consecutive consonants.

33 Problem (Stanford 1953) Bob has ten pockets and forty four silver dollars. He wants to put his dollars into his pockets so distributed that each pocket contains a different number of dollars.

1. Can he do so?
2. Generalise the problem, considering p pockets and n dollars. The problem is most interesting when
$$n=\frac{(p-1)(p-2)}{2}.$$
Why?

34 Problem Let M be a seventeen-digit positive integer and let N be the number obtained from M by writing the same digits in reversed order. Prove that at least one digit in the decimal representation of the number $M+N$ is even.

35 Problem No matter which fifty five integers may be selected from
$$\{1,2,\ldots,100\},$$
prove that you must select some two that differ by 9, some two that differ by 10, some two that differ by 12, and some two that differ by 13, but that you need not have any two that differ by 11.

36 Problem Let $mn+1$ different real numbers be given. Prove that there is either an increasing sequence with at least $n+1$ members, or a decreasing sequence with at least $m+1$ members.

37 Problem If the points of the plane are coloured with three colours, shew that there will always exist two points of the same colour which are one unit apart.

38 Problem Shew that if the points of the plane are coloured with two colours, there will always exist an equilateral triangle with all its vertices of the same colour. There is, however, a colouring of the points of the plane with two colours for which no equilateral triangle of side 1 has all its vertices of the same colour.

39 Problem (USAMO 1979) Nine mathematicians meet at an international conference and discover that amongst any three of them, at least two speak a common language. If each of the mathematicians can speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

40 Problem (USAMO 1982) In a party with 1982 persons, amongst any group of four there is at least one person who knows each of the other three. What is the minimum number of people in the party who know everyone else?

41 Problem (USAMO 1985) There are n people at a party. Prove that there are two people such that, of the remaining $n-2$ people, there are at least $\lfloor\lfloor n/2\rfloor\rfloor-1$ of them, each of whom knows both or else knows neither of the two. Assume that "knowing" is a symmetrical relationship.

42 Problem (USAMO 1986) During a certain lecture, each of five mathematicians fell asleep exactly twice. For each pair of these mathematicians, there was some moment when both were sleeping simultaneously. Prove that, at some moment, some three were sleeping simultaneously.

1.3 Parity

43 Example Two diametrically opposite corners of a chess board are deleted. Shew that it is impossible to tile the remaining 62 squares with 31 dominoes.

Solution: Each domino covers one red square and one black squares. But diametrically opposite corners are of the same colour, hence this tiling is impossible.

44 Example All the dominoes in a set are laid out in a chain according to the rules of the game. If one end of the chain is a 6, what is at the other end?

Solution: At the other end there must be a 6 also. Each number of spots must occur in a pair, so that we may put them end to end. Since there are eight 6's, this last 6 pairs off with the one at the beginning of the chain.

45 Example The numbers $1,2,\ldots,10$ are written in a row. Shew that no matter what choice of sign $\pm$ is put in between them, the sum will never be 0.

Solution: The sum $1+2+\cdots+10=55$, an odd integer. Since parity is not affected by the choice of sign, for any choice of sign $\pm 1\pm 2\pm\cdots\pm 10$ will never be even, in particular it will never be 0.

46 Definition A *lattice point* (m,n) on the plane is one having integer coordinates.

47 Definition The midpoint of the line joining (x,y) to (x_1,y_1) is the point

$$\left(\frac{x+x_1}{2},\frac{y+y_1}{2}\right).$$

48 Example Five lattice points are chosen at random. Prove that one can always find two so that the midpoint of the line joining them is also a lattice point.

Solution: There are four parity patterns: (even, even), (even, odd), (odd, odd), (odd, even). By the Pigeonhole Principle among five lattice points there must be two having the same parity pattern. Choose these two. It is clear that their midpoint is an integer.

For the next few examples we will need to know the names of the following tetrominoes.

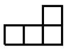

Figure 1.1: L-tetromino

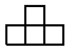

Figure 1.2: T-tetromino

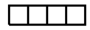

Figure 1.3: Straight-tetromino

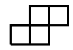

Figure 1.4: Skew-tetromino

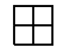

Figure 1.5: Square-tetromino

49 Example A single copy of each of the tetrominoes shewn above is taken. Shew that no matter how these are arranged, it is impossible to construct a rectangle.

Solution: If such a rectangle were possible, it would have 20 squares. Colour the rectangle like a chessboard. Then there are 10 red squares and 10 black squares. The T-tetromino always covers an odd number of red squares. The other tetrominoes always cover an even number of red squares. This means that the number of red squares covered is odd, a contradiction.

50 Example Shew that an 8×8 chessboard cannot be tiles with 15 straight tetrominoes and one L-tetromino.

Solution: Colour rows $1,3,5,7$ black and colour rows $2,4,6$, and 8 red. A straight tetromino will always cover an even number of red boxes and the L-tetromino will always cover an odd number of red squares. If the tiling were possible, then we would be covering an odd number of red squares, a contradiction.

Practice

51 Problem Twenty-five boys and girls are seated at a round table. Shew that both neighbours of at least one student are girls.

52 Problem A closed path is made of 2001 line segments. Prove that there is no line, not passing through a vertex of the path, intersecting each of the segments of the path.

53 Problem The numbers $1,2,\ldots,2001$ are written on a blackboard. One starts erasing any two of them and replacing the deleted ones with their difference. Will a situation arise where all the numbers on the blackboard be 0?

54 Problem Shew that a 10×10 chessboard cannot be tiled with 25 straight tetrominoes.

55 Problem Shew that an 8×8 chess board cannot be tiled with 15 T-tetrominoes and one square tetromino.

Chapter 2

Algebra

2.1 Identities with Squares

Recall that

$$(x+y)^2 = (x+y)(x+y) = x^2+y^2+2xy \tag{2.1}$$

If we substitute y by $y+z$ we obtain

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2xz+2yz \tag{2.2}$$

If we substitute z by $z+w$ we obtain

$$(x+y+z+w)^2 = x^2+y^2+z^2+w^2+2xy+2xz+2xw+2yz+2yw+2zw \tag{2.3}$$

56 Example The sum of two numbers is 21 and their product -7. Find (i) the sum of their squares, (ii) the sum of their reciprocals and (iii) the sum of their fourth powers.

Solution: If the two numbers are a and b, we are given that $a+b=21$ and $ab=-7$. Hence

$$a^2+b^2 = (a+b)^2-2ab = 21^2-2(-7) = 455$$

and

$$\frac{1}{a}+\frac{1}{b} = \frac{b+a}{ab} = \frac{21}{-7} = -3$$

Also

$$a^4+b^4 = (a^2+b^2)^2-2a^2b^2 = 455^2-2(-7)^2 = 357$$

57 Example Find positive integers a and b with

$$\sqrt{5+\sqrt{24}} = \sqrt{a}+\sqrt{b}.$$

Solution: Observe that

$$5+\sqrt{24} = 3+2\sqrt{2\cdot 3}+2 = (\sqrt{2}+\sqrt{3})^2.$$

Therefore

$$\sqrt{5+2\sqrt{6}} = \sqrt{2}+\sqrt{3}.$$

58 Example Compute

$$\sqrt{(1000000)(1000001)(1000002)(1000003)+1}$$

without using a calculator.

Solution: Let $x = 1\,000\,000 = 10^6$. Then

$$x(x+1)(x+2)(x+3) = x(x+3)(x+1)(x+2) = (x^2+3x)(x^2+3x+2).$$

Put $y = x^2+3x$. Then

$$x(x+1)(x+2)(x+3)+1 = (x^2+3x)(x^2+3x+2)+1 = y(y+2)+1 = (y+1)^2.$$

Thus

$$\begin{aligned}\sqrt{x(x+1)(x+2)(x+3)+1} &= y+1\\ &= x^2+3x+1\\ &= 10^{12}+3\cdot 10^6+1\\ &= 1\,000\,003\,000\,001.\end{aligned}$$

Another useful identity is the difference of squares:

$$x^2-y^2 = (x-y)(x+y) \tag{2.4}$$

59 Example Explain how to compute $123456789^2 - 123456790 \times 123456788$ mentally.

Solution: Put $x = 123456789$. Then

$$123456789^2 - 123456790 \times 123456788 = x^2-(x+1)(x-1) = 1.$$

60 Example Shew that

$$1+x+x^2+\cdots+x^{1023} = (1+x)(1+x^2)(1+x^4)\cdots(1+x^{256})(1+x^{512}).$$

Solution: Put $S = 1+x+x^2+\cdots+x^{1023}$. Then $xS = x+x^2+\cdots+x^{1024}$. This gives

$$S-xS = (1+x+x^2+\cdots+x^{1023})-(x+x^2+\cdots+x^{1024}) = 1-x^{1024}$$

or $S(1-x) = 1-x^{1024}$, from where

$$1+x+x^2+\cdots+x^{1023} = S = \frac{1-x^{1024}}{1-x}.$$

But

$$\begin{aligned}\frac{1-x^{1024}}{1-x} &= \left(\frac{1-x^{1024}}{1-x^{512}}\right)\left(\frac{1-x^{512}}{1-x^{256}}\right)\cdots\left(\frac{1-x^4}{1-x^2}\right)\left(\frac{1-x^2}{1-x}\right)\\ &= (1+x^{512})(1+x^{256})\cdots(1+x^2)(1+x),\end{aligned}$$

proving the assertion.

61 Example Given that

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}$$

is an integer, find it.

Solution: As $1 = n+1-n = (\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})$, we have

$$\frac{1}{\sqrt{n}+\sqrt{n+1}} = \sqrt{n+1}-\sqrt{n}.$$

Therefore

$$\begin{array}{rcl}\frac{1}{\sqrt{1}+\sqrt{2}} & = & \sqrt{2}-\sqrt{1}\\ \frac{1}{\sqrt{2}+\sqrt{3}} & = & \sqrt{3}-\sqrt{2}\\ \frac{1}{\sqrt{3}+\sqrt{4}} & = & \sqrt{4}-\sqrt{3}\\ \vdots & \vdots & \vdots\\ \frac{1}{\sqrt{99}+\sqrt{100}} & = & \sqrt{100}-\sqrt{99},\end{array}$$

and thus

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}=\sqrt{100}-\sqrt{1}=9.$$

Using the difference of squares identity,

$$\begin{array}{rcl}x^4+x^2y^2+y^4 & = & x^4+2x^2y^2+y^4-x^2y^2\\ & = & (x^2+y^2)^2-(xy)^2\\ & = & (x^2-xy+y^2)(x^2+xy+y^2).\end{array}$$

The following factorisation is credited to Sophie Germain.

$$\begin{array}{rcl}a^4+4b^4 & = & a^4+4a^2b^2+4b^4-4a^2b^2\\ & = & (a^2+2b^2)^2-(2ab)^2\\ & = & (a^2-2ab+2b^2)(a^2+2ab+2b^2)\end{array}$$

62 Example Prove that n^4+4 is a prime only when $n=1$ for $n\in\mathbb{N}$.

Solution: Using Sophie Germain's trick,

$$\begin{array}{rcl}n^4+4 & = & n^4+4n^2+4-4n^2\\ & = & (n^2+2)^2-(2n)^2\\ & = & (n^2+2-2n)(n^2+2+2n)\\ & = & ((n-1)^2+1)((n+1)^2+1).\end{array}$$

Each factor is greater than 1 for $n>1$, and so n^4+4 cannot be a prime if $n>1$.

63 Example Shew that the product of four consecutive integers, none of them 0, is never a perfect square.

Solution: Let $n-1, n, n+1, n+2$ be four consecutive integers. Then their product P is

$$P=(n-1)n(n+1)(n+2)=(n^3-n)(n+2)=n^4+2n^3-n^2-2n.$$

But

$$(n^2+n-1)^2=n^4+2n^3-n^2-2n+1=P+1>P.$$

As $P\neq 0$ and P is 1 more than a square, P cannot be a square.

64 Example Find infinitely many pairs of integers (m,n) such that m and n share their prime factors and $(m-1,n-1)$ share their prime factors.

Solution: Take $m = 2^k - 1, n = (2^k - 1)^2, k = 2, 3, \ldots$. Then m, n obviously share their prime factors and $m - 1 = 2(2^{k-1} - 1)$ shares its prime factors with $n - 1 = 2^{k+1}(2^{k-1} - 1)$.

65 Example Prove that if $r \geq s \geq t$ then

$$r^2 - s^2 + t^2 \geq (r - s + t)^2 \tag{2.5}$$

Solution: We have

$$(r - s + t)^2 - t^2 = (r - s + t - t)(r - s + t + t) = (r - s)(r - s + 2t).$$

Since $t - s \leq 0$, $r - s + 2t = r + s + 2(t - s) \leq r + s$ and so

$$(r - s + t)^2 - t^2 \leq (r - s)(r + s) = r^2 - s^2$$

which gives

$$(r - s + t)^2 \leq r^2 - s^2 + t^2.$$

Practice

66 Problem The sum of two numbers is -7 and their product 2. Find (i) the sum of their reciprocals, (ii) the sum of their squares.

67 Problem Write x^2 as a sum of powers of $x + 3$.

68 Problem Write $x^2 - 3x + 8$ as a sum of powers of $x - 1$.

69 Problem Prove that 3 is the only prime of the form $n^2 - 1$.

70 Problem Prove that there are no primes of the form $n^4 - 1$.

71 Problem Prove that $n^4 + 4^n$ is prime only for $n = 1$.

72 Problem Use Sophie Germain's trick to obtain

$$x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1),$$

and then find all the primes of the form $n^4 + n^2 + 1$.

73 Problem If a, b satisfy $\dfrac{2}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$, find $\dfrac{a^2}{b^2}$.

74 Problem If $\cot x + \tan x = a$, prove that $\cot^2 x + \tan^2 x = a^2 - 2$. .

75 Problem Prove that if a, b, c are positive integers, then

$$\begin{aligned}&(\sqrt{a} + \sqrt{b} + \sqrt{c})(-\sqrt{a} + \sqrt{b} + \sqrt{c})\\ &\cdot(\sqrt{a} - \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c})\end{aligned}$$

is an integer.

76 Problem By direct computation, shew that the product of sums of two squares is itself a sum of two squares:

$$(a^2 + b^2)(c^2 + d^2) = (ab + bd)^2 + (ad - bc)^2 \tag{2.6}$$

77 Problem Divide $x^{128} - y^{128}$ by

$$\begin{aligned}&(x + y)(x^2 + y^2)(x^4 + y^4)(x^8 + y^8)\\ &(x^{16} + y^{16})(x^{32} + y^{32})(x^{64} + y^{64}).\end{aligned}$$

78 Problem Solve the system

$$\begin{aligned}x + y &= 9,\\ x^2 + xy + y^2 &= 61.\end{aligned}$$

79 Problem Solve the system

$$\begin{aligned}x - y &= 10,\\ x^2 - 4xy + y^2 &= 52.\end{aligned}$$

80 Problem Find the sum of the prime divisors of $2^{16} - 1$.

81 Problem Find integers a, b with

$$\sqrt{11 + \sqrt{72}} = a + \sqrt{b}.$$

82 Problem Given that the difference

$$\sqrt{57 - 40\sqrt{2}} - \sqrt{57 + 40\sqrt{2}}$$

is an integer, find it.

83 Problem Solve the equation

$$\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1.$$

84 Problem Prove that if $a > 0$, $b > 0, a + b > c$, then $\sqrt{a} + \sqrt{b} > \sqrt{c}$

85 Problem Prove that if $1 < x < 2$, then

$$\frac{1}{\sqrt{x + 2\sqrt{x - 1}}} + \frac{1}{\sqrt{x - 2\sqrt{x - 1}}} = \frac{2}{2 - x}.$$

86 Problem If $x > 0$, from

$$\sqrt{x + 1} - \sqrt{x} = \frac{1}{\sqrt{x + 1} + \sqrt{x}},$$

prove that

$$\frac{1}{2\sqrt{x + 1}} < \sqrt{x + 1} - \sqrt{x} < \frac{1}{2\sqrt{x}}.$$

Use this to prove that if $n > 1$ is a positive integer, then

$$2\sqrt{n + 1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n} - 1$$

87 Problem Shew that

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{1024}) = \frac{1-x^{2048}}{1-x}.$$

88 Problem Shew that

$$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right).$$

89 Problem Prove that if $r \geq s \geq t \geq u \geq v$ then

$$r^2-s^2+t^2-u^2+v^2 \geq (r-s+t-u+v)^2 \tag{2.7}$$

90 Problem (AIME 1987) Compute

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

91 Problem Write $(a^2+a+1)^2$ as the sum of three squares.

2.2 Squares of Real Numbers

If x is a real number then $x^2 \geq 0$. Thus if $a \geq 0, b \geq 0$ then $(\sqrt{a}-\sqrt{b})^2 \geq 0$ gives, upon expanding the square, $a-2\sqrt{ab}+b \geq 0$, or

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

Since $\frac{a+b}{2}$ is the arithmetic mean of a, b and $\sqrt{ab}$ is the geometric mean of a, b the inequality

$$\sqrt{ab} \leq \frac{a+b}{2} \tag{2.8}$$

is known as the *Arithmetic-Mean-Geometric Mean* (AM-GM) Inequality.

92 Example Let u_1, u_2, u_3, u_4 be non-negative real numbers. By applying the preceding result twice, establish the AM-GM Inequality for four quantities:

$$(u_1u_2u_3u_4)^{1/4} \leq \frac{u_1+u_2+u_3+u_4}{4} \tag{2.9}$$

Solution: We have $\sqrt{u_1u_2} \leq \frac{u_1+u_2}{2}$ and $\sqrt{u_3u_4} \leq \frac{u_3+u_4}{2}$. Now, applying the AM-GM Inequality twice to $\sqrt{u_1u_2}$ and $\sqrt{u_3u_4}$ we obtain

$$\sqrt{\sqrt{u_1u_2}\sqrt{u_3u_4}} \leq \frac{\sqrt{u_1u_2}+\sqrt{u_3u_4}}{2} \leq \frac{\frac{u_1+u_2}{2}+\frac{u_3+u_4}{2}}{2}.$$

Simplification yields the desired result.

93 Example Let u, v, w be non-negative real numbers. By using the preceding result on the four quantities u, v, w, and $\frac{u+v+w}{3}$, establish the AM-GM Inequality for three quantities:

$$(uvw)^{1/3} \leq \frac{u+v+w}{3} \tag{2.10}$$

Solution: By the AM-GM Inequality for four values

$$\left(uvw\left(\frac{u+v+w}{3}\right)\right)^{1/4} \leq \frac{u+v+w+\frac{u+v+w}{3}}{4}.$$

Some algebraic manipulation makes this equivalent to

$$(uvw)^{1/4}\left(\frac{u+v+w}{3}\right)^{1/4} \leq \frac{u+v+w}{4}+\frac{u+v+w}{12}$$

or upon adding the fraction on the right

$$(uvw)^{1/4}\left(\frac{u+v+w}{3}\right)^{1/4} \leq \frac{u+v+w}{3}.$$

Multiplying both sides by $\left(\frac{u+v+w}{3}\right)^{-1/4}$ we obtain

$$(uvw)^{1/4} \leq \left(\frac{u+v+w}{3}\right)^{3/4},$$

from where the desired inequality follows.

94 Example Let $a > 0, b > 0$. Prove the *Harmonic-Mean-Geometric-Mean* Inequality

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \leq \sqrt{ab} \tag{2.11}$$

Solution: By the AM-HM Inequality

$$\sqrt{\frac{1}{a}\cdot\frac{1}{b}} \leq \frac{\frac{1}{a}+\frac{1}{b}}{2},$$

from where the desired inequality follows.

95 Example Prove that if a, b, c are non-negative real numbers then

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution: The result quickly follows upon multiplying the three inequalities $a+b \geq 2\sqrt{ab}$, $b+c \geq 2\sqrt{bc}$ and $c+a \geq 2\sqrt{ca}$.

96 Example If a, b, c, d, are real numbers such that $a^2+b^2+c^2+d^2 = ab+bc+cd+da$, prove that $a=b=c=d$.

Solution: Transposing,

$$a^2-ab+b^2-bc+c^2-dc+d^2-da = 0,$$

or

$$\frac{a^2}{2}-ab+\frac{b^2}{2}+\frac{b^2}{2}-bc+\frac{c^2}{2}+\frac{c^2}{2}-dc+\frac{d^2}{2}+\frac{d^2}{2}-da+\frac{a^2}{2} = 0.$$

Factoring,

$$\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-d)^2+\frac{1}{2}(d-a)^2 = 0.$$

As the sum of non-negative quantities is zero only when the quantities themselves are zero, we obtain $a=b, b=c, c=d, d=a$, which proves the assertion.

We note in passing that from the identity

$$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \tag{2.12}$$

it follows that

$$a^2+b^2+c^2 \geq ab+bc+ca \tag{2.13}$$

97 Example The values of a, b, c, and d are $1, 2, 3$ and 4 but not necessarily in that order. What is the largest possible value of $ab+bc+cd+da$?

Solution:

$$\begin{aligned} ab+bc+cd+da &= (a+c)(b+d) \\ &\leq \left(\frac{a+c+b+d}{2}\right)^2 \\ &= \left(\frac{1+2+3+4}{2}\right)^2 \\ &= 25, \end{aligned}$$

by AM-GM. Equality occurs when $a+c=b+d$. Thus one may choose, for example, $a=1, c=4, b=2, d=3$.

Practice

98 Problem If $0 < a \leq b$, shew that

$$\frac{1}{8}\cdot\frac{(b-a)^2}{b} \leq \frac{a+b}{2} - \sqrt{ab} \leq \frac{1}{8}\cdot\frac{(b-a)^2}{a}$$

99 Problem Prove that if a, b, c are non-negative real numbers then

$$(a^2+1)(b^2+1)(c^2+1) \geq 8abc$$

100 Problem The sum of two positive numbers is 100. Find their maximum possible product.

101 Problem Prove that if a, b, c are positive numbers then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3.$$

102 Problem Prove that of all rectangles with a given perimeter, the square has the largest area.

103 Problem Prove that if $0 \leq x \leq 1$ then $x - x^2 \leq \frac{1}{4}$.

104 Problem Let $0 \leq a, b, c, d \leq 1$. Prove that at least one of the products

$$a(1-b),\ b(1-c),\quad c(1-d),\ d(1-a)$$

is $\leq \frac{1}{4}$.

105 Problem Use the AM-GM Inequality for four non-negative real numbers to prove a version of the AM-GM for eight non-negative real numbers.

2.3 Identities with Cubes

By direct computation we find that

$$(x+y)^3 = (x+y)(x^2+y^2+2xy) = x^3+y^3+3xy(x+y) \tag{2.14}$$

106 Example The sum of two numbers is 2 and their product 5. Find the sum of their cubes.

Solution: If the numbers are x, y then $x^3+y^3 = (x+y)^3 - 3xy(x+y) = 2^3 - 3(5)(2) = -22$.

Two other useful identities are the sum and difference of cubes,

$$x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2) \tag{2.15}$$

107 Example Find all the prime numbers of the form $n^3 - 1$, n a positive integer.

Solution: As $n^3 - 1 = (n-1)(n^2+n+1)$ and as $n^2+n+1 > 1$, it must be the case that $n-1 = 1$, i.e., $n = 2$. Therefore, the only prime of this form is $2^3 - 1 = 7$.

108 Example Prove that

$$1+x+x^2+\cdots+x^{80} = (x^{54}+x^{27}+1)(x^{18}+x^9+1)(x^6+x^3+1)(x^2+x+1).$$

Solution: Put $S = 1+x+x^2+\cdots+x^{80}$. Then

$$S - xS = (1+x+x^2+\cdots+x^{80}) - (x+x^2+x^3+\cdots+x^{80}+x^{81}) = 1-x^{81},$$

or $S(1-x) = 1-x^{81}$. Hence

$$1+x+x^2+\cdots+x^{80} = \frac{x^{81}-1}{x-1}.$$

Therefore

$$\frac{x^{81}-1}{x-1} = \frac{x^{81}-1}{x^{27}-1}\cdot\frac{x^{27}-1}{x^9-1}\cdot\frac{x^9-1}{x^3-1}\cdot\frac{x^3-1}{x-1}.$$

Thus

$$1+x+x^2+\cdots+x^{80} = (x^{54}+x^{27}+1)(x^{18}+x^9+1)(x^6+x^3+1)(x^2+x+1).$$

109 Example Shew that

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \tag{2.16}$$

Solution: We use the identity

$$x^3+y^3=(x+y)^3-3xy(x+y)$$

twice. Then

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b)^3+c^3-3ab(a+b)-3abc \\ &= (a+b+c)^3-3(a+b)c(a+b+c)-3ab(a+b+c) \\ &= (a+b+c)((a+b+c)^2-3ac-3bc-3ab) \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \end{aligned}$$

If a,b,c are non-negative then $a+b+c\geq 0$ and also $a^2+b^2+c^2-ab-bc-ca\geq 0$ by (2.13). This gives

$$\frac{a^3+b^3+c^3}{3}\geq abc.$$

Letting $a^3=x, b^3=y, c^3=z$, for non-negative real numbers x,y,z, we obtain the AM-GM Inequality for three quantities.

Practice

110 Problem If $a^3-b^3=24, a-b=2$, find $(a+b)^2$.

111 Problem Shew that for integer $n\geq 2$, the expression

$$\frac{n^3+(n+2)^3}{4}$$

is a composite integer.

112 Problem If $\tan x+\cot x=a$, prove that $\tan^3 x+\cot^3 x=a^3-3a$.

113 Problem (AIME 1986) What is the largest positive integer n for which

$$(n+10)|(n^3+100)?$$

114 Problem Find all the primes of the form n^3+1.

115 Problem Solve the system

$$x^3+y^3=126,$$

$$x^2-xy+y^2=21.$$

116 Problem Evaluate the sum

$$\frac{1}{\sqrt[3]{1}+\sqrt[3]{2}+\sqrt[3]{4}}+\frac{1}{\sqrt[3]{4}+\sqrt[3]{6}+\sqrt[3]{9}}+\frac{1}{\sqrt[3]{9}+\sqrt[3]{12}+\sqrt[3]{16}}.$$

117 Problem Find a^6+a^{-6} given that $a^2+a^{-2}=4$.

118 Problem Prove that

$$(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a) \tag{2.17}$$

119 Problem (ITT 1994) Let a,b,c,d be complex numbers satisfying

$$a+b+c+d=a^3+b^3+c^3+d^3=0.$$

Prove that a pair of the a,b,c,d must add up to 0.

2.4 Miscellaneous Algebraic Identities

We have seen the identity

$$y^2-x^2=(y-x)(y+x). \tag{2.18}$$

We would like to deduce a general identity for y^n-x^n, where n is a positive integer. A few multiplications confirm that

$$y^3-x^3=(y-x)(y^2+yx+x^2), \tag{2.19}$$

$$y^4-x^4=(y-x)(y^3+y^2x+yx^2+x^3), \tag{2.20}$$

and

$$y^5-x^5=(y-x)(y^4+y^3x+y^2x^2+yx^3+x^4). \tag{2.21}$$

The general result is in fact the following theorem.

120 Theorem If n is a positive integer, then

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}).$$

Proof: *We first prove that for $a \neq 1$.*

$$1 + a + a^2 + \cdots a^{n-1} = \frac{1-a^n}{1-a}.$$

For, put $S = 1 + a + a^2 + \cdots + a^{n-1}$. Then $aS = a + a^2 + \cdots + a^{n-1} + a^n$. Thus $S - aS = (1 + a + a^2 + \cdots + a^{n-1}) - (a + a^2 + \cdots + a^{n-1} + a^n) = 1 - a^n$, and from $(1-a)S = S - aS = 1 - a^n$ we obtain the result. By making the substitution $a = \frac{x}{y}$ we see that

$$1 + \frac{x}{y} + \left(\frac{x}{y}\right)^2 + \cdots + \left(\frac{x}{y}\right)^{n-1} = \frac{1 - \left(\frac{x}{y}\right)^n}{1 - \frac{x}{y}}$$

we obtain

$$\left(1 - \frac{x}{y}\right)\left(1 + \frac{x}{y} + \left(\frac{x}{y}\right)^2 + \cdots + \left(\frac{x}{y}\right)^{n-1}\right) = 1 - \left(\frac{x}{y}\right)^n,$$

or equivalently,

$$\left(1 - \frac{x}{y}\right)\left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \cdots + \frac{x^{n-1}}{y^{n-1}}\right) = 1 - \frac{x^n}{y^n}.$$

Multiplying by y^n both sides,

$$y\left(1 - \frac{x}{y}\right)y^{n-1}\left(1 + \frac{x}{y} + \frac{x^2}{y^2} + \cdots + \frac{x^{n-1}}{y^{n-1}}\right) = y^n\left(1 - \frac{x^n}{y^n}\right),$$

which is

$$y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \cdots + yx^{n-2} + x^{n-1}),$$

yielding the result. ❑

☞ *The second factor has n terms and each term has degree (weight)* $n-1$.

As an easy corollary we deduce

121 Corollary If x, y are integers $x \neq y$ and n is a positive integer then $x - y$ divides $x^n - y^n$.

Thus without any painful calculation we see that $781 = 1996 - 1215$ divides $1996^5 - 1215^5$.

122 Example (Eőtvős 1899) Shew that for any positive integer n, the expression

$$2903^n - 803^n - 464^n + 261^n$$

is always divisible by 1897.

Solution: By the theorem above, $2903^n - 803^n$ is divisible by $2903 - 803 = 2100 = 7 \cdot 300$ and $261^n - 464^n$ is divisible by $-203 = (-29) \cdot 7$. This means that the given expression is divisible by 7. Furthermore, $2903^n - 464^n$ is divisible by $2903 - 464 = 2439 = 9 \cdot 271$ and $-803^n + 261^n$ is divisible by $-803 + 261 = -542 = -2 \cdot 271$. Therefore as the given expression is divisible by 7 and by 271 and as these two numbers have no common factors, we have that $2903^n - 803^n - 464^n + 261^n$ is divisible by $7 \cdot 271 = 1897$.

123 Example ($(UM)^2C^4$ 1987) Given that 1002004008016032 has a prime factor $p > 250000$, find it.

Solution: If $a = 10^3, b = 2$ then

$$1002004008016032 = a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5 = \frac{a^6 - b^6}{a - b}.$$

This last expression factorises as

$$\begin{aligned} \frac{a^6 - b^6}{a - b} &= (a+b)(a^2+ab+b^2)(a^2-ab+b^2) \\ &= 1002 \cdot 1002004 \cdot 998004 \\ &= 4 \cdot 4 \cdot 1002 \cdot 250501 \cdot k, \end{aligned}$$

where $k < 250000$. Therefore $p = 250501$.

Another useful corollary of Theorem 120 is the following.

124 Corollary If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is a polynomial with integral coefficients and if a, b are integers then $b - a$ divides $f(b) - f(a)$.

125 Example Prove that there is no polynomial p with integral coefficients with $p(2) = 3$ and $p(7) = 17$.

Solution: If the assertion were true then by the preceding corollary, $7 - 2 = 5$ would divide $p(7) - p(2) = 17 - 3 = 14$, which is patently false.

Theorem 120 also yields the following colloraries.

126 Corollary If n is an odd positive integer

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots + x^2y^{n-3} - xy^{n-2} + y^{n-1}) \tag{2.22}$$

127 Corollary Let x, y be integers, $x \neq y$ and let n be an odd positive number. Then $x + y$ divides $x^n + y^n$.

For example $129 = 2^7 + 1$ divides $2^{861} + 1$ and $1001 = 1000 + 1 = 999 + 2 = \cdots = 500 + 501$ divides

$$1^{1997} + 2^{1997} + \cdots + 1000^{1997}.$$

128 Example Prove the following identity of Catalan:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Solution: The quantity on the sinistral side is

$$\begin{aligned} &\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad -2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad -2 \cdot \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n-1} + \frac{1}{2n}\right) \\ &\quad -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right) \\ &= \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}, \end{aligned}$$

as we wanted to shew.

Practice

129 Problem Shew that 100 divides $11^{10}-1$.

130 Problem Shew that $27195^8-10887^8+10152^8$ is divisible by 26460.

131 Problem Shew that 7 divides

$$2222^{5555}+5555^{2222}.$$

132 Problem Shew that if k is an odd positive integer

$$1^k+2^k+\cdots+n^k$$

is divisible by

$$1+2+\cdots+n.$$

133 Problem Shew that

$$(x+y)^5-x^5-y^5=5xy(x+y)(x^2+xy+y^2).$$

134 Problem Shew that

$$(x+a)^7-x^7-a^7=7xa(x+a)(x^2+xa+a^2)^2.$$

135 Problem Shew that

$$A=x^{9999}+x^{8888}+x^{7777}+\cdots+x^{1111}+1$$

is divisible by $B=x^9+x^8+x^7+\cdots+x^2+x+1$.

136 Problem Shew that for any natural number n, there is another natural number x such that each term of the sequence

$$x+1, x^x+1, x^{x^x}+1, \ldots$$

is divisible by n.

137 Problem Shew that $1492^n-1770^n-1863^n+2141^n$ is divisible by 1946 for all positive integers n.

138 Problem Decompose $1+x+x^2+x^3+\cdots+x^{624}$ into factors.

139 Problem Shew that if 2^n-1 is prime, then n must be prime. Primes of this form are called *Mersenne* primes.

140 Problem Shew that if 2^n+1 is a prime, then n must be a power of 2. Primes of this form are called Fermat primes.

141 Problem Let n be a positive integer and $x>y$. Prove that

$$\frac{x^n-y^n}{x-y}>ny^{n-1}.$$

By choosing suitable values of x and y, further prove than

$$\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n+1}\right)^{n+1}$$

and

$$\left(1+\frac{1}{n}\right)^{n+1}>\left(1+\frac{1}{n+1}\right)^{n+2}$$

2.5 Logarithms

142 Definition Let $a>0, a\neq 1$ be a real number. A number x is called the *logarithm* of a number N to the base a if $a^x=N$. In this case we write $x=\log_a N$.

We enumerate some useful properties of logarithms. We assume that $a>0, a\neq 1, M>0, N>0$.

$$a^{\log_a N}=N \tag{2.23}$$

$$\log_a MN=\log_a M+\log_a N \tag{2.24}$$

$$\log_a \frac{M}{N}=\log_a M-\log_a N \tag{2.25}$$

$$\log_a N^\alpha=\alpha\log_a N,\ \ \alpha \text{ any real number} \tag{2.26}$$

$$\log_{a^\beta} N=\frac{1}{\beta}\log_a N,\ \ \beta\neq 0 \text{ a real number} \tag{2.27}$$

$$(\log_a b)(\log_b a)=1,\ b>0, b\neq 1. \tag{2.28}$$

143 Example Given that $\log_{8\sqrt{2}} 1024$ is a rational number, find it.

Solution: We have

$$\log_{8\sqrt{2}} 1024=\log_{2^{7/2}} 1024=\frac{2}{7}\log_2 2^{10}=\frac{20}{7}$$

144 Example Given that

$$(\log_2 3)\cdot(\log_3 4)\cdot(\log_4 5)\cdots(\log_{511} 512)$$

is an integer, find it.

Solution: Choose $a>0, a\neq 1$. Then

$$\begin{aligned}(\log_2 3)\cdot(\log_3 4)\cdot(\log_4 5)\cdots(\log_{511} 512) &= \frac{\log_a 3}{\log_a 2}\cdot\frac{\log_a 4}{\log_a 3}\cdot\frac{\log_a 5}{\log_a 4}\cdots\frac{\log_a 512}{\log_a 511}\\ &= \frac{\log_a 512}{\log_a 2}.\end{aligned}$$

But

$$\frac{\log_a 512}{\log_a 2} = \log_2 512 = \log_2 2^9 = 9,$$

so the integer sought is 9.

145 Example Simplify

$$S = \log\tan 1^\circ + \log\tan 2^\circ + \log\tan 3^\circ + \cdots + \log\tan 89^\circ.$$

Solution: Observe that $(90-k)^\circ + k^\circ = 90^\circ$. Thus adding the kth term to the $(90-k)$th term, we obtain

$$\begin{aligned}S &= \log(\tan 1^\circ)(\tan 89^\circ) + \log(\tan 2^\circ)(\tan 88^\circ)\\ &\quad + \log(\tan 3^\circ)(\tan 87^\circ) + \cdots + \log(\tan 44^\circ)(\tan 46^\circ) + \log\tan 45^\circ.\end{aligned}$$

As $\tan k^\circ = 1/\tan(90-k)^\circ$, we get

$$S = \log 1 + \log 1 + \cdots + \log 1 + \log\tan 45^\circ.$$

Finally, as $\tan 45^\circ = 1$, we gather that

$$S = \log 1 + \log 1 + \cdots + \log 1 = 0.$$

146 Example Which is greater $\log_5 7$ or $\log_8 3$?

Solution: Clearly $\log_5 7 > 1 > \log_8 3$.

147 Example Solve the system

$$5\left(\log_x y + \log_y x\right) = 26$$
$$xy = 64$$

Solution: Clearly we need $x>0, y>0, x\neq 1, y\neq 1$. The first equation may be written as $5\left(\log_x y + \frac{1}{\log_x y}\right) = 26$ which is the same as $(\log_x y - 5)(\log_y x - \frac{1}{5}) = 0$. Thus the system splits into the two equivalent systems (I) $\log_x y = 5, xy = 64$ and (II) $\log_x y = 1/5, xy = 64$. Using the conditions $x>0, y>0, x\neq 1, y\neq 1$ we obtain the two sets of solutions $x=2, y=32$ or $x=32, y=2$.

148 Example Let $\llbracket x\rrbracket$ be the unique integer satisfying $x-1 < \llbracket x\rrbracket \leq x$. For example $\llbracket 2.9\rrbracket = 2$, $\llbracket -\pi\rrbracket = -4$. Find

$$\llbracket \log_2 1\rrbracket + \llbracket \log_2 2\rrbracket + \llbracket \log_2 3\rrbracket + \cdots + \llbracket \log_2 1000\rrbracket.$$

Solution: First observe that $2^9 = 512 < 1000 < 1024 = 2^{10}$. We decompose the interval $[1;1000]$ into dyadic blocks

$$[1;1000] = [1;2[\ \bigcup\ [2;2^2[\ \bigcup\ [2^2;2^3[\ \bigcup\ \cdots\ \bigcup\ [2^8;2^9[\ \bigcup\ [2^9;1000].$$

If $x \in [2^k, 2^{k+1}[$ then $\lfloor\!\lfloor \log_2 x \rfloor\!\rfloor = k$. If a, b are integers, the interval $[a; b[$ contains $b - a$ integers. Thus

$$\begin{aligned}
\lfloor\!\lfloor \log_2 1 \rfloor\!\rfloor + \lfloor\!\lfloor \log_2 2 \rfloor\!\rfloor + \lfloor\!\lfloor \log_2 3 \rfloor\!\rfloor + \cdots + \lfloor\!\lfloor \log_2 1000 \rfloor\!\rfloor &= (2^1 - 2^0)0 + (2^2 - 2^1)1 \\
&\quad +(2^3 - 2^2)2 + \cdots \\
&\quad +(2^9 - 2^8)8 \\
&\quad +(1000 - 2^9)9 \\
&= 0 + 2 \cdot 1 + 4 \cdot 2 + 8 \cdot 3 \\
&\quad +16 \cdot 4 + 32 \cdot 5 + \\
&\quad +64 \cdot 6 + 128 \cdot 7 \\
&\quad +256 \cdot 8 + 489 \cdot 9 \\
&= 7987
\end{aligned}$$

(the last interval has $1000 - 512 + 1 = 489$ integers).

Practice

149 Problem Find the exact value of

$$\frac{1}{\log_2 1996!} + \frac{1}{\log_3 1996!} + \frac{1}{\log_4 1996!} + \cdots + \frac{1}{\log_{1996} 1996!}.$$

1a18

150 Problem Shew that $\log_{1/2} x > \log_{1/3} x$ only when $0 < x < 1$.

151 Problem Prove that $\log_3 \pi + \log_\pi 3 > 2$.

152 Problem Let $a > 1$. Shew that $\dfrac{1}{\log_a x} > 1$ only when $1 < x < a$.

153 Problem Let $A = \log_6 16, B = \log_{12} 27$. Find integers a, b, c such that $(A + a)(B + b) = c$.

Solution: Completing squares,

$$\begin{aligned} 2x^2+6x+5 &= 2x^2+6x+\frac{9}{2}+\frac{1}{2} \\ &= (\sqrt{2}x+\frac{3}{\sqrt{2}})^2-(i\frac{1}{\sqrt{2}})^2 \\ &= (\sqrt{2}x+\frac{3}{\sqrt{2}}-i\frac{1}{\sqrt{2}})(\sqrt{2}x+\frac{3}{\sqrt{2}}+i\frac{1}{\sqrt{2}}). \end{aligned}$$

Then $x=-\frac{3}{2}\pm i\frac{1}{2}$.

If a,b are real numbers then the object $a+bi$ is called a *complex number*. If $a+bi, c+di$ are complex numbers, then the sum of them is naturally defined as

$$(a+bi)+(c+di)=(a+c)+(b+d)i \tag{2.29}$$

The product of $a+bi$ and $c+di$ is obtained by multiplying the binomials:

$$(a+bi)(c+di)=ac+adi+bci+bdi^2=(ac-bd)+(ad+bc)i \tag{2.30}$$

160 Definition If a,b are real numbers, then the *conjugate* $\overline{a+bi}$ of $a+bi$ is defined by

$$\overline{a+bi}=a-bi \tag{2.31}$$

The *norm* $|a+bi|$ of $a+bi$ is defined by

$$|a+bi|=\sqrt{(a+bi)(\overline{a+bi})}=\sqrt{a^2+b^2} \tag{2.32}$$

161 Example Find $|7+3i|$.

Solution: $|7+3i|=\sqrt{(7+3i)(7-3i)}=\sqrt{7^2+3^2}=\sqrt{58}$.

162 Example Express the quotient $\frac{2+3i}{3-5i}$ in the form $a+bi$.

Solution: We have

$$\frac{2+3i}{3-5i}=\frac{2+3i}{3-5i}\cdot\frac{3+5i}{3+5i}==\frac{-9+19i}{34}=\frac{-9}{34}+\frac{19i}{34}$$

If z_1,z_2 are complex numbers, then their norms are multiplicative.

$$|z_1z_2|=|z_1||z_2| \tag{2.33}$$

163 Example Write $(2^2+3^2)(5^2+7^2)$ as the sum of two squares.

Solution: The idea is to write $2^2+3^2=|2+3i|^2, 5^2+7^2=|5+7i|^2$ and use the multiplicativity of the norm. Now

$$\begin{aligned} (2^2+3^2)(5^2+7^2) &= |2+3i|^2|5+7i|^2 \\ &= |(2+3i)(5+7i)|^2 \\ &= |-11+29i|^2 \\ &= 11^2+29^2 \end{aligned}$$

164 Example Find the roots of $x^3-1=0$.

Solution: $x^3-1=(x-1)(x^2+x+1)$. If $x\neq 1$, the two solutions to $x^2+x+1=0$ can be obtained using the quadratic formula, getting $x=1/2\pm i\sqrt{3}/2$. Traditionally one denotes $\omega=1/2+i\sqrt{3}/2$ and hence $\omega^2=1/2-i\sqrt{3}/2$. Clearly $\omega^3=1$ and $\omega^2+\omega+1=0$.

165 Example (AHSME 1992) Find the product of the real parts of the roots of $z^2-z=5-5i$.

Solution: By the quadratic formula,

$$\begin{aligned} z &= \frac{1}{2}\pm\frac{1}{2}\sqrt{21-20i} \\ &= \frac{1}{2}\pm\frac{1}{2}\sqrt{21-2\sqrt{-100}} \\ &= \frac{1}{2}\pm\frac{1}{2}\sqrt{25-2\sqrt{(25)(-4)}-4} \\ &= \frac{1}{2}\pm\frac{1}{2}\sqrt{(5-2i)^2} \\ &= \frac{1}{2}\pm\frac{5-2i}{2} \end{aligned}$$

The roots are thus $3-i$ and $-2+i$. The product of their real parts is therefore $(3)(-2)=-6$.

☞ *Had we chosen to write $21-20i=(-5+2i)^2$, we would have still gotten the same values of z.*

Practice

166 Problem Simplify
$$\frac{(1+i)^{2004}}{(1-i)^{2000}}.$$

167 Problem Prove that
$$\begin{aligned}&1+2i+3i^2+4i^3\\&+\cdots+1995i^{1994}+1996i^{1995}\\&=-998-998i.\end{aligned}$$

168 Problem Let
$$(1+x+x^2)^{1000}=a_0+a_1x+\cdots+a_{2000}x^{2000}.$$
Find
$$a_0+a_4+a_8+\cdots+a_{2000}.$$

Chapter 3

Arithmetic

3.1 Division Algorithm

169 Definition If $a \neq 0, b$ are integers, we say that a *divides* b if there is an integer c such that $ac = b$. We write this as $a|b$.

If a does not divide b we write $a \not| b$. It should be clear that if $a|b$ and $b \neq 0$ then $1 \leq |a| \leq |b|$.

170 Theorem The following are properties of divisibility.

- If c divides a and b then c divides any linear combination of a and b. That is, if a, b, c, m, n are integers with $c|a, c|b$, then $c|(am+nb)$.
- Division by an integer is transitive. That is, if x, y, z are integers with $x|y, y|z$ then $x|z$.

Proof: *There are integers s, t with $sc = a, tc = b$. Thus*

$$am + nb = c(sm + tn),$$

giving $c|(am+bn)$. Also, there are integers u, v with $xu = y, yv = z$. Hence $xuv = z$, giving $x|z$. ❑

A very useful property of the integers is the following:

171 Theorem (Division Algorithm) Let a, b be integers, $b > 0$. There exist unique integers q and r satisfying

$$a = bq + r, \quad 0 \leq r < b \tag{3.1}$$

Proof: *The set $S = \{a - bs : s \in \mathbb{Z}, b - as \geq 0\}$ is non-empty, since $a - b(-a^2) \geq 0$. Since S is a non-empty set of non-negative integers, it must contain a least element, say $r = a - bq$. To prove uniqueness, assume $a = bq + r = bq' + r'$ with $0 \leq r' < b$. Then $b(q - q') = r' - r$. This means that $b|(r' - r)$. Since $0 \leq |r' - r| < b$, we must have $r' = r$. But this also implies $q = q'$.* ❑

For example, $39 = 4 \cdot 9 + 3$. The Division Algorithm thus discriminates integers according to the remainder they leave upon division by a. For example, if $a = 2$, then according to the Division Algorithm, the integers may be decomposed into the two families

$$A_0 = \{\ldots -4, -2, 0, 2, 4, \ldots\},$$

$$A_1 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}.$$

Therefore, all integers have one of the forms $2k$ or $2k+1$. We mention in passing that every integer of the form $2k+1$ is also of the form $2t-1$, for $2k+1 = 2(k+1) - 1$, so it suffices to take $t = k+1$.

If $a = 4$ we may decompose the integers into the four families

$$B_0 = \{\ldots,-8,-4,0,4,8,\ldots\},$$

$$B_1 = \{\ldots,-7,-3,1,5,9,\ldots\},$$

$$B_2 = \{\ldots,-6,-2,2,6,10,\ldots\},$$

$$B_3 = \{\ldots,-5,-1,3,7,11,\ldots\}.$$

Therefore any integer will take one of the forms $4k, 4k+1, 4k+2$ or $4k+3$. Again, any integer of the form $4k+1$ is also of the form $4t-3$ and any integer of the form $4k+3$ is also of the form $4t-1$.

172 Example Shew that the square of any integer is of the form $4k$ or of the form $4k+1$. That is, the square of any integer is either divisible by 4 or leaves remainder 1 upon division by 4.

Solution: If n is even, that is $n = 2a$, then $n^2 = (2a)^2 = 4a^2$, which is of the form $4k$. If n is odd, say $n = 2t+1$, then $n^2 = (2t+1)^2 = 4(t^2+t)+1$, which is of the form $4k+1$.

173 Example Shew that no integer in the sequence

$$11, 111, 1111, 11111, \ldots$$

is a perfect square.

Solution: Clearly 11 is not a square, so assume, that an integer of this sequence has $n > 2$ digits. If $n > 2$,

$$\underbrace{11\ldots1}_{n\ 1's} = \underbrace{11\ldots11}_{n-2\ 1's}00+12-1 = 100\cdot\underbrace{11\ldots11}_{n-2\ 1's}+12-1.$$

Hence any integer in this sequence is of the form $4k-1$. By the preceding problem, no integer of the form $4k-1$ can be a square. This finishes the proof.

174 Example Shew that n^2+23 is divisible by 24 for infinitely many values of n.

Solution: Observe that $n^2+23 = n^2-1+24 = (n-1)(n+1)+24$. Therefore the families of integers $n = 24m\pm1, m = 0,\pm1,\pm2,\pm3,\ldots$ produce infinitely many values such that n^2+23 is divisible by 24.

175 Example Shew that the square of any prime greater than 3 leaves remainder 1 upon division by 12.

Solution: If $p > 3$ is prime, then p is of one of the forms $6k\pm1$.

Now,

$$(6k\pm1)^2 = 12(3k^2\pm k)+1,$$

proving the assertion.

176 Example Prove that if p is a prime, then one of $8p-1$ and $8p+1$ is a prime and the other is composite.

Solution: If $p = 3$, $8p-1 = 23$ and $8p+1 = 25$, then the assertion is true for $p = 3$. If $p > 3$, then either $p = 3k+1$ or $p = 3k+2$. If $p = 3k+1$, $8p-1 = 24k-7$ and $8p+1 = 24k-6$, which is divisible by 6 and hence not prime. If $p = 3k+2$, $8p-1 = 24k-15$ is not a prime, .

177 Example Shew that if $3n+1$ is a square, then $n+1$ is the sum of three squares.

Solution: Clearly $3n+1$ is not a multiple of 3, and so $3n+1=(3k\pm 1)^2$. Therefore

$$n+1=\frac{(3k\pm 1)^2-1}{3}+1=3k^2\pm 2k+1=k^2+k^2+(k\pm 1)^2,$$

as we wanted to shew.

178 Example (AHSME 1976) Let r be the common remainder when $1059, 1417$ and 2312 are divided by $d>1$. Find $d-r$.

Solution: By the division algorithm there are integers q_1,q_2,q_3 with $1059=dq_1+r, 1417=dq_2+r$ and $2312=dq_3+r$. Subtracting we get $1253=d(q_3-q_1), 895=d(q_3-q_2)$ and $358=d(q_2-q_1)$. Notice that d is a common divisor of $1253, 895,$ and 358. As $1253=7\cdot 179,\ 895=5\cdot 179$, and $358=2\cdot 179$, we see that 179 is the common divisor greater than 1 of all three quantities, and so $d=179$. Since $1059=179q_1+r$, and $1059=5\cdot 179+164$, we deduce that $r=164$. Finally, $d-r=15$.

179 Example Shew that from any three integers, one can always choose two so that a^3b-ab^3 is divisible by 10.

Solution: It is clear that $a^3b-ab^3=ab(a-b)(a+b)$ is always even, no matter which integers are substituted. If one of the three integers is of the form $5k$, then we are done. If not, we are choosing three integers that lie in the residue classes $5k\pm 1$ or $5k\pm 2$. By the Pigeonhole Principle, two of them must lie in one of these two groups, and so there must be two whose sum or whose difference is divisible by 5. The assertion follows.

Practice

180 Problem Find all positive integers n for which

$$n+1|n^2+1.$$

181 Problem If $7|3x+2$ prove that $7|(15x^2-11x-14.)$.

182 Problem Shew that the square of any integer is of the form $3k$ or $3k+1$.

183 Problem Prove that if $3|(a^2+b^2)$, then $3|a$ and $3|b$

184 Problem Shew that if the sides of a right triangle are all integers, then 3 divides one of the lengths of a side.

185 Problem Given that 5 divides $(n+2)$, which of the following are divisible by 5

$$n^2-4,\ n^2+8n+7,\ n^4-1,\ n^2-2n?$$

186 Problem Prove that there is no prime triplet of the form $p, p+2, p+4$, except for $3,5,7$.

187 Problem Find the largest positive integer n such that

$$(n+1)(n^4+2n)+3(n^3+57)$$

be divisible by n^2+2.

188 Problem Demonstrate that if n is a positive integer such that $2n+1$ is a square, then $n+1$ is the sum of two consecutive squares.

189 Problem Shew that the product of two integers of the form $4n+1$ is again of this form. Use this fact and an argument by contradiction similar to Euclid's to prove that there are infinitely many primes of the form $4n-1$.

190 Problem Prove that there are infinitely many primes of the form $6n-1$.

191 Problem Prove that there are infinitely many primes p such that $p-2$ is not prime.

192 Problem Demonstrate that there are no three consecutive odd integers such that each is the sum of two squares greater than zero.

193 Problem Let $n>1$ be a positive integer. Prove that if one of the numbers $2^n-1, 2^n+1$ is prime, then the other is composite.

194 Problem Prove that there are infinitely many integers n such that $4n^2+1$ is divisible by both 13 and 5.

195 Problem Prove that any integer $n>11$ is the sum of two positive composite numbers.

196 Problem Prove that 3 never divides n^2+1.

197 Problem Shew the existence of infinitely many natural numbers x,y such that $x(x+1)|y(y+1)$ but

$$x\not| y \text{ and } (x+1)\not| y,$$

and also

$$x\not|(y+1) \text{ and } (x+1)\not|(y+1).$$

3.2 The Decimal Scale

Any natural number n can be written in the form

$$n = a_0 10^k + a_1 10^{k-1} + a_2 10^{k-2} + \cdots + a_{k-1} 10 + a_k$$

where $1 \leq a_0 \leq 9, 0 \leq a_j \leq 9, j \geq 1$. This is the *decimal* representation of n. For example

$$65789 = 6 \cdot 10^4 + 5 \cdot 10^3 + 7 \cdot 10^2 + 8 \cdot 10 + 9.$$

198 Example Find a reduced fraction equivalent to the repeating decimal $0.\overline{123} = 0.123123123\ldots$.

Solution: Let $N = 0.123123123\ldots$. Then $1000N = 123.123123123\ldots$. Hence $1000N - N = 123$, whence $N = \dfrac{123}{999} = \dfrac{41}{333}$.

199 Example What are all the two-digit positive integers in which the difference between the integer and the product of its two digits is 12?

Solution: Let such an integer be $10a + b$, where a, b are digits. Solve $10a + b - ab = 12$ for a getting

$$a = \frac{12 - b}{10 - b} = 1 + \frac{2}{10 - b}.$$

Since a is an integer, $10 - b$ must be a positive integer that divides 2. This gives $b = 8, a = 2$ or $b = 9, a = 3$. Thus 28 and 39 are the only such integers.

200 Example Find all the integers with initial digit 6 such that if this initial integer is suppressed, the resulting number is $1/25$ of the original number.

Solution: Let x be the integer sought. Then $x = 6 \cdot 10^n + y$ where y is a positive integer. The given condition stipulates that

$$y = \frac{1}{25}\left(6 \cdot 10^n + y\right),$$

that is,

$$y = \frac{10^n}{4} = 25 \cdot 10^{n-2}.$$

This requires $n \geq 2$, whence $y = 25, 250, 2500, 25000$, etc.. Therefore $x = 625, 6250, 62500, 625000$, etc..

201 Example (IMO 1968) Find all natural numbers x such that the product of their digits (in decimal notation) equals $x^2 - 10x - 22$.

Solution: Let x have the form

$$x = a_0 + a_1 10 + a_2 10^2 + \cdots + a_n 10^n,\ a_k \leq 9,\ a_n \neq 0.$$

Let $P(x)$ be the product of the digits of $x, P(x) = x^2 - 10x - 22$. Now $P(x) = a_0 a_1 \cdots a_n \leq 9^n a_n < 10^n a_n \leq x$ (strict inequality occurs when x has more than one digit). This means that $x^2 - 10x - 22 \leq x$ which entails that $x < 13$, whence x has one digit or $x = 10, 11$ or 12. Since $x^2 - 10x - 22 = x$ has no integral solutions, x cannot have one digit. If $x = 10, P(x) = 0$, but $x^2 - 10x - 22 \neq 0$. If $x = 11, P(x) = 1$, but $x^2 - 10x - 22 \neq 1$. The only solution is seen to be $x = 12$.

202 Example A whole number decreases an integral number of times when its last digit is deleted. Find all such numbers.

Solution: Let $0 \leq y \leq 9$, and $10x + y = mx$, where m, x are natural numbers. This requires $10 + \dfrac{y}{x} = m$, an integer. Hence, x must divide y. If $y = 0$, any natural number x will do, as we obtain multiples of 10. If $y = 1$ then $x = 1$, and we obtain 11. Continuing in this fashion, the sought number are the multiples of 10, together with the numbers 11, 12, 13, 14, 15, 16, 17, 18, 19, 22, 24, 26, 28, 33, 36, 39, 44, 55, 77, 88, and 99.

203 Example Shew that all integers in the sequence

$$49, 4489, 444889, 44448889, \underbrace{44\ldots44}_{n\ 4's}\underbrace{88\ldots88}_{n-1\ 8's}9$$

are perfect squares.

Solution: Observe that

$$\begin{aligned}\underbrace{44\ldots44}_{n\ 4's}\underbrace{88\ldots88}_{n-1\ 8's}9 &= \underbrace{44\ldots44}_{n\ 4's}\cdot 10^n + \underbrace{88\ldots88}_{n-1\ 8's}\cdot 10 + 9\\ &= \frac{4}{9}\cdot(10^n - 1)\cdot 10^n + \frac{8}{9}\cdot(10^{n-1}-1)\cdot 10 + 9\\ &= \frac{4}{9}\cdot 10^{2n} + \frac{4}{9}\cdot 10^n + \frac{1}{9}\\ &= \frac{1}{9}(2\cdot 10^n + 1)^2\\ &= \left(\frac{2\cdot 10^n + 1}{3}\right)^2\end{aligned}$$

We must shew that this last quantity is an integer, that is, that 3 divides $2\cdot 10^n + 1 = 2\underbrace{00\ldots00}_{n-1\ 0's}1$. But the sum of the digits of this last quantity is 3, which makes it divisible by 3. In fact, $\dfrac{2\cdot 10^n+1}{3} = \underbrace{6\ldots6}_{n-1\ 6's}7$

204 Example (AIME 1987) An ordered pair (m,n) of non-negative integers is called *simple* if the addition $m+n$ requires no carrying. Find the number of simple ordered pairs of non-negative integers that add to 1492.

Solution: Observe that there are $d+1$ solutions to $x+y=d$, where x,y are positive integers and d is a digit. These are

$$(0+d),\ (1+d-1),\ (2+d-2),\ldots,\ (d+0)$$

Since there is no carrying, we search for the numbers of solutions of this form to $x+y=1$, $u+v=4$, $s+t=9$, and $a+b=2$. Since each separate solution may combine with any other, the total number of simple pairs is

$$(1+1)(4+1)(9+1)(2+1) = 300.$$

205 Example (AIME 1992) For how many pairs of consecutive integers in

$$\{1000, 1001, \ldots, 2000\}$$

is no carrying required when the two integers are added?

Solution: Other than 2000, a number on this list has the form $n = 1000 + 100a + 10b + c$, where a,b,c are digits. If there is no carrying in $n+n+1$ then n has the form

$$1999,\ 1000+100a+10b+9,\ 1000+100a+99, 1000+100a+10b+c$$

with $0\leq a,b,c\leq 4$, i.e., five possible digits. There are $5^3 = 125$ integers of the form $1000+100a+10b+c, 0\leq a,b,c\leq 4$, $5^2 = 25$ integers of the form $1000+100a+10b+9, 0\leq a,b\leq 4$, and 5 integers of the form $1000+100a+99, 0\leq a\leq 4$. The total of integers sought is thus $125+25+5+1 = 156$.

206 Example (AIME 1994) Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + \cdots + p(999).$$

Find S.

Solution: If $x=0$, put $m(x)=1$, otherwise put $m(x)=x$. We use three digits to label all the integers, from 000 to 999 If a,b,c are digits, then clearly $p(100a+10b+c)=m(a)m(b)m(c)$. Thus

$$\begin{aligned}
p(000)+p(001)+p(002)+\cdots+p(999) &= m(0)m(0)m(0)+m(0)m(0)m(1) \\
&\quad +m(0)m(0)m(2)+\cdots+m(9)m(9)m(9) \\
&= (m(0)+m(1)+\cdots+m(9))^3 \\
&= (1+1+2+\cdots+9)^3 \\
&= 46^3 \\
&= 97336.
\end{aligned}$$

Hence

$$\begin{aligned}
S &= p(001)+p(002)+\cdots+p(999) \\
&= 97336-p(000) \\
&= 97336-m(0)m(0)m(0) \\
&= 97335.
\end{aligned}$$

207 Example (AIME 1992) Let S be the set of all rational numbers r, $0<r<1$, that have a repeating decimal expansion of the form

$$0.abcabcabc\ldots=0.\overline{abc},$$

where the digits a,b,c are not necessarily distinct. To write the elements of S as fractions in lowest terms, how many different numerators are required?

Solution: Observe that $0.abcabcabc\ldots=\dfrac{abc}{999}$, and that $999=3^3\cdot 37$. If abc is neither divisible by 3 nor by 37, the fraction is already in lowest terms. By Inclusion-Exclusion there are

$$999-\left(\frac{999}{3}+\frac{999}{37}\right)+\frac{999}{3\cdot 37}=648$$

such fractions. Also, fractions of the form $\dfrac{s}{37}$ where s is divisible by 3 but not by 37 are in S. There are 12 fractions of this kind (with s = 3, 6, 9, 12, ..., 36). We do not consider fractions of the form $\dfrac{l}{3^t}, t\leq 3$ with l divisible by 37 but not by 3, because these fractions are >1 and hence not in S. The total number of distinct numerators in the set of reduced fractions is thus $640+12=660$.

Practice

208 Problem Find an equivalent fraction for the repeating decimal $0.3\overline{172}$.

209 Problem A two-digit number is divided by the sum of its digits. What is the largest possible remainder?

210 Problem Shew that the integer

$$\underbrace{11\ldots 11}_{221\ 1's}$$

is a composite number.

211 Problem Let a and b be the integers

$$a = \underbrace{111\ldots 1}_{m\ 1's}$$

$$b = 1\underbrace{000\ldots 0}_{m-1\ 0's}5.$$

Shew that $ab+1$ is a perfect square.

212 Problem What digits appear on the product

$$\underbrace{3\ldots 3}_{666\ 3's}\cdot\underbrace{6\ldots 6}_{666\ 6's}?$$

213 Problem Shew that there exist no integers with the following property: if the initial digit is suppressed, the resulting integer is $1/35$ of the original number.

214 Problem Shew that the sum of all the integers of n digits, $n \geq 3$, is

$$494\underbrace{99\ldots 9}_{n-3\ 9's}55\underbrace{00\ldots 0}_{n-2\ 0's}.$$

215 Problem Shew that for any positive integer n,

$$\underbrace{11\ldots 1}_{2n\ 1's}-\underbrace{22\ldots 2}_{n\ 2's}$$

is a perfect square.

216 Problem A whole number is equal to the arithmetic mean of all the numbers obtained from the given number with the aid of all possible permutation of its digits. Find all whole numbers with that property.

217 Problem The integer n is the smallest multiple of 15 such that every digit of n is either 0 or 8. Compute $\frac{n}{15}$.

218 Problem Shew that *Champernowne's number*

$$0.12345678910111213141516171819202122\ldots,$$

which is the sequence of natural numbers written after the decimal point, is irrational.

219 Problem Given that

$$\frac{1}{49} = 0.0204081632653061224489795918367346938775 51,$$

find the last thousand digits of

$$1+50+50^2+\cdots+50^{999}.$$

220 Problem Let t be a positive real number. Prove that there is a positive integer n such that the decimal expansion of nt contains a 7.

221 Problem (AIME 1989) Suppose that n is a positive integer and d is a single digit in base-ten. Find n if

$$\frac{n}{810} = 0.d25d25d25d25d25\ldots$$

222 Problem (AIME 1988) Find the smallest positive integer whose cube ends in 888.

223 Problem (AIME 1986) In the parlour game, the "magician" asks one of the participants to think of a three-digit number abc, where a,b,c represent the digits of the number in the order indicated. The magician asks his victim to form the numbers

$$acb, bac, cab, cba,$$

to add these numbers and to reveal their sum N. If told the value of N, the magician can identify abc. Play the magician and determine abc if $N = 319$.

224 Problem (AIME 1988) For any positive integer k, let $f_1(k)$ denote the square of the sums of the digits of k. For $n \geq 2$, let $f_n(k) = f_1(f_{n-1}(k))$. Find $f_{1988}(11)$.

225 Problem (IMO 1969) Determine all three-digit numbers N that are divisible by 11 and such that $\frac{N}{11}$ equals the sum of the squares of the digits of N.

226 Problem (IMO 1962) Find the smallest natural number having the last digit 6 and if this 6 is erased and put in from of the other digits, the resulting number is four times as large as the original number.

3.3 Non-decimal Scales

The fact that most people have ten fingers has fixed our scale of notation to the decimal. Given any positive integer $r > 1$, we can, however, express any number x in base r.

If n is a positive integer, and $r > 1$ is an integer, then n has the base-r representation

$$n = a_0 + a_1 r + a_2 r^2 + \cdots + a_k r^k,\ 0 \leq a_t \leq r-1,\ a_k \neq 0,\ r^k \leq n < r^{k+1}.$$

We use the convention that we shall refer to a decimal number without referring to its base, and to a base-r number by using the subindex $_r$.

227 Example Express the decimal number 5213 in base-seven.

Solution: Observe that $5213 < 7^5$. We thus want to find $0 \le a_0, \ldots, a_4 \le 6, a_4 \ne 0$ such that

$$5213 = a_4 7^4 + a_3 7^3 + a_2 7^2 + a_1 7 + a_0.$$

Dividing by 7^4, we obtain $2+$ proper fraction $= a_4+$ proper fraction. This means that $a_4 = 2$. Thus $5213 = 2 \cdot 7^4 + a_3 7^3 + a_2 7^2 + a_1 7 + a_0$ or $411 = 5213 = a_3 7^3 + a_2 7^2 + a_1 7 + a_0$. Dividing by 7^3 this last equality we obtain $1+$ proper fraction $= a_3+$ proper fraction, and so $a_3 = 1$. Continuing in this way we deduce that $5213 = 21125_7$.

The method of successive divisions used in the preceding problem can be conveniently displayed as

7	5212	5
7	744	2
7	106	1
7	15	1
7	2	2

The central column contains the successive quotients and the rightmost column contains the corresponding remainders. Reading from the last remainder up, we recover $5213 = 21125_7$.

228 Example Write 562_7 in base-five.

Solution: $562_7 = 5 \cdot 7^2 + 6 \cdot 7 + 2 =$ in decimal scale, so the problem reduces to convert 289 to base-five. Doing successive divisions,

5	289	4
5	57	2
5	11	1
5	2	2

Thus $562_7 = 289 = 2124_5$.

229 Example Express the fraction $\frac{13}{16}$ in base-six.

Solution: Write

$$\frac{13}{16} = \frac{a_1}{6} + \frac{a_2}{6^2} + \frac{a_3}{6^3} + \frac{a_4}{6^4} + \cdots$$

Multiplying by 6, we obtain $4+$ proper fraction $= a_1+$ proper fraction, so $a_1 = 4$. Hence

$$\frac{13}{16} - \frac{4}{6} = \frac{7}{48} = \frac{a_2}{6^2} + \frac{a_3}{6^3} + \frac{a_4}{6^4} + \cdots$$

Multiply by 6^2 we obtain $5+$ proper fraction $= a_2+$ proper fraction, and so $a_2 = 5$. Continuing in this fashion

$$\frac{13}{16} = \frac{4}{6} + \frac{5}{6^2} + \frac{1}{6^3} + \frac{3}{6^4} = 0.4513_6.$$

We may simplify this procedure of successive multiplications by recurring to the following display:

6	$\frac{13}{16}$	4
6	$\frac{7}{8}$	5
6	$\frac{1}{4}$	1
6	$\frac{1}{2}$	3

The third column contains the integral part of the products of the first column and the second column. Each term of the second column from the second on is the fractional part of the product obtained in the preceding row. Thus $6\cdot\frac{13}{16}-4=\frac{7}{8}$, $6\cdot\frac{7}{8}-5=\frac{1}{4}$, etc..

230 Example Prove that 4.41_r is a perfect square in any scale of notation.

Solution:

$$4.41_r = 4+\frac{4}{r}+\frac{4}{r^2} = \left(2+\frac{1}{r}\right)^2$$

231 Example (AIME 1986) The increasing sequence

$$1,3,4,9,10,12,13,\ldots$$

consists of all those positive integers which are powers of 3 or sums of distinct powers or 3. Find the hundredth term of the sequence.

Solution: If the terms of the sequence are written in base-three, they comprise the positive integers which do not contain the digit 2. Thus the terms of the sequence in ascending order are

$$1_3, 10_3, 11_3, 100_3, 101_3, 110_3, 111_3, \ldots$$

In the *binary* scale these numbers are, of course, the ascending natural numbers $1,2,3,4,\ldots$. Therefore to obtain the 100th term of the sequence we write 100 in binary and then translate this into ternary: $100 = 1100100_2$ and $1100100_3 = 3^6+3^5+3^2 = 981$.

232 Example (AHSME 1993) Given $0 \leq x_0 < 1$, let

$$x_n = \begin{cases} 2x_{n-1} & \text{if} \quad 2x_{n-1} < 1, \\ 2x_{n-1}-1 & \text{if} \quad 2x_{n-1} \geq 1. \end{cases}$$

for all integers $n > 0$. For how many x_0 is it true that $x_0 = x_5$?

Solution: Write x_0 in binary,

$$x_0 = \sum_{k=1}^{\infty} \frac{a_k}{2^k}, \quad a_k = 0 \text{ or } 1.$$

The algorithm given moves the binary point one unit to the right. For x_0 to equal x_5 we need $(0.a_1a_2a_3a_4a_5a_6a_7\ldots)_2 = (0.a_6a_7a_8a_9a_{10}a_{11}a_{12}\ldots)_2$. This will happen if and only if x_0 has a repeating expansion with $a_1a_2a_3a_4a_5$ as the repeating block. There are $2^5 = 32$ such blocks. But if $a_1 = a_2 = \cdots = a_5 = 1$ then $x_0 = 1$, which lies outside $]0,1[$. The total number of values for which $x_0 = x_5$ is therefore $32-1 = 31$.

Practice

233 Problem Express the decimal number 12345 in every scale from binary to base-nine.

234 Problem Distribute the 27 weights of $1^2, 2^2, 3^2, \ldots, 27^2$ lbs each into three separate piles, each of equal weight.

235 Problem Let $\mathscr{C}$ denote the class of positive integers which, when written in base-three, do not require the digit 2. Prove that no three integers in $\mathscr{C}$ are in arithmetic progression.

236 Problem What is the largest integer that I should be permitted to choose so that you may determine my number in twenty "yes" or "no" questions?

237 Problem Let $\llbracket x \rrbracket$ denote the greatest integer less than or equal to x. Does the equation

$$\llbracket x \rrbracket + \llbracket 2x \rrbracket + \llbracket 4x \rrbracket + \llbracket 8x \rrbracket + \llbracket 16x \rrbracket + \llbracket 32x \rrbracket = 12345$$

have a solution?

3.4 Well-Ordering Principle

The set $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$ of natural numbers is endowed with two operations, addition and multiplication, that satisfy the following properties for natural number a, b, and c:

1. **Closure:** $a+b$ and ab are also natural numbers,
2. **Commutativity:** $a+b = b+a$ and $ab = ba$,
3. **Associative Laws:** $(a+b)+c = a+(b+c)$ and $(ab)c = a(bc)$,
4. **Distributive Law:** $a(b+c) = ab+ac$
5. **Additive Identity:** $0+a = a$.
6. **Multiplicative Identity:** $1a = a$.

One further property of the natural numbers is the following.

Well-Ordering Axiom: *Every non-empty subset $\mathscr{S}$ of the natural numbers has a least element.*
As an example of the use of the Well-Ordering Axiom let us prove that there is no integer between 0 and 1.

238 Example Prove that there is no integer in the open interval $]0;1[$.

Solution: Assume to the contrary that the set $\mathscr{S}$ of integers in $]0;1[$ is non-empty. As a set of positive integers, by Well-Ordering it must contain a least element, say m. Since $0 < m < 1$, we have $0 < m^2 < m < 1$. But this last string of inequalities says that m^2 is an integer in $]0;1[$ which is smaller than m, the smallest integer in $]0;1[$. This contradiction shews that m cannot exist.
Recall that an *irrational* number is one that cannot be represented as the ratio of two integers.

239 Example Prove that $\sqrt{2}$ is irrational.

Solution: The proof is by contradiction. Suppose that $\sqrt{2}$ were rational, i.e., that $\sqrt{2} = \dfrac{a}{b}$ for some integers $a, b, b \neq 0$. This implies that the set

$$\mathscr{A} = \{n\sqrt{2} : \text{ both } n \text{ and } n\sqrt{2} \text{ positive integers}\}$$

is non-empty since it contains a. By Well-Ordering, $\mathscr{A}$ has a smallest element, say $j = k\sqrt{2}$. As $\sqrt{2} - 1 > 0$, $j(\sqrt{2}-1) = j\sqrt{2} - k\sqrt{2} = \sqrt{2}(j-k)$, is a positive integer. Since $2 < 2\sqrt{2}$ implies $2 - \sqrt{2} < \sqrt{2}$ and also $j\sqrt{2} = 2k$, we see that

$$(j-k)\sqrt{2} = k(2-\sqrt{2}) < k\sqrt{2} = j.$$

Thus $(j-k)\sqrt{2}$ is a positive integer in $\mathscr{A}$ which is smaller than j. This contradicts the choice of j as the smallest integer in $\mathscr{A}$ and hence, finishes the proof.

240 Example Let a, b, c be integers such that $a^6 + 2b^6 = 4c^6$. Shew that $a = b = c = 0$.

Solution: Clearly we can restrict ourselves to non-negative numbers. Choose a triplet of non-negative integers a,b,c satisfying this equation and with

$$\max(a,b,c) > 0$$

as small as possible. If $a^6+2b^6=4c^6$, then a must be even, $a=2a_1$. This leads to $32a_1^6+b^6=2c^6$. This implies that b is even, $b=2b_1$ and so $16a_1^6+32b_1^6=c^6$. This implies that c is even, $c=2c_1$ and so $a_1^6+2b_1^6=4c_1^6$. But clearly $\max(a_1,b_1,c_1) < \max(a,b,c)$. We have produce a triplet of integers with a maximum smaller than the smallest possible maximum, a contradiction.

241 Example (IMO 1988) If a,b are positive integers such that $\dfrac{a^2+b^2}{1+ab}$ is an integer, then shew that $\dfrac{a^2+b^2}{1+ab}$ must be a square.

Solution: Suppose that $\dfrac{a^2+b^2}{1+ab}=k$ is a counterexample of an integer which is not a perfect square, with $\max(a,b)$ as small as possible. We may assume without loss of generality that $a<b$ for if $a=b$ then

$$0 < k = \frac{2a^2}{a^2+1} = 2 - \frac{2}{a^2+1} < 2,$$

which forces $k=1$, a square.

Now, $a^2+b^2-k(ab+1)=0$ is a quadratic in b with sum of roots ka and product of roots a^2-k. Let b_1,b be its roots, so $b_1+b=ka, bb_1=a^2-k$.

As a,k are positive integers, supposing $b_1<0$ is incompatible with $a^2+b_1^2=k(ab_1+1)$. As k is not a perfect square, supposing $b_1=0$ is incompatible with $a^2+0^2=k(0\cdot a+1)$. Also

$$b_1 = \frac{a^2-k}{b} < \frac{b^2-k}{b} = b - \frac{k}{b} < b.$$

Thus we have shewn b_1 to be a positive integer with $\dfrac{a^2+b_1^2}{1+ab_1}=k$ smaller than b. This is a contradiction to the choice of b. Such a counterexample k cannot exist, and so $\dfrac{a^2+b^2}{1+ab}$ must be a perfect square. In fact, it can be shewn that $\dfrac{a^2+b^2}{1+ab}$ is the square of the greatest common divisor of a and b.

Practice

242 Problem Find all integers solutions of $a^3+2b^3=4c^3$.

243 Problem Prove that the equality $x^2+y^2+z^2=2xyz$ can hold for whole numbers x,y,z only when $x=y=z=0$.

244 Problem Shew that the series of integral squares does not contain an infinite arithmetic progression.

245 Problem Prove that $x^2+y^2=3(z^2+w^2)$ does not have a positive integer solution.

3.5 Mathematical Induction

The Principle of Mathematical Induction is based on the following fairly intuitive observation. Suppose that we are to perform a task that involves a certain finite number of steps. Suppose that these steps are sequential. Finally, suppose that we know how to perform the n-th step provided we have accomplished the $n-1$-th step. Thus if we are ever able to start the task (that is, if we have a base case), then we should be able to finish it (because starting with the base we go to the next case, and then to the case following that, etc.).
We formulate the Principle of Mathematical Induction (PMI) as follows:

Principle of Mathematical Induction Suppose we have an assertion $P(n)$ concerning natural numbers satisfying the following two properties:

(**PMI I**) $P(k_0)$ is true for some natural number k_0,

(**PMI II**) If $P(n-1)$ is true then $P(n)$ is true.

Then the assertion $P(n)$ is true for every $n \geq k_0$.

246 Example Prove that the expression $3^{3n+3}-26n-27$ is a multiple of 169 for all natural numbers n.

Let $P(n)$ be the assertion "$3^{3n+3}-26n-27$ is a multiple of 169." Observe that $3^{3(1)+3}-26(1)-27=676=4(169)$ so $P(1)$ is true. Assume the truth of $P(n-1)$, that is, that there is an integer M such that

$$3^{3(n-1)+3}-26(n-1)-27=169M.$$

This entails

$$3^{3n}-26n-1=169M.$$

Now

$$\begin{aligned} 3^{3n+3}-26n-27 &= 27\cdot 3^{3n}-26n-27 \\ &= 27(3^{3n}-26n-1)+676n \\ &= 27(169M)+169\cdot 4n \\ &= 169(27M+4n), \end{aligned}$$

and so the truth of $P(n-1)$ implies the truth of $P(n)$. The assertion then follows for all $n \geq 1$ by PMI.

247 Example Prove that

$$(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}$$

is an even integer and that

$$(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}=b\sqrt{2}$$

for some positive integer b, for all integers $n \geq 1$.

Solution: Let $P(n)$ be the assertion: "

$$(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n}$$

is an even integer and that

$$(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n}=b\sqrt{2}$$

for some positive integer b." We see that $P(1)$ is true since

$$(1+\sqrt{2})^{2}+(1-\sqrt{2})^{2}=6,$$

and

$$(1+\sqrt{2})^{2}-(1-\sqrt{2})^{2}=4\sqrt{2}.$$

Assume now that $P(n-1)$, i.e., assume that

$$(1+\sqrt{2})^{2(n-1)}+(1-\sqrt{2})^{2(n-1)}=2N$$

for some integer N and that

$$(1+\sqrt{2})^{2(n-1)}-(1-\sqrt{2})^{2(n-1)}=a\sqrt{2}$$

for some positive integer a. Consider now the quantity

$$\begin{aligned}(1+\sqrt{2})^{2n}+(1-\sqrt{2})^{2n} &= (1+\sqrt{2})^{2}(1+\sqrt{2})^{2n-2}+(1-\sqrt{2})^{2}(1-\sqrt{2})^{2n-2}\\ &= (3+2\sqrt{2})(1+\sqrt{2})^{2n-2}+(3-2\sqrt{2})(1-\sqrt{2})^{2n-2}\\ &= 12N+4a\\ &= 2(6n+2a),\end{aligned}$$

an even integer. Similarly

$$\begin{aligned}(1+\sqrt{2})^{2n}-(1-\sqrt{2})^{2n} &= (1+\sqrt{2})^{2}(1+\sqrt{2})^{2n-2}-(1-\sqrt{2})^{2}(1-\sqrt{2})^{2n-2}\\ &= (3+2\sqrt{2})(1+\sqrt{2})^{2n-2}-(3-2\sqrt{2})(1-\sqrt{2})^{2n-2}\\ &= 3a\sqrt{2}+2\sqrt{2}(2N)\\ &= (3a+4N)\sqrt{2},\end{aligned}$$

which is of the form $b\sqrt{2}$. This implies that $P(n)$ is true. The statement of the problem follows by PMI.

248 Example Prove that if k is odd, then 2^{n+2} divides

$$k^{2^n}-1$$

for all natural numbers n.

Solution: The statement is evident for $n=1$, as $k^2-1=(k-1)(k+1)$ is divisible by 8 for any odd natural number k since $k-1$ and $k+1$ are consecutive even integers. Assume that $2^{n+2}a=k^{2^n}-1$ for some integer a. Then

$$k^{2^{n+1}}-1=(k^{2^n}-1)(k^{2^n}+1)=2^{n+2}a(k^{2^n}+1).$$

Since k is odd, $k^{2^n}+1$ is even and so $k^{2^n}+1=2b$ for some integer b. This gives

$$k^{2^{n+1}}-1=2^{n+2}a(k^{2^n}+1)=2^{n+3}ab,$$

and so the assertion follows by PMI.

249 Example Let s be a positive integer. Prove that every interval $[s, 2s]$ contains a power of 2.

Solution: If s is a power of 2, then there is nothing to prove. If s is not a power of 2 then it must lie between two consecutive powers of 2, say $2^r < s < 2^{r+1}$. This yields $2^{r+1} < 2s$. Hence $s < 2^{r+1} < 2s$, which yields the result.

250 Definition The *Fibonacci Numbers* are given by $f_0 = 0,\ f_1 = 1,\ f_{n+1} = f_n + f_{n-1}, n \geq 1$, that is every number after the second one is the sum of the preceding two.

The Fibonacci sequence then goes like $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$.

251 Example Prove that for integer $n \geq 1$,

$$f_{n-1}f_{n+1} = f_n^2 + (-1)^{n+1}.$$

Solution: If $n = 1$, then $2 = f_0 f_2 = 1^2 + (-1)^2 = f_1^2 + (-1)^{1+1}$. If $f_{n-1}f_{n+1} = f_n^2 + (-1)^{n+1}$ then using the fact that $f_{n+2} = f_n + f_{n+1}$,

$$\begin{aligned} f_n f_{n+2} &= f_n(f_n + f_{n+1}) \\ &= f_n^2 + f_n f_{n+1} \\ &= f_{n-1}f_{n+1} - (-1)^{n+1} + f_n f_{n+1} \\ &= f_{n+1}(f_{n-1} + f_n) + (-1)^{n+2} \\ &= f_{n+1}^2 + (-1)^{n+2}, \end{aligned}$$

which establishes the assertion by induction.

252 Example Prove that a given square can be decomposed into n squares, not necessarily of the same size, for all $n = 4, 6, 7, 8, \ldots$.

Solution: A quartering of a subsquare increases the number of squares by three (four new squares are gained but the original square is lost). Figure 3.1 below shews that $n = 4$ is achievable. If n were achievable, a quartering would make

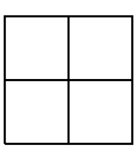

Figure 3.1: Example 252.

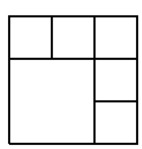

Figure 3.2: Example 252.

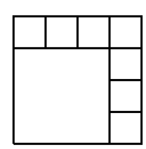

Figure 3.3: Example 252.

$\{n, n+3, n+6, n+9, \ldots\}$ also achievable. We will shew now that $n = 6$ and $n = 8$ are achievable. But this is easily seen from figures 3.2 and 3.3, and this finishes the proof.

Sometimes it is useful to use the following version of PMI, known as the Principle of Strong Mathematical Induction (PSMI).

Principle of Strong Mathematical Induction Suppose we have an assertion $P(n)$ concerning natural numbers satisfying the following two properties:

- **(PSMI I)** $P(k_0)$ is true for some natural number k_0,

- (**PSMI II**) If $m < n$ and $P(m), P(m+1), \ldots, P(n-1)$ are true then $P(n)$ is true.

Then the assertion $P(n)$ is true for every $n \geq k_0$.

253 Example In the country of SmallPesia coins only come in values of 3 and 5 pesos. Shew that any quantity of pesos greater than or equal to 8 can be paid using the available coins.

Solution: We use PSMI. Observe that $8 = 3+5, 9 = 3+3+3, 10 = 5+5$, so, we can pay $8, 9$, or 10 pesos with the available coinage. Assume that we are able to pay $n-3, n-2$, and $n-1$ pesos, that is, that $3x+5y = k$ has non-negative solutions for $k = n-3, n-2$ and $n-1$. We will shew that we may also obtain solutions for $3x+5y = k$ for $k = n, n+1$ and $n+2$. Now

$$3x+5y = n-3 \Longrightarrow 3(x+1)+5y = n,$$

$$3x_1+5y_1 = n-2 \Longrightarrow 3(x_1+1)+5y_1 = n+1,$$

$$3x_2+5y_2 = n-1 \Longrightarrow 3(x_2+1)+5y_2 = n+2,$$

and so if the amounts $n-3, n-2, n-1$ can be paid so can $n, n+1, n+2$. The statement of the problem now follows from PSMI.

254 Example (USAMO 1978) An integer n will be called *good* if we can write

$$n = a_1 + a_2 + \cdots + a_k,$$

where the integers $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_k} = 1.$$

Given the information that the integers 33 through 73 are good, prove that every integer ≥ 33 is good.

Solution: We first prove that if n is good, then $2n+8$ and $2n+9$ are also good. For assume that $n = a_1 + a_2 + \cdots + a_k$, and

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots \frac{1}{a_k} = 1.$$

Then $2n+8 = 2(a_1 + a_2 + \cdots + a_k) + 4 + 4$ and

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots \frac{1}{2a_k} + \frac{1}{4} + \frac{1}{4} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1.$$

Also $2n+9 = 2(a_1 + a_2 + \cdots + a_k) + 3 + 6$ and

$$\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots \frac{1}{2a_k} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1.$$

Therefore

$$\text{if } n \text{ is good then } 2n+8 \text{ and } 2n+9 \text{ are good} \quad (*)$$

We now establish the truth of the assertion of the problem by induction on n. Let $P(n)$ be the proposition "all the integers $n, n+1, n+2, \ldots, 2n+7$" are good. By the statement of the problem, we see that $P(33)$ is true. But (*) implies the truth of $P(n+1)$ whenever $P(n)$ is true. The assertion is thus proved by induction.

Practice

255 Problem Use Sophie Germain's trick to shew that $x^4+x^2+1=(x^2-x+1)(x^2+x+1)$. Use this to shew that if n is a positive integer then
$$2^{2^{n+1}}+2^{2^n}+1$$
has at least n different prime factors.

256 Problem Prove that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9.

257 Problem Let $n\in\mathbb{N}$. Prove the inequality
$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{3n+1}>1.$$

258 Problem Prove that for all positive integers n and all real numbers x,
$$|\sin nx|\leq n|\sin x| \tag{3.2}$$

259 Problem Prove that
$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n \text{ radical signs}}=2\cos\frac{\pi}{2^{n+1}}$$
for $n\in\mathbb{N}$.

260 Problem Let $a_1=3, b_1=4$, and $a_n=3^{a_{n-1}}$, $b_n=4^{b_{n-1}}$ when $n>1$. Prove that $a_{1000}>b_{999}$.

261 Problem Let $n\in\mathbb{N}, n>1$. Prove that
$$\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}<\frac{1}{\sqrt{3n+1}}.$$

262 Problem Prove that for all natural number $n>1$,
$$\frac{4^n}{n+1}<\frac{(2n)!}{(n!)^2}.$$

263 Problem Let k be a positive integer Prove that if $x+\frac{1}{x}$ is an integer then $x^k+\frac{1}{x^k}$ is also an integer.

264 Problem Prove that for all natural numbers $n>1$,
$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}<2-\frac{1}{n}.$$

265 Problem Let $n\geq 2$ be an integer. Prove that $f_1+f_2+\cdots+f_n=f_{n+2}-1$.

266 Problem Let $n,m\geq 0$ be integers. Prove that
$$f_{n+m}=f_{n-1}f_m+f_nf_{m+1} \tag{3.3}$$

267 Problem This problem uses the argument of A. Cauchy's to prove the AM-GM Inequality. It consists in shewing that AM-GM is true for all powers of 2 and then deducing its truth for the numbers between two consecutive powers of 2. Let $a_1,a_2,\ldots,a_l$ be non-negative real numbers. Let $P(l)$ be the assertion the AM-GM Inequality
$$\frac{a_1+a_2+\cdots+a_l}{l}\geq\sqrt[l]{a_1a_2\cdots a_l}$$
holds for the l given numbers.

1. Prove that $P(2)$ is true.

2. Prove that the truth of $P(2^{k-1})$ implies that of $P(2^k)$.

3. Let $2^{k-1}<n<2^k$. By considering the 2^k quantities
$$a_1=y_1, a_2=y_2,\ldots,a_n=y_n,$$
$$a_{n+1}=a_{n+1}=\cdots=a_{2^k}=\frac{y_1+y_2+\cdots+y_n}{n},$$
prove that $P(n)$ is true.

3.6 Congruences

268 Definition The notation $a\equiv b \mod n$ is due to Gauß, and it means that $n|(a-b)$.

Thus if $a\equiv b \mod n$ then a and b leave the same remainder upon division by n. For example, since 8 and 13 leave the same remainder upon division by 5, we have $8\equiv 13 \mod 5$. Also observe that $5|(8-13)$. As a further example, observe that $-8\equiv -1\equiv 6\equiv 13 \mod 7$.

Consider all the integers and arrange them in five columns as follows.

$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
-10	-9	-8	-7	-6
-5	-4	-3	-2	-1
0	1	2	3	4
5	6	7	8	9
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

The arrangement above shews that any integer comes in one of 5 flavours: those leaving remainder 0 upon division by 5, those leaving remainder 1 upon division by 5, etc..
Since $n|(a-b)$ implies that $\exists k \in \mathbb{Z}$ such that $nk = a-b$, we deduce that $a \equiv b \mod n$ if and only if there is an integer k such that $a = b+nk$.
The following theorem is quite useful.

269 Theorem Let $n \geq 2$ be an integer. If $x \equiv y \mod n$ and $u \equiv v \mod n$ then

$$ax+bu \equiv ay+bv \mod n.$$

Proof: *As* $n|(x-y)$, $n|(u-v)$ *then there are integers* s,t *with* $ns = x-y$, $nt = u-v$. *This implies that*

$$a(x-y)+b(u-v) = n(as+bt),$$

which entails that,

$$n|(ax+bu-ay-bv).$$

This last assertion is equivalent to saying

$$ax+bu \equiv ay+bv \mod n.$$

This finishes the proof. ❑

270 Corollary Let $n \geq 2$ be an integer. If $x \equiv y \mod n$ and $u \equiv v \mod n$ then

$$xu \equiv yv \mod n.$$

Proof: *Let* $a = u, b = y$ *in Theorem 269.* ❑

271 Corollary Let $n > 1$ be an integer, $x \equiv y \mod n$ and j a positive integer. Then $x^j \equiv y^j \mod n$.

Proof: *Use repeteadly Corollary 270 with* $u = x, v = y$. ❑

272 Corollary Let $n > 1$ be an integer, $x \equiv y \mod n$. If f is a polynomial with integral coefficients then $f(x) \equiv f(y) \mod n$.

273 Example Find the remainder when 6^{1987} is divided by 37.

Solution: $6^2 \equiv -1 \mod 37$. Thus

$$6^{1987} \equiv 6 \cdot 6^{1986} \equiv 6(6^2)^{993} \equiv 6(-1)^{993} \equiv -6 \equiv 31 \mod 37$$

and the remainder sought is 31.

274 Example Find the remainder when

$$12233 \cdot 455679 + 87653^3$$

is divided by 4.

Solution: $12233 = 12200 + 32 + 1 \equiv 1 \mod 4$. Similarly, $455679 = 455600 + 76 + 3 \equiv 3$, $87653 = 87600 + 52 + 1 \equiv 1 \mod 4$. Thus

$$12233 \cdot 455679 + 87653^3 \equiv 1 \cdot 3 + 1^3 \equiv 4 \equiv 0 \mod 4.$$

This means that $12233 \cdot 455679 + 87653^3$ is divisible by 4.

275 Example Prove that 7 divides $3^{2n+1} + 2^{n+2}$ for all natural numbers n.

Solution: Observe that

$$3^{2n+1} \equiv 3 \cdot 9^n \equiv 3 \cdot 2^n \mod 7$$

and

$$2^{n+2} \equiv 4 \cdot 2^n \mod 7$$

. Hence

$$3^{2n+1} + 2^{n+2} \equiv 7 \cdot 2^n \equiv 0 \mod 7,$$

for all natural numbers n.

276 Example Prove the following result of Euler: $641 | (2^{32} + 1)$.

Solution: Observe that $641 = 2^7 \cdot 5 + 1 = 2^4 + 5^4$. Hence $2^7 \cdot 5 \equiv -1 \mod 641$ and $5^4 \equiv -2^4 \mod 641$. Now, $2^7 \cdot 5 \equiv -1 \mod 641$ yields

$$5^4 \cdot 2^{28} = (5 \cdot 2^7)^4 \equiv (-1)^4 \equiv 1 \mod 641.$$

This last congruence and

$$5^4 \equiv -2^4 \mod 641$$

yield

$$-2^4 \cdot 2^{28} \equiv 1 \mod 641,$$

which means that $641 | (2^{32} + 1)$.

277 Example Prove that $7 | (2222^{5555} + 5555^{2222})$.

Solution: $2222 \equiv 3 \mod 7$, $5555 \equiv 4 \mod 7$ and $3^5 \equiv 5 \mod 7$. Now

$$2222^{5555} + 5555^{2222} \equiv 3^{5555} + 4^{2222} \equiv (3^5)^{1111} + (4^2)^{1111} \equiv 5^{1111} - 5^{1111} \equiv 0 \mod 7.$$

278 Example Find the units digit of 7^{7^7}.

Solution: We must find $7^{7^7} \mod 10$. Now, $7^2 \equiv -1 \mod 10$, and so $7^3 \equiv 7^2 \cdot 7 \equiv -7 \equiv 3 \mod 10$ and $7^4 \equiv (7^2)^2 \equiv 1 \mod 10$. Also, $7^2 \equiv 1 \mod 4$ and so $7^7 \equiv (7^2)^3 \cdot 7 \equiv 3 \mod 4$, which means that there is an integer t such that $7^7 = 3 + 4t$. Upon assembling all this,

$$7^{7^7} \equiv 7^{4t+3} \equiv (7^4)^t \cdot 7^3 \equiv 1^t \cdot 3 \equiv 3 \mod 10.$$

Thus the last digit is 3.

279 Example Find infinitely many integers n such that 2^n+27 is divisible by 7.

Solution: Observe that $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 1, 2^4 \equiv 2, 2^5 \equiv 4, 2^6 \equiv 1 \mod 7$ and so $2^{3k} \equiv 1 \mod 3$ for all positive integers k. Hence $2^{3k}+27 \equiv 1+27 \equiv 0 \mod 7$ for all positive integers k. This produces the infinitely many values sought.

280 Example Prove that $2^k-5, k=0,1,2,\ldots$ never leaves remainder 1 when divided by 7.

Solution: $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 1 \mod 7$, and this cycle of three repeats. Thus 2^k-5 can leave only remainders 3, 4, or 6 upon division by 7.

281 Example (AIME 1994) The increasing sequence

$$3, 15, 24, 48, \ldots,$$

consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994-th term of the sequence is divided by 1000?

Solution: We want $3|n^2-1=(n-1)(n+1)$. Since 3 is prime, this requires $n=3k+1$ or $n=3k-1, k=1,2,3,\ldots$. The sequence $3k+1, k=1,2,\ldots$ produces the terms $n^2-1=(3k+1)^2-1$ which are the terms at even places of the sequence of $3,15,24,48,\ldots$. The sequence $3k-1, k=1,2,\ldots$ produces the terms $n^2-1=(3k-1)^2-1$ which are the terms at odd places of the sequence $3,15,24,48,\ldots$. We must find the 997th term of the sequence $3k+1, k=1,2,\ldots$. Finally, the term sought is $(3(997)+1)^2-1 \equiv (3(-3)+1)^2-1 \equiv 8^2-1 \equiv 63 \mod 1000$. The remainder sought is 63.

Practice

282 Problem Prove that $0,1,3,4,9,10,$ and 12 are the only perfect squares modulo 13.

(Hint: It is enough to consider $0^2, 1^2, 2^2, \ldots, 12^2$. In fact, by observing that $r^2 \equiv (13-r)^2 \mod n$, you only have to go half way.)

283 Problem Prove that there are no integers with $x^2-5y^2=2$.

(Hint: Find all the perfect squares mod 5.)

284 Problem Which digits must we substitute for a and b in $30a0b03$ so that the resulting integer be divisible by 13?

285 Problem Find the number of all $n, 1 \le n \le 25$ such that $n^2+15n+122$ is divisible by 6.

(Hint: $n^2+15n+122 \equiv n^2+3n+2=(n+1)(n+2) \mod 6$.)

286 Problem (AIME 1983) Let $a_n = 6^n + 8^n$. Determine the remainder when a_{83} is divided by 49.

287 Problem (Polish Mathematical Olympiad) What digits should be put instead of x and y in $30x0y03$ in order to give a number divisible by 13?

288 Problem Prove that if $9|(a^3+b^3+c^3)$, then $3|abc$, for integers a,b,c.

289 Problem Describe all integers n such that $10|n^{10}+1$.

290 Problem Find the last digit of 3^{100}.

291 Problem (AHSME 1992) What is the size of the largest subset S of $\{1,2,\ldots,50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

292 Problem Prove that there are no integer solutions to the equation $x^2-7y=3$.

293 Problem Prove that if $7|a^2+b^2$ then $7|a$ and $7|b$.

294 Problem Prove that there are no integers with

$$800000007 = x^2+y^2+z^2.$$

295 Problem Prove that the sum of the decimal digits of a perfect square cannot be equal to 1991.

296 Problem Prove that

$$7|4^{2^n}+2^{2^n}+1$$

for all natural numbers n.

297 Problem Find the last two digits of 3^{100}.

298 Problem (USAMO 1986) What is the smallest integer $n>1$, for which the root-mean-square of the first n positive integers is an integer?

Note. The root mean square of n numbers $a_1, a_2, \ldots, a_n$ is defined to be

$$\left(\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}\right)^{1/2}.$$

299 Problem If $62ab427$ is a multiple of 99, find the digits a and b.

300 Problem Shew that an integer is divisible by $2^n, n = 1,2,3,\ldots$ if the number formed by its last n digits is divisible by 2^n.

301 Problem Find the last digit of

$$2333333334 \cdot 9987737 + 12 \cdot 21327 + 12123 \cdot 99987.$$

302 Problem (AIME 1994) The increasing sequence

$$3, 15, 24, 48, \ldots,$$

consists of all those multiples of 3 which are one less than a square. Find the remainder when the 1994th term is divided by 1000.

303 Problem (AIME 1983) Let $a_n = 6^n + 8^n$. Find the remainder when a_{83} is divided by 49.

304 Problem Shew that if $9|(a^3 + b^3 + c^3)$, then $3|abc$, for the integers a, b, c.

3.7 Miscellaneous Problems Involving Integers

Recall that $\llbracket x \rrbracket$ is the unique integer satisfying

$$x - 1 < \llbracket x \rrbracket \leq x \tag{3.4}$$

Thus $\llbracket x \rrbracket$ is x is an integer, or the integer just to the left of x if x is not an integer. For example $\llbracket 3.9 \rrbracket = 1, \llbracket -3.1 \rrbracket = -4$.
Let p be a prime and n a positive integer. In the product $n! = 1 \cdot 2 \cdot 3 \cdots n$ the number of factors contributing a factor of of p is $\llbracket \frac{n}{p} \rrbracket$ the number of factors contributing a factor of p^2 is $\llbracket \frac{n}{p^2} \rrbracket$, etc.. This proves the following theorem.

305 Theorem (De Polignac-Legendre) The highest power of a prime p diving $n!$ is given by

$$\sum_{k=1}^{\infty} \llbracket \frac{n}{p^k} \rrbracket \tag{3.5}$$

306 Example How many zeroes are there at the end of $999! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots 998 \cdot 999$?

Solution: The number of zeroes is determined by the highest power of 10 dividing 999!. As there are fewer multiples of 5 amongst $\{1, 2, \ldots, 999\}$ that multiples of 2, the number of zeroes is the determined by the highest power of 5 dividing 999!. But the highest power of 5 dividing 999! is given by

$$\llbracket \frac{999}{5} \rrbracket + \llbracket \frac{999}{5^2} \rrbracket + \llbracket \frac{999}{5^3} \rrbracket + \llbracket \frac{999}{5^4} \rrbracket = 199 + 39 + 7 + 1 = 246.$$

Therefore 999! ends in 246 zeroes.

307 Example Let m, n be non-negative integers. Prove that

$$\frac{(m+n)!}{m!n!} \quad \text{is an integer.} \tag{3.6}$$

Solution: Let p be a prime and k a positive integer. By the De Polignac-Legendre Theorem, it suffices to shew that

$$\llbracket \frac{m+n}{p^k} \rrbracket \geq \llbracket \frac{m}{p^k} \rrbracket + \llbracket \frac{n}{p^k} \rrbracket.$$

This inequality in turn will follow from the inequality

$$\llbracket \alpha \rrbracket + \llbracket \beta \rrbracket \leq \llbracket \alpha + \beta \rrbracket \tag{3.7}$$

which we will shew valid for all real numbers α, β.
Adding the inequalities $\llbracket \alpha \rrbracket \leq \alpha$, $\llbracket \beta \rrbracket \leq \beta$, we obtain $\llbracket \alpha \rrbracket + \llbracket \beta \rrbracket \leq \alpha + \beta$. Since $\llbracket \alpha \rrbracket + \llbracket \beta \rrbracket$ is an integer less than or equal to $\alpha + \beta$, it must be less than or equal than the integral part of $\alpha + \beta$, that is $\llbracket \alpha \rrbracket + \llbracket \beta \rrbracket \leq \llbracket \alpha + \beta \rrbracket$, as we wanted to shew.
Observe that $(m+n)! = m!(m+1)(m+2)\cdots(m+n)$. Thus cancelling a factor of $m!$,

$$\frac{(m+n)!}{m!n!} = \frac{(m+1)(m+2)\cdots(m+n)}{n!}$$

we see that the product of n consecutive positive integers is divisible by $n!$. If all the integers are negative, we may factor out a $(-1)^n$, or if they include 0, their product is 0. This gives the following theorem.

308 Theorem The product of n consecutive integers is divisible by $n!$.

309 Example Prove that n^5-5n^3+4n is always divisible by 120 for all integers n.

Solution: We have
$$n^5-5n^3+4n = n(n^2-4)(n^2-1) = (n-2)(n-1)(n)(n+1)(n+2),$$
the product of 5 consecutive integers and hence divisible by $5! = 120$.

310 Example Let A be a positive integer and let A' be the resulting integer after a specific permutation of the digits of A. Shew that if $A+A' = 10^{10}$ then A s divisible by 10.

Solution: Clearly, A and A' must have 10 digits each. Put
$$A = \overline{a_{10}a_9a_8\ldots a_1}$$
and
$$A' = \overline{b_{10}b_9b_8\ldots b_1},$$
where $a_k, b_k, k = 1,2,\ldots,10$ are the digits of A and A' respectively. As $A+A' = 10000000000$, we must have $a_1+b_1 = a_2+b_2 = \cdots = a_i+b_i = 0$ and
$$a_{i+1}+b_{i+1} = 10, a_{i+2}+b_{i+2} = \cdots = a_{10}+b_{10} = 9,$$
for some subindex $i, 0 \leq i \leq 9$. Notice that if $i = 9$ there are no sums $a_{i+2}+b_{i+2}, a_{i+3}+b_{i+3},\ldots$ and if $i = 0$ there are no sums $a_1+b_1,\ldots,a_i+b_i$.
Adding,
$$a_1+b_1+a_2+b_2+\cdots+a_i+b_i+a_{i+1}+b_{i+1}+\cdots+a_{10}+b_{10} = 10+9(9-i).$$
If i is even, $10+9(9-i)$ is odd and if i is odd $10+9(9-i)$ is even. As
$$a_1+a_2+\cdots+a_{10} = b_1+b_2+\cdots+b_{10},$$
we have
$$a_1+b_1+a_2+b_2+\cdots+a_i+b_i+a_{i+1}+b_{i+1}+\cdots+a_{10}+b_{10} = 2(a_1+a_2+\cdots+a_{10}),$$
an even integer. We gather that i is odd, which entails that $a_1 = b_1 = 0$, that is , A and A' are both divisible by 10.

311 Example (Putnam 1956) Prove that every positive integer has a multiple whose decimal representation involves all 10 digits.

Solution: Let n be an arbitrary positive integer with k digits. Let $m = 1234567890\cdot 10^{k+1}$. Then all of the n consecutive integers
$$m+1, m+2, \ldots, m+n$$
begin with 1234567890 and one of them is divisible by n.

312 Example (Putnam 1966) Let $0 < a_1 < a_2 < \ldots < a_{mn+1}$ be $mn+1$ integers. Prove that you can find either $m+1$ of them no one of which divides any other, or $n+1$ of them, each dividing the following.

Solution: Let, for each $1 \leq k \leq mn+1, n_k$ denote the length of the longest chain, starting with a_k and each dividing the following one, that can be selected from $a_k, a_{k+1}, \ldots, a_{mn+1}$. If no n_k is greater than n, then the are at least $m+1$ n_k's that are the same. However, the integers a_k corresponding to these n_k's cannot divide each other, because $a_k|a_l$ implies that $n_k \geq n_l+1$.

313 Theorem If $k|n$ then $f_k|f_n$.

Proof Letting $s = kn, t = n$ in the identity $f_{s+t} = f_{s-1}f_t + f_s f_{t+1}$ we obtain

$$f_{(k+1)n} = f_{kn+n} = f_{n-1}f_{kn} + f_n f_{kn+1}.$$

It is clear that if $f_n|f_{kn}$ then $f_n|f_{(k+1)n}$. Since $f_n|f_{n\cdot 1}$, the assertion follows.

314 Example Prove that if p is an odd prime and if

$$\frac{a}{b} = 1 + 1/2 + \cdots + 1/(p-1),$$

then p divides a.

Solution: Arrange the sum as

$$1 + \frac{1}{p-1} + \frac{1}{2} + \frac{1}{p-2} + \cdots + \frac{1}{(p-1)/2} + \frac{1}{(p+1)/2}.$$

After summing consecutive pairs, the numerator of the resulting fractions is p. Each term in the denominator is $< p$. Since p is a prime, the p on the numerator will not be thus cancelled out.

315 Example The sum of some positive integers is 1996. What is their maximum product?

Solution: We are given some positive integers $a_1, a_2, \ldots, a_n$ with $a_1 + a_2 + \cdots + a_n = 1996$. To maximise $a_1 a_2 \cdots a_n$, none of the a_k's can be 1. Let us shew that to maximise this product, we make as many possible $a_k = 3$ and at most two $a_j = 2$. Suppose that $a_j > 4$. Substituting a_j by the two terms $a_j - 3$ and 3 the sum is not changed, but the product increases since $a_j < 3(a_j - 3)$. Thus the a_k's must equal 2, 3 or 4. But $2+2+2 = 3+3$ and $2\times 2\times 2 < 3\times 3$, thus if there are more than two 2's we may substitute them by 3's. As $1996 = 3(665)+1 = 3(664)+4$, the maximum product sought is $3^{664}\times 4$.

316 Example Find all the positive integers of the form

$$r + \frac{1}{r},$$

where r is a rational number.

Solution: We will shew that the expression $r + 1/r$ is a positive integer only if $r = 1$, in which case $r + 1/r = 2$. Let

$$r + \frac{1}{r} = k,$$

k a positive integer. Then

$$r = \frac{k \pm \sqrt{k^2-4}}{2}.$$

Since k is an integer, r will be an integer if and only $k^2 - 4$ is a square of the same parity as k. Now, if $k \geq 3$,

$$(k-1)^2 < k^2 - 4 < k^2,$$

that is, $k^2 - 4$ is strictly between two consecutive squares and so it cannot be itself a square. If $k = 1$, $\sqrt{k^2-4}$ is not a real number. If $k = 2$, $k^2 - 4 = 0$. Therefore, $r + 1/r = 2$, that is, $r = 1$. This finishes the proof.

317 Example For how many integers n in $\{1, 2, 3, \ldots, 100\}$ is the tens digit of n^2 odd?

Solution: In the subset $\{1, 2, \ldots 10\}$ there are only two values of n (4 and 6) for which the digits of the tens of n^2 is odd. Now, the tens digit of $(n+10)^2 = n^2 + 20n + 100$ has the same parity as the tens digit of n^2. Thus there are only 20 n for which the prescribed condition is verified.

Practice

318 Problem Find the sum

$$5+55+555+\cdots+\underbrace{5\ldots5}_{n\ 5's}.$$

319 Problem Shew that for all numbers $a\neq 0, a\neq \pm i\sqrt{3}$ the following formula of Reyley (1825) holds.

$$a=\left(\frac{a^6+45a^5-81a^2+27}{6a(a^2+3)^2}\right)^3 +\left(\frac{-a^2+30a^2-9}{6a(a^2+3)}\right)^3 +\left(\frac{-6a^3+18a}{(a^2+3)^2}\right)^3.$$

If a is rational this shews that every rational number is expressible as the sum of the cubes of three rational numbers.

320 Problem What is the largest power of 7 that divides 1000!?

321 Problem Demonstrate that for all integer values n,

$$n^9-6n^7+9n^5-4n^3$$

is divisible by 8640.

322 Problem Prove that if $n>4$ is composite, then n divides $(n-1)!$.

323 Problem Find all real numbers satisfying the equation

$$\lfloor x^2-x-2\rfloor=\lfloor x\rfloor.$$

324 Problem Solve the equation

$$\left\lfloor\frac{x}{1999}\right\rfloor=\left\lfloor\frac{x}{2000}\right\rfloor$$

325 Problem (Putnam 1948) Let n be a positive integer. Prove that

$$\lfloor\sqrt{n}+\sqrt{n+1}\rfloor=\lfloor\sqrt{4n+2}\rfloor$$

(Hint: Prove that $\sqrt{4n+1}<\sqrt{n}+\sqrt{n+1}<\sqrt{4n+3}$. Argue that neither $4n+2$ nor $4n+3$ are perfect squares.)

326 Problem Prove that $6|n^3-n$, for all integers n.

327 Problem (Polish Mathematical Olympiad) Prove that if n is an even natural number, then the number 13^n+6 is divisible by 7.

328 Problem Find, with proof, the unique square which is the product of four consecutive odd numbers.

329 Problem (Putnam 1989) How many primes amongst the positive integers, written as usual in base-ten are such that their digits are alternating 1's and 0's, beginning and ending in 1?

330 Problem Let a,b,c be the lengths of the sides of a triangle. Shew that

$$3(ab+bc+ca)\leq(a+b+c)^2\leq 4(ab+bc+ca).$$

331 Problem Let $k\geq 2$ be an integer. Shew that if n is a positive integer, then n^k can be represented as the sum of n successive odd numbers.

332 Problem (IMO 1979) If a,b are natural numbers such that

$$\frac{a}{b}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1318}+\frac{1}{1319},$$

prove that $1979|a$.

333 Problem (Polish Mathematical Olympiad) A *triangular number* is one of the form $1+2+\ldots+n, n\in\mathbb{N}$. Prove that none of the digits $2,4,7,9$ can be the last digit of a triangular number.

334 Problem Demonstrate that there are infinitely many square triangular numbers.

335 Problem (Putnam 1975) Supposing that an integer n is the sum of two triangular numbers,

$$n=\frac{a^2+a}{2}+\frac{b^2+b}{2},$$

write $4n+1$ as the sum of two squares, $4n+1=x^2+y^2$ where x and y are expressed in terms of a and b.
Conversely, shew that if $4n+1=x^2+y^2$, then n is the sum of two triangular numbers.

336 Problem (Polish Mathematical Olympiad) Prove that
amongst ten successive natural numbers, there are always at least one and at most four numbers that are not divisible by any of the numbers $2,3,5,7$.

337 Problem Are there five consecutive positive integers such that the sum of the first four, each raised to the fourth power, equals the fifth raised to the fourth power?

338 Problem Prove that

$$\frac{(2m)!(3n)!}{(m!)^2(n!)^3}$$

is always an integer.

339 Problem Prove that for $n\in\mathbb{N}$, $(n!)!$ is divisible by $n!^{(n-1)!}$

340 Problem (Olimpíada matemática española, 1985) If n is a positive integer, prove that $(n+1)(n+2)\cdots(2n)$ is divisible by 2^n.

Chapter 4

Sums, Products, and Recursions

4.1 Telescopic cancellation

We could sum the series

$$a_1 + a_2 + a_3 + \cdots + a_n$$

if we were able to find $\{v_k\}$ satisfying $a_k = v_k - v_{k-1}$. For

$$a_1 + a_2 + a_3 + \cdots + a_n = v_1 - v_0 + v_2 - v_1 + \cdots + v_{n-1} - v_{n-2} + v_n - v_{n-1} = v_n - v_0.$$

If such sequence v_n exists, we say that $a_1 + a_2 + \cdots + a_n$ is a *telescopic series*.

341 Example Simplify

$$\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{99}\right).$$

Solution: Adding each fraction:

$$\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{100}{99},$$

which simplifies to $100/2 = 50$.

342 Example Find integers a, b so that

$$(2+1)\cdot\left(2^2+1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right) = 2^a + b.$$

Solution: Using the identity $x^2-y^2=(x-y)(x+y)$ and letting P be the sought product:

$$\begin{aligned}(2-1)P &= (2-1)(2+1)\cdot\left(2^2+1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)\\ &= \left(2^2-1\right)\cdot\left(2^2+1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)\\ &= \left(2^{2^2}-1\right)\cdot\left(2^{2^2}+1\right)\cdot\left(2^{2^3}+1\right)\cdots\left(2^{2^{99}}+1\right)\\ &= \left(2^{2^3}-1\right)\cdot\left(2^{2^3}+1\right)\cdot\left(2^{2^4}+1\right)\cdots\left(2^{2^{99}}+1\right)\\ &\vdots \qquad\qquad \vdots\\ &= (2^{2^{99}}-1)(2^{2^{99}}+1)\\ &= 2^{2^{100}}-1,\end{aligned}$$

whence

$$P=2^{2^{100}}-1.$$

343 Example Find the exact value of the product

$$P=\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}.$$

Solution: Multiplying both sides by $\sin\frac{\pi}{7}$ and using $\sin 2x=2\sin x\cos x$ we obtain

$$\begin{aligned}\sin\frac{\pi}{7}P &= (\sin\frac{\pi}{7}\cos\frac{\pi}{7})\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{4\pi}{7}\\ &= \frac{1}{2}(\sin\frac{2\pi}{7}\cos\frac{2\pi}{7})\cdot\cos\frac{4\pi}{7}\\ &= \frac{1}{4}(\sin\frac{4\pi}{7}\cos\frac{4\pi}{7})\\ &= \frac{1}{8}\sin\frac{8\pi}{7}.\end{aligned}$$

As $\sin\frac{\pi}{7}=-\sin\frac{8\pi}{7}$, we deduce that

$$P=-\frac{1}{8}.$$

344 Example Shew that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000}<\frac{1}{100}.$$

Solution: Let

$$A=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000}$$

and

$$B=\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{10000}{10001}.$$

Clearly, $x^2-1<x^2$ for all real numbers x. This implies that

$$\frac{x-1}{x}<\frac{x}{x+1}$$

whenever these four quantities are positive. Hence

$$\begin{array}{ccc} 1/2 & < & 2/3 \\ 3/4 & < & 4/5 \\ 5/6 & < & 6/7 \\ \vdots & \vdots & \vdots \\ 9999/10000 & < & 10000/10001 \end{array}$$

As all the numbers involved are positive, we multiply both columns to obtain

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000} < \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{10000}{10001},$$

or $A < B$. This yields $A^2 = A\cdot A < A\cdot B$. Now

$$A\cdot B = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{7}{8}\cdots\frac{9999}{10000}\cdot\frac{10000}{10001} = \frac{1}{10001},$$

and consequently, $A^2 < A\cdot B = 1/10001$. We deduce that $A < 1/\sqrt{10001} < 1/100$.
For the next example we recall that $n!$ (*n factorial*) means

$$n! = 1\cdot 2\cdot 3\cdots n.$$

For example, $1! = 1, 2! = 1\cdot 2 = 2$, $3! = 1\cdot 2\cdot 3 = 6, 4! = 1\cdot 2\cdot 3\cdot 4 = 24$. Observe that $(k+1)! = (k+1)k!$. We make the convention $0! = 1$.

345 Example Sum

$$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + 99\cdot 99!.$$

Solution: From $(k+1)! = (k+1)k! = k\cdot k! + k!$ we deduce $(k+1)! - k! = k\cdot k!$. Thus

$$\begin{array}{ccc} 1\cdot 1! & = & 2! - 1! \\ 2\cdot 2! & = & 3! - 2! \\ 3\cdot 3! & = & 4! - 3! \\ \vdots & \vdots & \vdots \\ 98\cdot 98 & = & 99! - 98! \\ 99\cdot 99! & = & 100! - 99! \end{array}$$

Adding both columns,

$$1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + 99\cdot 99! = 100! - 1! = 100! - 1.$$

Practice

346 Problem Find a closed formula for

$$D_n = 1 - 2 + 3 - 4 + \cdots + (-1)^{n-1}n.$$

347 Problem Simplify

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{99^2}\right).$$

348 Problem Simplify

$$\log_2\left(1+\frac{1}{2}\right)+\log_2\left(1+\frac{1}{3}\right) +\log_2\left(1+\frac{1}{4}\right)+\cdots+\log_2\left(1+\frac{1}{1023}\right).$$

349 Problem Prove that for all positive integers n, $2^{2^n}+1$ divides

$$2^{2^{2^n}+1}-2.$$

4.2 Arithmetic Sums

An *arithmetic progression* is one of the form

$$a, a+d, a+2d, a+3d, \ldots, a+(n-1)d, \ldots$$

One important arithmetic sum is

$$1+2+\cdots+n = \frac{n(n+1)}{2}.$$

To obtain a closed form, we utilise Gauss' trick:
If

$$A_n = 1+2+3+\cdots+n$$

then

$$A_n = n+(n-1)+\cdots+1.$$

Adding these two quantities,

$$\begin{array}{lllllllll} A_n & = & 1 & + & 2 & + & \cdots & + & n \\ A_n & = & n & + & (n-1) & + & \cdots & + & 1 \\ 2A_n & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) \\ & = & n(n+1), & & & & & & \end{array}$$

since there are n summands. This gives $A_n = \dfrac{n(n+1)}{2}$, that is,

$$1+2+\cdots+n = \frac{n(n+1)}{2}. \tag{4.1}$$

For example,

$$1+2+3+\cdots+100 = \frac{100(101)}{2} = 5050.$$

Applying Gauss's trick to the general arithmetic sum

$$(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d)$$

we obtain

$$(a)+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \frac{n(2a+(n-1)d)}{2} \tag{4.2}$$

350 Example Find the sum of all the integers from 1 to 1000 inclusive, which are not multiples of 3 or 5.

Solution: One computes the sum of all integers from 1 to 1000 and weeds out the sum of the multiples of 3 and the sum of the multiples of 5, but puts back the multiples of 15, which one has counted twice. Put

$$A_n = 1+2+3+\cdots+n,$$

$$B = 3+6+9+\cdots+999 = 3A_{333},$$
$$C = 5+10+15+\cdots+1000 = 5A_{200},$$
$$D = 15+30+45+\cdots+990 = 15A_{66}.$$

The desired sum is

$$\begin{aligned} A_{1000}-B-C+D &= A_{1000}-3A_{333}-5A_{200}+15A_{66} \\ &= 500500-3\cdot 55611-5\cdot 20100+15\cdot 2211 \\ &= 266332. \end{aligned}$$

351 Example Each element of the set $\{10,11,12,\ldots,19,20\}$ is multiplied by each element of the set $\{21,22,23,\ldots,29,30\}$. If all these products are added, what is the resulting sum?

Solution: This is asking for the product $(10+11+\cdots+20)(21+22+\cdots+30)$ after all the terms are multiplied. But

$$10+11+\cdots+20 = \frac{(20+10)(11)}{2} = 165$$

and

$$21+22+\cdots+30 = \frac{(30+21)(10)}{2} = 255.$$

The required total is $(165)(255) = 42075$.

352 Example The sum of a certain number of consecutive positive integers is 1000. Find these integers.

Solution: Let the the sum of integers be $S = (l+1)+(l+2)+(l+n)$. Using Gauss' trick we obtain $S = \dfrac{n(2l+n+1)}{2}$. As $S = 1000$, $2000 = n(2l+n+1)$. Now $2000 = n^2+2ln+n > n^2$, whence $n \leq \llbracket\sqrt{2000}\rrbracket = 44$. Moreover, n and $2l+n+1$ divisors of 2000 and are of opposite parity. Since $2000 = 2^4 5^3$, the odd factors of 2000 are 1, 5, 25, and 125. We then see that the problem has te following solutions:

$$n = 1,\ l = 999,$$

$$n = 5,\ l = 197,$$

$$n = 16,\ l = 54,$$

$$n = 25,\ l = 27.$$

353 Example Find the sum of all integers between 1 and 100 that leave remainder 2 upon division by 6.

Solution: We want the sum of the integers of the form $6r+2, r = 0,1,\ldots,16$. But this is

$$\sum_{r=0}^{16}(6r+2) = 6\sum_{r=0}^{16} r + \sum_{r=0}^{16} 2 = 6\frac{16(17)}{2} + 2(17) = 850.$$

Practice

354 Problem Shew that

$$1+2+3+\cdots+(n^2-1)+n^2=\frac{n^2(n^2+1)}{2}.$$

355 Problem Shew that

$$1+3+5+\cdots+2n-1=n^2.$$

356 Problem (AHSME 1994) Sum the series

$$20+20\frac{1}{5}+20\frac{2}{5}+\cdots+40.$$

357 Problem Shew that

$$\frac{1}{1996}+\frac{2}{1996}+\frac{3}{1996}+\cdots+\frac{1995}{1996}$$

is an integer.

358 Problem (AHSME 1991) Let $T_n=1+2+3+\cdots+n$ and

$$P_n=\frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdots\frac{T_n}{T_n-1}.$$

Find P_{1991}.

359 Problem Given that

$$\frac{1}{a+b},\frac{1}{b+c},\frac{1}{c+a}$$

are consecutive terms in an arithmetic progression, prove that

$$b^2,a^2,c^2$$

are also consecutive terms in an arithmetic progression.

360 Problem Consider the following table:

$$1=1$$

$$2+3+4=1+8$$

$$5+6+7+8+9=8+27$$

$$10+11+12+13+14+15+16=27+64$$

Conjecture the law of formation and prove your answer.

361 Problem The odd natural numbers are arranged as follows:

$$(1)$$

$$(3,5)$$

$$(7,9,11)$$

$$(13,15,17,19)$$

$$(21,23,25,27,29)$$

....

Find the sum of the nth row.

362 Problem Sum

$$1000^2-999^2+998^2-997^2+\cdots+4^4-3^2+2^2-1^1.$$

363 Problem The first term of an arithmetic progression is 14 and its 100th term is -16. Find (i) its 30th term and (ii) the sum of all the terms from the first to the 100th.

4.3 Geometric Sums

A *geometric progression* is one of the form

$$a,ar,ar^2,ar^3,\ldots,ar^{n-1},\ldots,$$

364 Example Find the following geometric sum:

$$1+2+4+\cdots+1024.$$

Solution: Let

$$S=1+2+4+\cdots+1024.$$

Then

$$2S=2+4+8+\cdots+1024+2048.$$

Hence

$$S=2S-S=(2+4+8\cdots+2048)-(1+2+4+\cdots+1024)=2048-1=2047.$$

365 Example Find the geometric sum

$$x=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{99}}.$$

Solution: We have

$$\frac{1}{3}x=\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{99}}+\frac{1}{3^{100}}.$$

Then

$$\begin{aligned}\frac{2}{3}x &= x-\frac{1}{3}x\\ &= (\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{99}})\\ &\quad -(\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{99}}+\frac{1}{3^{100}})\\ &= \frac{1}{3}-\frac{1}{3^{100}}.\end{aligned}$$

From which we gather

$$x=\frac{1}{2}-\frac{1}{2\cdot 3^{99}}.$$

The following example presents an *arithmetic-geometric* sum.

366 Example Sum

$$a=1+2\cdot 4+3\cdot 4^2+\cdots+10\cdot 4^9.$$

Solution: We have

$$4a=4+2\cdot 4^2+3\cdot 4^3+\cdots+9\cdot 4^9+10\cdot 4^{10}.$$

Now, $4a-a$ yields

$$3a=-1-4-4^2-4^3-\cdots-4^9+10\cdot 4^{10}.$$

Adding this last geometric series,

$$a=\frac{10\cdot 4^{10}}{3}-\frac{4^{10}-1}{9}.$$

367 Example Find the sum

$$S_n=1+1/2+1/4+\cdots+1/2^n.$$

Interpret your result as $n\to\infty$.

Solution: We have

$$S_n-\frac{1}{2}S_n=(1+1/2+1/4+\cdots+1/2^n)-(1/2+1/4+\cdots+1/2^n+1/2^{n+1})=1-1/2^n.$$

Whence

$$S_n=2-1/2^n.$$

So as n varies, we have:

$$\begin{aligned}S_1 &=2-1/2^0 &=& \quad 1\\ S_2 &=2-1/2 &=& \quad 1.5\\ S_3 &=2-1/2^2 &=& \quad 1.875\\ S_4 &=2-1/2^3 &=& \quad 1.875\\ S_5 &=2-1/2^4 &=& \quad 1.9375\\ S_6 &=2-1/2^5 &=& \quad 1.96875\\ S_{10} &=2-1/2^9 &=& \quad 1.998046875\end{aligned}$$

Thus the farther we go in the series, the closer we get to 2.

Let us sum now the geometric series

$$S = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Plainly, if $r = 1$ then $S = na$, so we may assume that $r \neq 1$. We have

$$rS = ar + ar^2 + \cdots + ar^n.$$

Hence

$$S - rS = a + ar + ar^2 + \cdots + ar^{n-1} - ar - ar^2 - \cdots - ar^n = a - ar^n.$$

From this we deduce that

$$S = \frac{a - ar^n}{1 - r},$$

that is,

$$a + ar + \cdots + ar^{n-1} = \frac{a - ar^n}{1 - r} \tag{4.3}$$

If $|r| < 1$ then $r^n \to 0$ as $n \to \infty$.
For $|r| < 1$, we obtain the sum of the infinite geometric series

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r} \tag{4.4}$$

368 Example A fly starts at the origin and goes 1 unit up, $1/2$ unit right, $1/4$ unit down, $1/8$ unit left, $1/16$ unit up, etc., *ad infinitum.* In what coordinates does it end up?

Solution: Its x coordinate is

$$\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \cdots = \frac{\frac{1}{2}}{1 - \frac{-1}{4}} = \frac{2}{5}.$$

Its y coordinate is

$$1 - \frac{1}{4} + \frac{1}{16} - \cdots = \frac{1}{1 - \frac{-1}{4}} = \frac{4}{5}.$$

Therefore, the fly ends up in $(\frac{2}{5}, \frac{4}{5})$.

Practice

369 Problem The 6th term of a geometric progression is 20 and the 10th is 320. Find (i) its 15th term, (ii) the sum of its first 30 terms.

4.4 Fundamental Sums

In this section we compute several sums using telescoping cancellation.
We start with the sum of the first n positive integers, which we have already computed using Gauss' trick.

370 Example Find a closed formula for

$$A_n = 1 + 2 + \cdots + n.$$

Solution: Observe that

$$k^2 - (k-1)^2 = 2k - 1.$$

From this

$$\begin{array}{rcl} 1^2-0^2 & = & 2\cdot 1-1 \\ 2^2-1^2 & = & 2\cdot 2-1 \\ 3^2-2^2 & = & 2\cdot 3-1 \\ \vdots & \vdots & \vdots \\ n^2-(n-1)^2 & = & 2\cdot n-1 \end{array}$$

Adding both columns,

$$n^2-0^2=2(1+2+3+\cdots+n)-n.$$

Solving for the sum,

$$1+2+3+\cdots+n=n^2/2+n/2=\frac{n(n+1)}{2}.$$

371 Example Find the sum

$$1^2+2^2+3^2+\cdots+n^2.$$

Solution: Observe that

$$k^3-(k-1)^3=3k^2-3k+1.$$

Hence

$$\begin{array}{rcl} 1^3-0^3 & = & 3\cdot 1^2-3\cdot 1+1 \\ 2^3-1^3 & = & 3\cdot 2^2-3\cdot 2+1 \\ 3^3-2^3 & = & 3\cdot 3^2-3\cdot 3+1 \\ \vdots & \vdots & \vdots \\ n^3-(n-1)^3 & = & 3\cdot n^2-3\cdot n+1 \end{array}$$

Adding both columns,

$$n^3-0^3=3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+3+\cdots+n)+n.$$

From the preceding example $1+2+3+\cdots+n=\cdot n^2/2+n/2=\dfrac{n(n+1)}{2}$ so

$$n^3-0^3=3(1^2+2^2+3^2+\cdots+n^2)-\frac{3}{2}\cdot n(n+1)+n.$$

Solving for the sum,

$$1^2+2^2+3^2+\cdots+n^2=\frac{n^3}{3}+\frac{1}{2}\cdot n(n+1)-\frac{n}{3}.$$

After simplifying we obtain

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \tag{4.5}$$

372 Example Add the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}.$$

Solution: Observe that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus

$$\begin{array}{lll}
\frac{1}{1\cdot 2} & = & \frac{1}{1} - \frac{1}{2} \\
\frac{1}{2\cdot 3} & = & \frac{1}{2} - \frac{1}{3} \\
\frac{1}{3\cdot 4} & = & \frac{1}{3} - \frac{1}{4} \\
\vdots & \vdots & \vdots \\
\frac{1}{99\cdot 100} & = & \frac{1}{99} - \frac{1}{100}
\end{array}$$

Adding both columns,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{99\cdot 100} = 1 - \frac{1}{100} = \frac{99}{100}.$$

373 Example Add

$$\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{31\cdot 34}.$$

Solution: Observe that

$$\frac{1}{(3n+1)\cdot(3n+4)} = \frac{1}{3}\cdot\frac{1}{3n+1} - \frac{1}{3}\cdot\frac{1}{3n+4}.$$

Thus

$$\begin{array}{lll}
\frac{1}{1\cdot 4} & = & \frac{1}{3} - \frac{1}{12} \\
\frac{1}{4\cdot 7} & = & \frac{1}{12} - \frac{1}{21} \\
\frac{1}{7\cdot 10} & = & \frac{1}{21} - \frac{1}{30} \\
\frac{1}{10\cdot 13} & = & \frac{1}{30} - \frac{1}{39} \\
\vdots & \vdots & \vdots \\
\frac{1}{34\cdot 37} & = & \frac{1}{102} - \frac{1}{111}
\end{array}$$

Summing both columns,

$$\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \frac{1}{7\cdot 10} + \cdots + \frac{1}{31\cdot 34} = \frac{1}{3} - \frac{1}{111} = \frac{12}{37}.$$

374 Example Sum

$$\frac{1}{1\cdot 4\cdot 7} + \frac{1}{4\cdot 7\cdot 10} + \frac{1}{7\cdot 10\cdot 13} + \cdots + \frac{1}{25\cdot 28\cdot 31}.$$

Solution: Observe that

$$\frac{1}{(3n+1)\cdot(3n+4)\cdot(3n+7)} = \frac{1}{6}\cdot\frac{1}{(3n+1)(3n+4)} - \frac{1}{6}\cdot\frac{1}{(3n+4)(3n+7)}.$$

Therefore

$$\begin{aligned}
\frac{1}{1\cdot 4\cdot 7} &= \frac{1}{6\cdot 1\cdot 4}-\frac{1}{6\cdot 4\cdot 7}\\
\frac{1}{4\cdot 7\cdot 10} &= \frac{1}{6\cdot 4\cdot 7}-\frac{1}{6\cdot 7\cdot 10}\\
\frac{1}{7\cdot 10\cdot 13} &= \frac{1}{6\cdot 7\cdot 10}-\frac{1}{6\cdot 10\cdot 13}\\
\vdots &\ \vdots\ \vdots\\
\frac{1}{25\cdot 28\cdot 31} &= \frac{1}{6\cdot 25\cdot 28}-\frac{1}{6\cdot 28\cdot 31}
\end{aligned}$$

Adding each column,

$$\frac{1}{1\cdot 4\cdot 7}+\frac{1}{4\cdot 7\cdot 10}+\frac{1}{7\cdot 10\cdot 13}+\cdots+\frac{1}{25\cdot 28\cdot 31}=\frac{1}{6\cdot 1\cdot 4}-\frac{1}{6\cdot 28\cdot 31}=\frac{9}{217}.$$

375 Example Find the sum

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots+99\cdot 100.$$

Solution: Observe that

$$k(k+1)=\frac{1}{3}(k)(k+1)(k+2)-\frac{1}{3}(k-1)(k)(k+1).$$

Therefore

$$\begin{aligned}
1\cdot 2 &= \frac{1}{3}\cdot 1\cdot 2\cdot 3-\frac{1}{3}\cdot 0\cdot 1\cdot 2\\
2\cdot 3 &= \frac{1}{3}\cdot 2\cdot 3\cdot 4-\frac{1}{3}\cdot 1\cdot 2\cdot 3\\
3\cdot 4 &= \frac{1}{3}\cdot 3\cdot 4\cdot 5-\frac{1}{3}\cdot 2\cdot 3\cdot 4\\
\vdots &\ \vdots\ \vdots\\
99\cdot 100 &= \frac{1}{3}\cdot 99\cdot 100\cdot 101-\frac{1}{3}\cdot 98\cdot 99\cdot 100
\end{aligned}$$

Adding each column,

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots+99\cdot 100=\frac{1}{3}\cdot 99\cdot 100\cdot 101-\frac{1}{3}\cdot 0\cdot 1\cdot 2=333300.$$

Practice

376 Problem Shew that

$$1^3+2^3+3^3+\cdots+n^3=\left(\frac{n(n+1)}{2}\right)^2 \tag{4.6}$$

377 Problem Let $a_1,a_2,\ldots,a_n$ be arbitrary numbers. Shew that

$$\begin{aligned}
&a_1+a_2(1+a_1)+a_3(1+a_1)(1+a_2)\\
&\quad+a_4(1+a_1)(1+a_2)(1+a_3)+\cdots\\
&\quad+a_{n-1}(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_{n-2})\\
&\qquad=(1+a_1)(1+a_2)(1+a_3)\cdots(1+a_n)-1.
\end{aligned}$$

378 Problem Shew that

$$\csc 2+\csc 4+\csc 8+\cdots+\csc 2^n=\cot 1-\cot 2^n.$$

379 Problem Let $0<x<1$. Shew that

$$\sum_{n=1}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}}=\frac{x}{1-x}.$$

380 Problem Shew that

$$\tan\frac{\pi}{2^{100}}+2\tan\frac{\pi}{2^{99}}+2^2\tan\frac{\pi}{2^{98}}+\cdots+2^{98}\tan\frac{\pi}{2^2}=\cot\frac{\pi}{2^{100}}.$$

381 Problem Shew that

$$\sum_{k=1}^{n}\frac{k}{k^4+k^2+1}=\frac{1}{2}\cdot\frac{n^2+n}{n^2+n+1}.$$

382 Problem Evaluate

$$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+3\cdot 6\cdot 12+\cdots}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+3\cdot 9\cdot 27+\cdots}\right)^{1/3}.$$

383 Problem Shew that

$$\sum_{n=1}^{\infty}\arctan\frac{1}{1+n+n^2}=\frac{\pi}{4}.$$

Hint: From

$$\tan x-\tan y=\frac{\tan x-\tan y}{1+\tan x\tan y}$$

deduce that

$$\arctan a-\arctan b=\arctan\frac{a-b}{1+ab}$$

for suitable a and b.

384 Problem Prove the following result due to Gramm

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3}.$$

4.5 First Order Recursions

We have already seen the Fibonacci numbers, defined by the recursion $f_0=0$, $f_1=1$ and

$$f_{n+1}=f_n+f_{n-1},\ n\geq 1.$$

The *order* of the recurrence is the difference between the highest and the lowest subscripts. For example

$$u_{n+2}-u_{n+1}=2$$

is of the first order, and

$$u_{n+4}+9u_n^2=n^5$$

is of the fourth order.

A recurrence is *linear* if the subscripted letters appear only to the first power. For example

$$u_{n+2}-u_{n+1}=2$$

is a linear recurrence and

$$x_n^2+nx_{n-1}=1 \text{ and } x_n+2^{x_{n-1}}=3$$

are not linear recurrences.

A recursion is *homogeneous* if all its terms contain the subscripted variable to the same power. Thus

$$x_{m+3}+8x_{m+2}-9x_m=0$$

is homogeneous. The equation

$$x_{m+3}+8x_{m+2}-9x_m=m^2-3$$

is not homogeneous.

A *closed form* of a recurrence is a formula that permits us to find the n-th term of the recurrence without having to know a priori the terms preceding it.

We outline a method for solving first order linear recurrence relations of the form

$$x_n=ax_{n-1}+f(n),a\neq 1,$$

where f is a polynomial.

1. First solve the homogeneous recurrence $x_n = ax_{n-1}$ by "raising the subscripts" in the form $x^n = ax^{n-1}$. This we call the *characteristic equation*. Cancelling this gives $x = a$. The solution to the homogeneous equation $x_n = ax_{n-1}$ will be of the form $x_n = Aa^n$, where A is a constant to be determined.

2. Test a solution of the form $x_n = Aa^n + g(n)$, where g is a polynomial of the same degree as f.

385 Example Let $x_0 = 7$ and $x_n = 2x_{n-1}, n \geq 1$. Find a closed form for x_n.

Solution: Raising subscripts we have the characteristic equation $x^n = 2x^{n-1}$. Cancelling, $x = 2$. Thus we try a solution of the form $x_n = A2^n$, were A is a constant. But $7 = x_0 = A2^0$ and so $A = 7$. The solution is thus $x_n = 7(2)^n$.
Aliter: We have

$$\begin{array}{lcl} x_0 & = & 7 \\ x_1 & = & 2x_0 \\ x_2 & = & 2x_1 \\ x_3 & = & 2x_2 \\ \vdots & \vdots & \vdots \\ x_n & = & 2x_{n-1} \end{array}$$

Multiplying both columns,

$$x_0 x_1 \cdots x_n = 7 \cdot 2^n x_0 x_1 x_2 \cdots x_{n-1}.$$

Cancelling the common factors on both sides of the equality,

$$x_n = 7 \cdot 2^n.$$

386 Example Let $x_0 = 7$ and $x_n = 2x_{n-1} + 1, n \geq 1$. Find a closed form for x_n.

Solution: By raising the subscripts in the homogeneous equation we obtain $x^n = 2x^{n-1}$ or $x = 2$. A solution to the homogeneous equation will be of the form $x_n = A(2)^n$. Now $f(n) = 1$ is a polynomial of degree 0 (a constant) and so we test a particular constant solution C. The general solution will have the form $x_n = A2^n + B$. Now, $7 = x_0 = A2^0 + B = A + B$. Also, $x_1 = 2x_0 + 7 = 15$ and so $15 = x_1 = 2A + B$. Solving the simultaneous equations

$$A + B = 7,$$

$$2A + B = 15,$$

we find $A = 8, B = -1$. So the solution is $x_n = 8(2^n) - 1 = 2^{n+3} - 1$.
Aliter: We have:

$$\begin{array}{lcl} x_0 & = & 7 \\ x_1 & = & 2x_0 + 1 \\ x_2 & = & 2x_1 + 1 \\ x_3 & = & 2x_2 + 1 \\ \vdots & \vdots & \vdots \\ x_{n-1} & = & 2x_{n-2} + 1 \\ x_n & = & 2x_{n-1} + 1 \end{array}$$

Multiply the kth row by 2^{n-k}. We obtain

$$\begin{array}{lcl} 2^n x_0 & = & 2^n \cdot 7 \\ 2^{n-1} x_1 & = & 2^n x_0 + 2^{n-1} \\ 2^{n-2} x_2 & = & 2^{n-1} x_1 + 2^{n-2} \\ 2^{n-3} x_3 & = & 2^{n-2} x_2 + 2^{n-3} \\ \vdots & \vdots & \vdots \\ 2^2 x_{n-2} & = & 2^3 x_{n-3} + 2^2 \\ 2x_{n-1} & = & 2^2 x_{n-2} + 2 \\ x_n & = & 2x_{n-1} + 1 \end{array}$$

Adding both columns, cancelling, and adding the geometric sum,

$$x_n = 7 \cdot 2^n + (1 + 2 + 2^2 + \cdots + 2^{n-1}) = 7 \cdot 2^n + 2^n - 1 = 2^{n+3} - 1.$$

Aliter: Let $u_n = x_n + 1 = 2x_{n-1} + 2 = 2(x_{n-1} + 1) = 2u_{n-1}$. We solve the recursion $u_n = 2u_{n-1}$ as we did on our first example: $u_n = 2^n u_0 = 2^n (x_0 + 1) = 2^n \cdot 8 = 2^{n+3}$. Finally, $x_n = u_n - 1 = 2^{n+3} - 1$.

387 Example Let $x_0 = 2, x_n = 9x_{n-1} - 56n + 63$. Find a closed form for this recursion.

Solution: By raising the subscripts in the homogeneous equation we obtain the characteristic equation $x^n = 9x^{n-1}$ or $x = 9$. A solution to the homogeneous equation will be of the form $x_n = A(9)^n$. Now $f(n) = -56n + 63$ is a polynomial of degree 1 and so we test a particular solution of the form $Bn + C$. The general solution will have the form $x_n = A9^n + Bn + C$. Now $x_0 = 2, x_1 = 9(2) - 56 + 63 = 25, x_2 = 9(25) - 56(2) + 63 = 176$. We thus solve the system

$$2 = A + C,$$

$$25 = 9A + B + C,$$

$$176 = 81A + 2B + C.$$

We find $A = 2, B = 7, C = 0$. The general solution is $x_n = 2(9^n) + 7n$.

388 Example Let $x_0 = 1, x_n = 3x_{n-1} - 2n^2 + 6n - 3$. Find a closed form for this recursion.

Solution: By raising the subscripts in the homogeneous equation we obtain the characteristic equation $x^n = 3x^{n-1}$ or $x = 9$. A solution to the homogeneous equation will be of the form $x_n = A(3)^n$. Now $f(n) = -2n^2 + 6n - 3$ is a polynomial of degree 2 and so we test a particular solution of the form $Bn^2 + Cn + D$. The general solution will have the form $x_n = A3^n + Bn^2 + Cn + D$. Now $x_0 = 1, x_1 = 3(1) - 2 + 6 - 3 = 4, x_2 = 3(4) - 2(2)^2 + 6(2) - 3 = 13, x_3 = 3(13) - 2(3)^2 + 6(3) - 3 = 36$. We thus solve the system

$$1 = A + D,$$

$$4 = 3A + B + C + D,$$

$$13 = 9A + 4B + 2C + D,$$

$$36 = 27A + 9B + 3C + D.$$

We find $A = B = 1, C = D = 0$. The general solution is $x_n = 3^n + n^2$.

389 Example Find a closed form for $x_n = 2x_{n-1} + 3^{n-1}, x_0 = 2$.

Solution: We test a solution of the form $x_n = A2^n + B3^n$. Then $x_0 = 2, x_1 = 2(2) + 3^0 = 5$. We solve the system

$$2 = A + B,$$

$$7 = 2A + 3B.$$

We find $A = 1, B = 1$. The general solution is $x_n = 2^n + 3^n$.
We now tackle the case when $a = 1$. In this case, we simply consider a polynomial g of degree 1 higher than the degree of f.

390 Example Let $x_0 = 7$ and $x_n = x_{n-1} + n, n \geq 1$. Find a closed formula for x_n.

Solution: By raising the subscripts in the homogeneous equation we obtain the characteristic equation $x^n = x^{n-1}$ or $x = 1$. A solution to the homogeneous equation will be of the form $x_n = A(1)^n = A$, a constant. Now $f(n) = n$ is a polynomial of degree 1 and so we test a particular solution of the form $Bn^2 + Cn + D$, one more degree than that of f. The general solution will have the form $x_n = A + Bn^2 + Cn + D$. Since A and D are constants, we may combine them to obtain $x_n = Bn^2 + Cn + E$. Now, $x_0 = 7, x_1 = 7 + 1 = 8, x_2 = 8 + 2 = 10$. So we solve the system

$$7 = E,$$

$$8 = B + C + E,$$

$$10 = 4B + 2C + E.$$

We find $B = C = \frac{1}{2}, E = 7$. The general solution is $x_n = \frac{n^2}{2} + \frac{n}{2} + 7$.
Aliter: We have

$$\begin{array}{rcl} x_0 & = & 7 \\ x_1 & = & x_0 + 1 \\ x_2 & = & x_1 + 2 \\ x_3 & = & x_2 + 3 \\ \vdots & \vdots & \vdots \\ x_n & = & x_{n-1} + n \end{array}$$

Adding both columns,

$$x_0 + x_1 + x_2 + \cdots + x_n = 7 + x_0 + x_2 + \cdots + x_{n-1} + (1 + 2 + 3 + \cdots + n).$$

Cancelling and using the fact that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$,

$$x_n = 7 + \frac{n(n+1)}{2}.$$

Some non-linear first order recursions maybe reduced to a linear first order recursion by a suitable transformation.

391 Example A recursion satisfies $u_0 = 3, u_{n+1}^2 = u_n, n \geq 1$. Find a closed form for this recursion.

Solution: Let $v_n = \log u_n$. Then $v_n = \log u_n = \log u_{n-1}^{1/2} = \frac{1}{2} \log u_{n-1} = \frac{v_{n-1}}{2}$. As $v_n = v_{n-1}/2$, we have $v_n = v_0/2^n$, that is, $\log u_n = (\log u_0)/2^n$. Therefore, $u_n = 3^{1/2^n}$.

Practice

392 Problem Find a closed form for $x_0 = 3, x_n = \frac{x_{n-1}+4}{3}$.

393 Problem Find a closed form for $x_0 = 1, x_n = 5x_{n-1} - 20n + 25$.

394 Problem Find a closed form for $x_0 = 1, x_n = x_{n-1} + 12n$.

395 Problem Find a closed form for $x_n = 2x_{n-1} + 9(5^{n-1}), x_0 = 5$.

396 Problem Find a closed form for

$$a_0 = 5, a_{j+1} = a_j^2 + 2a_j, j \geq 0.$$

397 Problem (AIME, 1994) If $n \geq 1$,

$$x_n + x_{n-1} = n^2.$$

Given that $x_{19} = 94$, find the remainder when x_{94} is divided by 1000.

398 Problem Find a closed form for

$$x_0 = -1;\ x_n = x_{n-1} + n^2, n > 0.$$

399 Problem If $u_0 = 1/3$ and $u_{n+1} = 2u_n^2 - 1$, find a closed form for u_n.

400 Problem Let $x_1 = 1, x_{n+1} = x_n^2 - x_n + 1, n > 0$. Shew that

$$\sum_{n=1}^{\infty} \frac{1}{x_n} = 1.$$

4.6 Second Order Recursions

All the recursions that we have so far examined are first order recursions, that is, we find the next term of the sequence given the preceding one. Let us now briefly examine how to solve some second order recursions.
We now outline a method for solving second order homogeneous linear recurrence relations of the form

$$x_n = ax_{n-1} + bx_{n-2}.$$

1. Find the characteristic equation by "raising the subscripts" in the form $x^n = ax^{n-1} + bx^{n-2}$. Cancelling this gives $x^2 - ax - b = 0$. This equation has two roots r_1 and r_2.
2. If the roots are different, the solution will be of the form $x_n = A(r_1)^n + B(r_2)^n$, where A, B are constants.
3. If the roots are identical, the solution will be of the form $x_n = A(r_1)^n + Bn(r_1)^n$.

401 Example Let $x_0 = 1, x_1 = -1, x_{n+2} + 5x_{n+1} + 6x_n = 0$.

Solution: The characteristic equation is $x^2 + 5x + 6 = (x+3)(x+2) = 0$. Thus we test a solution of the form $x_n = A(-2)^n + B(-3)^n$. Since $1 = x_0 = A + B, -1 = -2A - 3B$, we quickly find $A = 2, B = -1$. Thus the solution is $x_n = 2(-2)^n - (-3)^n$.

402 Example Find a closed form for the Fibonacci recursion $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$.

Solution: The characteristic equation is $f^2 - f - 1 = 0$, whence a solution will have the form $f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$. The initial conditions give

$$0 = A + B,$$

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{2}(A+B) + \frac{\sqrt{5}}{2}(A-B) = \frac{\sqrt{5}}{2}(A-B)$$

This gives $A = \frac{1}{\sqrt{5}}, B = -\frac{1}{\sqrt{5}}$. We thus have the *Cauchy-Binet Formula:*

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{4.7}$$

403 Example Solve the recursion $x_0 = 1, x_1 = 4, x_n = 4x_{n-1} - 4x_{n-2} = 0$.

Solution: The characteristic equation is $x^2 - 4x + 4 = (x-2)^2 = 0$. There is a multiple root and so we must test a solution of the form $x_n = A2^n + Bn2^n$. The initial conditions give

$$1 = A,$$

$$4 = 2A + 2B.$$

This solves to $A = 1, B = 1$. The solution is thus $x_n = 2^n + n2^n$.

Practice

404 Problem Solve the recursion $x_0 = 0, x_1 = 1, x_n = 10x_{n-1} - 21x_{n-2}$.

405 Problem Solve the recursion $x_0 = 0, x_1 = 1, x_n = 10x_{n-1} - 25x_{n-2}$.

406 Problem Solve the recursion $x_0 = 0, x_1 = 1, x_n = 10x_{n-1} - 21x_{n-2} + n$.

407 Problem Solve the recursion $x_0 = 0, x_1 = 1, x_n = 10x_{n-1} - 21x_{n-2} + 2^n$.

4.7 Applications of Recursions

408 Example Find the recurrence relation for the number of n digit binary sequences with no pair of consecutive 1's.

Solution: It is quite easy to see that $a_1 = 2, a_2 = 3$. To form $a_n, n \geq 3$, we condition on the last digit. If it is 0, the number of sequences sought is a_{n-1}. If it is 1, the penultimate digit must be 0, and the number of sequences sought is a_{n-2}. Thus

$$a_n = a_{n-1} + a_{n-2}, a_1 = 2,\ a_2 = 3.$$

409 Example Let there be drawn n ovals on the plane. If an oval intersects each of the other ovals at exactly two points and no three ovals intersect at the same point, find a recurrence relation for the number of regions into which the plane is divided.

Solution: Let this number be a_n. Plainly $a_1 = 2$. After the $n-1$th stage, the nth oval intersects the previous ovals at $2(n-1)$ points, i.e. the nth oval is divided into $2(n-1)$ arcs. This adds $2(n-1)$ regions to the a_{n-1} previously existing. Thus

$$a_n = a_{n-1} + 2(n-1),\ a_1 = 2.$$

410 Example Find a recurrence relation for the number of regions into which the plane is divided by n straight lines if every pair of lines intersect, but no three lines intersect.

Solution: Let a_n be this number. Clearly $a_1 = 2$. The nth line is cut by he previous $n-1$ lines at $n-1$ points, adding n new regions to the previously existing a_{n-1}. Hence

$$a_n = a_{n-1} + n,\ a_1 = 2.$$

411 Example *(Derangements)* An absent-minded secretary is filling n envelopes with n letters. Find a recursion for the number D_n of ways in which she never stuffs the right letter into the right envelope.

Solution: Number the envelopes $1, 2, 3, \cdots, n$. We condition on the last envelope. Two events might happen. Either n and $r(1 \leq r \leq n-1)$ trade places or they do not.
In the first case, the two letters r and n are misplaced. Our task is just to misplace the other $n-2$ letters, $(1, 2, \cdots, r-1, r+1, \cdots, n-1)$ in the slots $(1, 2, \cdots, r-1, r+1, \cdots, n-1)$. This can be done in D_{n-2} ways. Since r can be chosen in $n-1$ ways, the first case can happen in $(n-1)D_{n-2}$ ways.
In the second case, let us say that letter r, $(1 \leq r \leq n-1)$ moves to the n-th position but n moves not to the r-th position. Since r has been misplaced, we can just ignore it. Since n is not going to the r-th position, we may relabel n as r. We now have $n-1$ numbers to misplace, and this can be done in D_{n-1} ways.
As r can be chosen in $n-1$ ways, the total number of ways for the second case is $(n-1)D_{n-1}$. Thus
$D_n = (n-1)D_{n-2} + (n-1)D_{n-1}$.

412 Example There are two urns, one is full of water and the other is empty. On the first stage, half of the contains of urn I is passed into urn II. On the second stage 1/3 of the contains of urn II is passed into urn I. On stage three, 1/4 of the contains of urn I is passed into urn II. On stage four 1/5 of the contains of urn II is passed into urn I, and so on. What fraction of water remains in urn I after the 1978th stage?

Solution: Let $x_n, y_n, n = 0, 1, 2, \ldots$ denote the fraction of water in urns I and II respectively at stage n. Observe that $x_n + y_n = 1$ and that

$$x_0 = 1; y_0 = 0$$

$$x_1 = x_0 - \frac{1}{2}x_0 = \frac{1}{2}; y_1 = y_1 + \frac{1}{2}x_0 = \frac{1}{2}$$
$$x_2 = x_1 + \frac{1}{3}y_1 = \frac{2}{3}; y_2 = y_1 - \frac{1}{3}y_1 = \frac{1}{3}$$
$$x_3 = x_2 - \frac{1}{4}x_2 = \frac{1}{2}; y_1 = y_1 + \frac{1}{4}x_2 = \frac{1}{2}$$
$$x_4 = x_3 + \frac{1}{5}y_3 = \frac{3}{5}; y_1 = y_1 - \frac{1}{5}y_3 = \frac{2}{5}$$
$$x_5 = x_4 - \frac{1}{6}x_4 = \frac{1}{2}; y_1 = y_1 + \frac{1}{6}x_4 = \frac{1}{2}$$
$$x_6 = x_5 + \frac{1}{7}y_5 = \frac{4}{7}; y_1 = y_1 - \frac{1}{7}y_5 = \frac{3}{7}$$
$$x_7 = x_6 - \frac{1}{8}x_6 = \frac{1}{2}; y_1 = y_1 + \frac{1}{8}x_6 = \frac{1}{2}$$
$$x_8 = x_7 + \frac{1}{9}y_7 = \frac{5}{9}; y_1 = y_1 - \frac{1}{9}y_7 = \frac{4}{9}$$

A pattern emerges (which may be proved by induction) that at each odd stage n we have $x_n = y_n = \frac{1}{2}$ and that at each even stage we have (if $n = 2k$) $x_{2k} = \frac{k+1}{2k+1}, y_{2k} = \frac{k}{2k+1}$. Since $\frac{1978}{2} = 989$ we have $x_{1978} = \frac{990}{1979}$.

Practice

413 Problem At the *Golem Gambling Casino Research Institute* an experiment is performed by rolling a die until two odd numbers have appeared (and then the experiment stops). The tireless researchers wanted to find a recurrence relation for the number of ways to do this. Help them!

414 Problem Mrs. Rosenberg has \$8 000 000 in one of her five savings accounts. In this one, she earns 15% interest per year. Find a recurrence relation for the amount of money after n years.

415 Problem Find a recurrence relation for the number of ternary n-digit sequences with no consecutive 2's.

Chapter 5

Counting

5.1 Inclusion-Exclusion

In this section we investigate a tool for counting unions of events. It is known as *The Principle of Inclusion-Exclusion* or Sylvester-Poincaré Principle.

416 Theorem (Two set Inclusion-Exclusion)

$$\mathrm{card}\,(A \cup B) = \mathrm{card}\,(A) + \mathrm{card}\,(B) - \mathrm{card}\,(A \cap B)$$

Proof: *We have*

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B),$$

and this last expression is a union of disjoint sets. Hence

$$\mathrm{card}\,(A \cup B) = \mathrm{card}\,(A \setminus B) + \mathrm{card}\,(B \setminus A) + \mathrm{card}\,(A \cap B)\,.$$

But

$$A \setminus B = A \setminus (A \cap B) \implies \mathrm{card}\,(A \setminus B) = \mathrm{card}\,(A) - \mathrm{card}\,(A \cap B)\,,$$
$$B \setminus A = B \setminus (A \cap B) \implies \mathrm{card}\,(B \setminus A) = \mathrm{card}\,(B) - \mathrm{card}\,(A \cap B)\,,$$

from where we deduce the result. ❑

In the Venn diagram 5.1, we mark by R_1 the number of elements which are simultaneously in both sets (i.e., in $A \cap B$), by R_2 the number of elements which are in A but not in B (i.e., in $A \setminus B$), and by R_3 the number of elements which are B but not in A (i.e., in $B \setminus A$). We have $R_1 + R_2 + R_3 = \mathrm{card}\,(A \cup B)$, which illustrates the theorem.

417 Example Of 40 people, 28 smoke and 16 chew tobacco. It is also known that 10 both smoke and chew. How many among the 40 neither smoke nor chew?

Solution: Let A denote the set of smokers and B the set of chewers. Then

$$\mathrm{card}\,(A \cup B) = \mathrm{card}\,(A) + \mathrm{card}\,(B) - \mathrm{card}\,(A \cap B) = 28 + 16 - 10 = 34,$$

meaning that there are 34 people that either smoke or chew (or possibly both). Therefore the number of people that neither smoke nor chew is $40 - 34 = 6$.

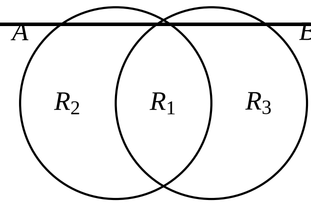

Figure 5.1: Two-set Inclusion-Exclusion

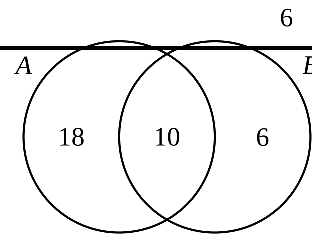

Figure 5.2: Example 417.

Aliter: We fill up the Venn diagram in figure 5.2 as follows. Since $\text{card}\,(A \cap B) = 10$, we put a 10 in the intersection. Then we put a $28 - 10 = 18$ in the part that A does not overlap B and a $16 - 10 = 6$ in the part of B that does not overlap A. We have accounted for $10 + 18 + 6 = 34$ people that are in at least one of the set. The remaining $40 - 34 = 6$ are outside these sets.

418 Example How many integers between 1 and 1000 inclusive, do not share a common factor with 1000, that is, are relatively prime to 1000?

Solution: Observe that $1000 = 2^3 5^3$, and thus from the 1000 integers we must weed out those that have a factor of 2 or of 5 in their prime factorisation. If A_2 denotes the set of those integers divisible by 2 in the interval $[1;1000]$ then clearly $\text{card}\,(A_2) = \lfloor\!\lfloor \frac{1000}{2} \rfloor\!\rfloor = 500$. Similarly, if A_5 denotes the set of those integers divisible by 5 then $\text{card}\,(A_5) = \lfloor\!\lfloor \frac{1000}{5} \rfloor\!\rfloor = 200$. Also $\text{card}\,(A_2 \cap A_5) = \lfloor\!\lfloor \frac{1000}{10} \rfloor\!\rfloor = 100$. This means that there are $\text{card}\,(A_2 \cup A_5) = 500 + 200 - 100 = 600$ integers in the interval $[1;1000]$ sharing at least a factor with 1000, thus there are $1000 - 600 = 400$ integers in $[1;1000]$ that do not share a factor prime factor with 1000.

We now deduce a formula for counting the number of elements of a union of three events.

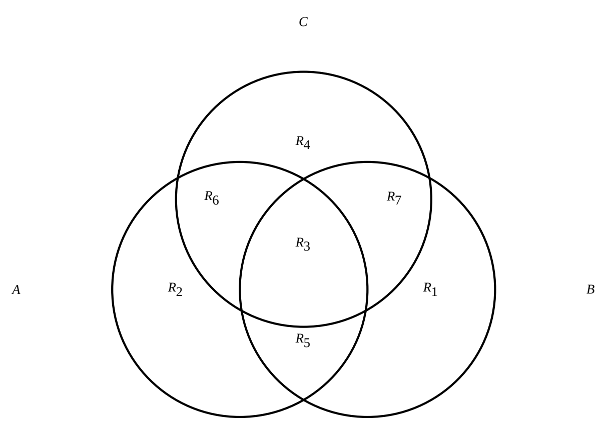

Figure 5.3: Three-set Inclusion-Exclusion

419 Theorem (Three set Inclusion-Exclusion) Let A, B, C be events of the same sample space Ω. Then

$$\begin{aligned}\text{card}\,(A \cup B \cup C) &= \text{card}\,(A) + \text{card}\,(B) + \text{card}\,(C) \\ &\quad - \text{card}\,(A \cap B) - \text{card}\,(B \cap C) - \text{card}\,(C \cap A) \\ &\quad + \text{card}\,(A \cap B \cap C)\end{aligned}$$

Proof: *Using the associativity and distributivity of unions of sets, we see that*

$$\begin{aligned}
\text{card}\,(A\cup B\cup C) &= \text{card}\,(A\cup(B\cup C)) \\
&= \text{card}\,(A)+\text{card}\,(B\cup C)-\text{card}\,(A\cap(B\cup C)) \\
&= \text{card}\,(A)+\text{card}\,(B\cup C)-\text{card}\,((A\cap B)\cup(A\cap C)) \\
&= \text{card}\,(A)+\text{card}\,(B)+\text{card}\,(C)-\text{card}\,(B\cap C) \\
&\quad -\text{card}\,(A\cap B)-\text{card}\,(A\cap C) \\
&\quad +\text{card}\,((A\cap B)\cap(A\cap C)) \\
&= \text{card}\,(A)+\text{card}\,(B)+\text{card}\,(C)-\text{card}\,(B\cap C) \\
&\quad -(\text{card}\,(A\cap B)+\text{card}\,(A\cap C)-\text{card}\,(A\cap B\cap C)) \\
&= \text{card}\,(A)+\text{card}\,(B)+\text{card}\,(C) \\
&\quad -\text{card}\,(A\cap B)-\text{card}\,(B\cap C)-\text{card}\,(C\cap A) \\
&\quad +\text{card}\,(A\cap B\cap C).
\end{aligned}$$

This gives the Inclusion-Exclusion Formula for three sets. See also figure 5.3.

❑

☞ *In the Venn diagram in figure 5.3 there are* 8 *disjoint regions: the 7 that form $A\cup B\cup C$ and the outside region, devoid of any element belonging to $A\cup B\cup C$.*

420 Example How many integers between 1 and 600 inclusive are not divisible by neither 3, nor 5, nor 7?

Solution: Let A_k denote the numbers in $[1;600]$ which are divisible by k. Then

$$\begin{aligned}
\text{card}\,(A_3) &= \left\lfloor\frac{600}{3}\right\rfloor = 200, \\
\text{card}\,(A_5) &= \left\lfloor\frac{600}{5}\right\rfloor = 120, \\
\text{card}\,(A_7) &= \left\lfloor\frac{600}{7}\right\rfloor = 85, \\
\text{card}\,(A_{15}) &= \left\lfloor\frac{600}{15}\right\rfloor = 40 \\
\text{card}\,(A_{21}) &= \left\lfloor\frac{600}{21}\right\rfloor = 28 \\
\text{card}\,(A_{35}) &= \left\lfloor\frac{600}{35}\right\rfloor = 17 \\
\text{card}\,(A_{105}) &= \left\lfloor\frac{600}{105}\right\rfloor = 5
\end{aligned}$$

By Inclusion-Exclusion there are $200+120+85-40-28-17+5=325$ integers in $[1;600]$ divisible by at least one of 3, 5, or 7. Those not divisible by these numbers are a total of $600-325=275$.

421 Example In a group of 30 people, 8 speak English, 12 speak Spanish and 10 speak French. It is known that 5 speak English and Spanish, 5 Spanish and French, and 7 English and French. The number of people speaking all three languages is 3. How many do not speak any of these languages?

Solution: Let A be the set of all English speakers, B the set of Spanish speakers and C the set of French speakers in our group. We fill-up the Venn diagram in figure 5.4 successively. In the intersection of all three we put 8. In the region common to A and B which is not filled up we put $5-2=3$. In the region common to A and C which is not already filled up we put $5-3=2$. In the region common to B and C which is not already filled up, we put $7-3=4$. In the remaining part of A we put $8-2-3-2=1$, in the remaining part of B we put $12-4-3-2=3$, and in the remaining part of C we put $10-2-3-4=1$. Each of the mutually disjoint regions comprise a total of $1+2+3+4+1+2+3=16$ persons. Those outside these three sets are then $30-16=14$.

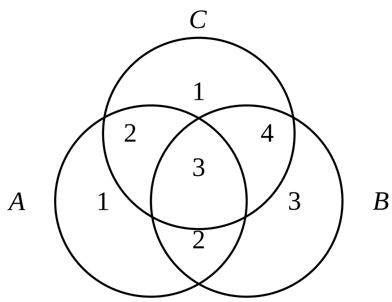

Figure 5.4: Example 421.

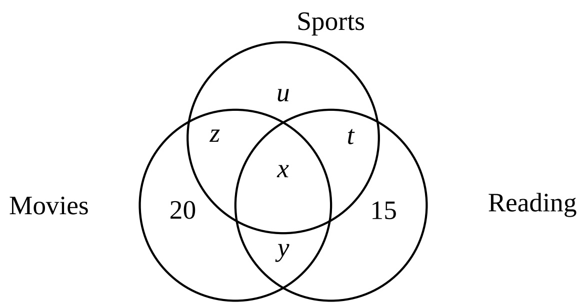

Figure 5.5: Example 422.

422 Example A survey shews that 90% of high-schoolers in Philadelphia like at least one of the following activities: going to the movies, playing sports, or reading. It is known that 45% like the movies, 48% like sports, and 35% like reading. Also, it is known that 12% like both the movies and reading, 20% like only the movies, and 15% only reading. What percent of high-schoolers like all three activities?

Solution: We make the Venn diagram in as in figure 5.5. From it we gather the following system of equations

$$\begin{array}{ccccccccccccccccc}
x & + & y & + & z & & & & & & & + & 20 & = & 45 \\
x & & & + & z & + & t & + & u & & & & & = & 48 \\
x & + & y & & & + & t & & & + & 15 & & & = & 35 \\
x & + & y & & & & & & & & & & & = & 12 \\
x & + & y & + & z & + & t & + & u & + & 15 & + & 20 & = & 90
\end{array}$$

The solution of this system is seen to be $x=5$, $y=7$, $z=13$, $t=8$, $u=22$. Thus the percent wanted is 5%.

Practice

423 Problem Consider the set

$$A=\{2,4,6,\ldots,114\}.$$

❶ How many elements are there in A?

❷ How many are divisible by 3?

❸ How many are divisible by 5?

❹ How many are divisible by 15?

❺ How many are divisible by either 3, 5 or both?

❻ How many are neither divisible by 3 nor 5?

❼ How many are divisible by exactly one of 3 or 5?

424 Problem Consider the set of the first 100 positive integers:

$$A=\{1,2,3,\ldots,100\}.$$

❶ How many are divisible by 2?

❷ How many are divisible by 3?

❸ How many are divisible by 7?

❹ How many are divisible by 6?

❺ How many are divisible by 14?

❻ How many are divisible by 21?

❼ How many are divisible by 42?

❽ How many are relatively prime to 42?

❾ How many are divisible by 2 and 3 but not by 7?

❿ How many are divisible by exactly one of 2, 3 and 7?

425 Problem A survey of a group's viewing habits over the last year revealed the following information:

❶ 28% watched gymnastics

❷ 29% watched baseball

❸ 19% watched soccer

❹ 14% watched gymnastics and baseball

❺ 12% watched baseball and soccer

❻ 10% watched gymnastics and soccer

❼ 8% watched all three sports.

Calculate the percentage of the group that watched none of the three sports during the last year.

426 Problem Out of 40 children, 30 can swim, 27 can play chess, and only 5 can do neither. How many children can swim and play chess?

427 Problem At *Medieval High* there are forty stu528(d)-257.1(s)-6.76355(1427(r)6.098]TJ -171.48 -7.92 Td [(l)2.68547(a)-3.38598(s)-6.76408(t)-238.215(y)0.992477(e)-3.38598(a)-3.38598(r)57.68

any predecessors $E_1, E_2, \ldots, E_{i-1}$. Then E_1 **and** E_2 **and** $\ldots$ **and** E_k can occur simultaneously in $n_1 n_2 \cdots n_k$ ways.

439 Example In a group of 8 men and 9 women we can pick one man **and** one woman in $8 \cdot 9 = 72$ ways. Notice that we are choosing two persons.

440 Example A red die and a blue die are tossed. In how many ways can they land?

Solution: If we view the outcomes as an ordered pair (r,b) then by the multiplication principle we have the $6 \cdot 6 = 36$ possible outcomes

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}$$

The red die can land in any of 6 ways,

6	

and also, the blue die may land in any of 6 ways

6	6

.

441 Example A multiple-choice test consists of 20 questions, each one with 4 choices. There are 4 ways of answering the first question, 4 ways of answering the second question, etc., hence there are $4^{20} = 1099511627776$ ways of answering the exam.

442 Example There are $9 \cdot 10 \cdot 10 = 900$ positive 3-digit integers:

$$100, 101, 102, \ldots, 998, 999.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1,2,3,4,5,6,7,8,9\}$ for it,

9		

.

There are 10 choices for the second digit

9	10	

,

and also 10 choices for the last digit

9	10	10

.

443 Example There are $9 \cdot 10 \cdot 5 = 450$ even positive 3-digit integers:

$$100, 102, 104, \ldots, 996, 998.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1,2,3,4,5,6,7,8,9\}$ for it,

9		

.

There are 10 choices for the second digit

9	10	

Since the integer must be even, the last digit must be one of the 5 choices $\{0,2,4,6,8\}$

9	10	5

444 Definition A *palindromic integer* or *palindrome* is a positive integer whose decimal expansion is symmetric and that is not divisible by 10. In other words, one reads the same integer backwards or forwards.[1]

445 Example The following integers are all palindromes:

$$1, 8, 11, 99, 101, 131, 999, 1234321, 9987899.$$

446 Example How many palindromes are there of 5 digits?
Solution: There are 9 ways of choosing the leftmost digit.

9				

Once the leftmost digit is chosen, the last digit must be identical to it, so we have

9				1

There are 10 choices for the second digit from the left

9	10			1

Once this digit is chosen, the second digit from the right must be identical to it, so we have only 1 choice for it,

9	10		1	1

Finally, there are 10 choices for the third digit from the right,

9	10	10	1	1

which give us 900 palindromes of 5-digits.

447 Example How many palindromes of 5 digits are even?

Solution: A five digit even palindrome has the form *ABCBA*, where A belongs to $\{2,4,6,8\}$, and B, C belong to $\{0,1,2,3,4,5,6,7,8,9\}$. Thus there are 4 choices for the first digit, 10 for the second, and 10 for the third. Once these digits are chosen, the palindrome is completely determined. Therefore, there are $4 \times 10 \times 10 = 400$ even palindromes of 5 digits.

448 Example How many positive divisors does 300 have?

[1] A palindrome in common parlance, is a word or phrase that reads the same backwards to forwards. The Philadelphia street name *Camac* is a palindrome. So are the phrases (if we ignore punctuation) (a) "A man, a plan, a canal, Panama!" (b) "Sit on a potato pan!, Otis." (c) "Able was I ere I saw Elba." This last one is attributed to Napoleon, though it is doubtful that he knew enough English to form it.

Solution: We have $300 = 3 \cdot 2^2 5^2$. Thus every factor of 300 is of the form $3^a 2^b 5^c$, where $0 \leq a \leq 1$, $0 \leq b \leq 2$, and $0 \leq c \leq 2$. Thus there are 2 choices for a, 3 for b and 3 for c. This gives $2 \cdot 3 \cdot 3 = 18$ positive divisors.

449 Example How many paths consisting of a sequence of horizontal and/or vertical line segments, each segment connecting a pair of adjacent letters in figure 5.6 spell *BIPOLAR*?

						B						
					B	*I*	*B*					
				B	*I*	*P*	*I*	*B*				
			B	*I*	*P*	*O*	*P*	*I*	*B*			
		B	*I*	*P*	*O*	*L*	*O*	*P*	*I*	*B*		
	B	*I*	*P*	*O*	*L*	*A*	*L*	*O*	*P*	*I*	*B*	
B	*I*	*P*	*O*	*L*	*A*	*R*	*A*	*L*	*O*	*P*	*I*	*B*

Figure 5.6: Problem 449.

						B
					B	*I*
				B	*I*	*P*
			B	*I*	*P*	*O*
		B	*I*	*P*	*O*	*L*
	B	*I*	*P*	*O*	*L*	*A*
B	*I*	*P*	*O*	*L*	*A*	*R*

Figure 5.7: Problem 449.

Solution: Split the diagram, as in figure 5.7. Since every required path must use the R, we count paths starting from R and reaching up to a B. Since there are six more rows that we can travel to, and since at each stage we can go either up or left, we have $2^6 = 64$ paths. The other half of the figure will provide 64 more paths. Since the middle column is shared by both halves, we have a total of $64 + 64 - 1 = 127$ paths.

We now prove that if a set A has n elements, then it has 2^n subsets. To motivate the proof, consider the set $\{a, b, c\}$. To each element we attach a binary code of length 3. We write 0 if a particular element is not in the set and 1 if it is. We then have the following associations:

$\varnothing \leftrightarrow 000,$	$\{a, b\} \leftrightarrow 110,$
$\{a\} \leftrightarrow 100,$	$\{a, c\} \leftrightarrow 101,$
$\{b\} \leftrightarrow 010,$	$\{b, c\} \leftrightarrow 011,$
$\{c\} \leftrightarrow 001,$	$\{a, b, c\} \leftrightarrow 111.$

Thus there is a one-to-one correspondence between the subsets of a finite set of 3 elements and binary sequences of length 3.

450 Theorem (Cardinality of the Power Set) Let A be a finite set with $\mathrm{card}(A) = n$. Then A has 2^n subsets.

Proof: *We attach a binary code to each element of the subset,* 1 *if the element is in the subset and* 0 *if the element is not in the subset. The total number of subsets is the total number of such binary codes, and there are* 2^n *in number.* ❑

Homework

451 Problem A true or false exam has ten questions. How many possible answer keys are there?

452 Problem Out of nine different pairs of shoes, in how many ways could I choose a right shoe and a left shoe, which should not form a pair?

453 Problem In how many ways can the following prizes be given away to a class of twenty boys: first and second Classical, first and second Mathematical, first Science, and first French?

454 Problem Under old hardware, a certain programme accepted passwords of the form

$$eell$$

where

$$e \in \{0,2,4,6,8\}, \qquad l \in \{a,b,c,d,u,v,w,x,y,z\}.$$

The hardware was changed and now the software accepts passwords of the form

$$eeelll.$$

How many more passwords of the latter kind are there than of the former kind?

455 Problem A license plate is to be made according to the following provision: it has four characters, the first two characters can be any letter of the English alphabet and the last two characters can be any digit. One is allowed to repeat letters and digits. How many different license plates can be made?

456 Problem In problem 455, how many different license plates can you make if (i) you may repeat letters but not digits?, (ii) you may repeat digits but not letters?, (iii) you may repeat neither letters nor digits?

457 Problem An alphabet consists of the **five** consonants {p, v, t, s, k} and the **three** vowels {a, e, o}. A license plate is to be made using **four** letters of this alphabet.

❶ How many letters does this alphabet have?

❷ If a license plate is of the form *CCVV* where *C* denotes a consonant and *V* denotes a vowel, how many possible license plates are there, assuming that you may repeat both consonants and vowels?

❸ If a license plate is of the form *CCVV* where *C* denotes a consonant and *V* denotes a vowel, how many possible license plates are there, assuming that you may repeat consonants but not vowels?

❹ If a license plate is of the form *CCVV* where *C* denotes a consonant and *V* denotes a vowel, how many possible license plates are there, assuming that you may repeat vowels but not consonants?

❺ If a license plate is of the form *LLLL* where *L* denotes any letter of the alphabet, how many possible license plates are there, assuming that you may not repeat letters?

458 Problem A man lives within reach of three boys' schools and four girls' schools. In how many ways can he send his three sons and two daughters to school?

459 Problem How many distinct four-letter words can be made with the letters of the set $\{c,i,k,t\}$

❶ if the letters are not to be repeated?

❷ if the letters can be repeated?

460 Problem How many distinct six-digit numbers that are multiples of 5 can be formed from the list of digits $\{1,2,3,4,5,6\}$ if we allow repetition?

461 Problem Telephone numbers in *Land of the Flying Camels* have 7 digits, and the only digits available are $\{0,1,2,3,4,5,7,8\}$. No telephone number may begin in 0, 1 or 5. Find the number of telephone numbers possible that meet the following criteria:

❶ You may repeat all digits.

❷ You may not repeat any of the digits.

❸ You may repeat the digits, but the phone number must be even.

❹ You may repeat the digits, but the phone number must be odd.

❺ You may not repeat the digits and the phone numbers must be odd.

462 Problem How many 5-lettered words can be made out of 26 letters, repetitions allowed, but not consecutive repetitions (that is, a letter may not follow itself in the same word)?

463 Problem How many positive integers are there having $n \geq 1$ digits?

464 Problem How many n-digits integers ($n \geq 1$) are there which are even?

465 Problem How many n-digit nonnegative integers do not contain the digit 5?

466 Problem How many n-digit numbers do not have the digit 0?

467 Problem There are m different roads from town A to town B. In how many ways can Dwayne travel from town A to town B and back if (a) he may come back the way he went?, (b) he must use a different road of return?

468 Problem How many positive divisors does $2^8 3^9 5^2$ have? What is the sum of these divisors?

469 Problem How many factors of 2^{95} are larger than $1,000,000$?

470 Problem How many positive divisors does 360 have? How many are even? How many are odd? How many are perfect squares?

471 Problem (AHSME 1988) At the end of a professional bowling tournament, the top 5 bowlers have a play-off. First # 5 bowls #4. The loser receives the 5th prize and the winner bowls # 3 in another game. The loser of this game receives the 4th prize and the winner bowls # 2. The loser of this game receives the 3rd prize and the winner bowls # 1. The loser of this game receives the 2nd prize and the winner the 1st prize. In how many orders can bowlers #1 through #5 receive the prizes?

472 Problem The password of the anti-theft device of a car is a four digit number, where one can use any digit in the set

$$\{0,1,2,3,4,5,6,7,8,9\}.$$

A. ❶ How many such passwords are possible?
   ❷ How many of the passwords have all their digits distinct?

B. After an electrical failure, the owner must reintroduce the password in order to deactivate the anti-theft device. He knows that the four digits of the code are $2,0,0,3$ but does not recall the order.

   ❶ How many such passwords are possible using only these digits?
   ❷ If the first attempt at the password fails, the owner must wait two minutes before a second attempt, if the second attempt fails he must wait four minutes before a third attempt, if the third attempt fails he must wait eight minutes before a fourth attempt, etc. (the time doubles from one attempt to the next). How many passwords can the owner attempt in a period of 24 hours?

473 Problem The number 3 can be expressed as a sum of one or more positive integers in four ways, namely, as 3, $1+2$, $2+1$, and $1+1+1$. Shew that any positive integer n can be so expressed in 2^{n-1} ways.

474 Problem Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

475 Problem Let $n \geq 3$. Find the number of n-digit ternary sequences that contain at least one 0, one 1 and one 2.

476 Problem In how many ways can one decompose the set

$$\{1,2,3,\dots,100\}$$

into subsets A,B,C satisfying

$$A \cup B \cup C = \{1,2,3,\dots,100\} \quad \text{and} \quad A \cap B \cap C = \varnothing$$

5.3 The Sum Rule

477 Rule (Sum Rule: Disjunctive Form) Let $E_1, E_2, \ldots, E_k$, be pairwise mutually exclusive events. If E_i can occur in n_i

Solution: The integers desired have the form $D_1D_2D_3D_4$ with $D_1 \neq 0$. Under the stipulated constraints, we must have

$$D_1+D_2+D_3+D_4 \in \{6,9,12\}.$$

We thus consider three cases.

Case I: $D_1+D_2+D_3+D_4 = 6$. Here we have $\{D_1,D_2,D_3,D_4\} = \{0,1,2,3,4\}, D_1 \neq 0$. There are then 3 choices for D_1. After D_1 is chosen, D_2 can be chosen in 3 ways, D_3 in 2 ways, and D_1 in 1 way. There are thus $3\times 3\times 2\times 1 = 3\cdot 3! = 18$ integers satisfying case I.

Case II: $D_1+D_2+D_3+D_4 = 9$. Here we have $\{D_1,D_2,D_3,D_4\} = \{0,2,3,4\}, D_1 \neq 0$ or $\{D_1,D_2,D_3,D_4\} = \{0,1,3,5\}, D_1 \neq 0$. Like before, there are $3\cdot 3! = 18$ numbers in each possibility, thus we have $2\times 18 = 36$ numbers in case II.

Case III: $D_1+D_2+D_3+D_4 = 12$. Here we have $\{D_1,D_2,D_3,D_4\} = \{0,3,4,5\}, D_1 \neq 0$ or $\{D_1,D_2,D_3,D_4\} = \{1,2,4,5\}$. In the first possibility there are $3\cdot 3! = 18$ numbers, and in the second there are $4! = 24$. Thus we have $18+24 = 42$ numbers in case III.

The desired number is finally $18+36+42 = 96$.

Homework

484 Problem How many different sums can be thrown with two dice, the faces of each die being numbered $0,1,3,7,15,31$?

485 Problem How many different sums can be thrown with three dice, the faces of each die being numbered $1,4,13,40,121,364$?

486 Problem How many two or three letter initials for people are available if at least one of the letters must be a D and one allows repetitions?

487 Problem How many strictly positive integers have all their digits distinct?

488 Problem The Morse code consists of points and dashes. How many letters can be in the Morse code if no letter contains more than four signs, but all must have at least one?

489 Problem An $n\times n\times n$ wooden cube is painted blue and then cut into n^3 $1\times 1\times 1$ cubes. How many cubes (a) are painted on exactly three sides, (b) are painted in exactly two sides, (c) are painted in exactly one side, (d) are not painted?

490 Problem (AIME 1993) How many even integers between 4000 and 7000 have four different digits?

491 Problem All the natural numbers, starting with 1, are listed consecutively

$$123456789101112131415161718192021\ldots$$

Which digit occupies the 1002nd place?

492 Problem All the positive integers are written in succession.

$$123456789101112131415161718192021222324\ldots$$

Which digit occupies the 206790th place?

493 Problem All the positive integers with initial digit 2 are written in succession:

$$2,20,21,22,23,24,25,26,27,28,29,200,201,\ldots,$$

Find the 1978-th digit written.

494 Problem (AHSME 1998) Call a 7-digit telephone number $d_1d_2d_3-d_4d_5d_6d_7$ *memorable* if the prefix sequence $d_1d_2d_3$ is exactly the same as either of the sequences $d_4d_5d_6$ or $d_5d_6d_7$ or possibly both. Assuming that each d_i can be any of the ten decimal digits $0,1,2,\ldots,9$, find the number of different memorable telephone numbers.

495 Problem Three-digit numbers are made using the digits $\{1,3,7,8,9\}$.

1. How many of these integers are there?
2. How many are even?
3. How many are palindromes?
4. How many are divisible by 3?

496 Problem (AHSME 1989) Five people are sitting at a round table. Let $f\geq 0$ be the number of people sitting next to at least one female, and let $m\geq 0$ be the number of people sitting next to at least one male. Find the number of possible ordered pairs (f,m).

497 Problem How many integers less than 10000 can be made with the eight digits $0,1,2,3,4,5,6,7$?

498 Problem (ARML 1999) In how many ways can one arrange the numbers $21,31,41,51,61,71$, and 81 such that the sum of every four consecutive numbers is divisible by 3?

499 Problem Let S be the set of all natural numbers whose digits are chosen from the set $\{1,3,5,7\}$ such that no digits are repeated. Find the sum of the elements of S.

500 Problem Find the number of ways to choose a pair $\{a,b\}$ of distinct numbers from the set $\{1,2,\ldots,50\}$ such that

1. $|a-b| = 5$
2. $|a-b|\leq 5$.

501 Problem (AIME 1994) Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1)+p(2)+\cdots+p(999).$$

Find S.

5.4 Permutations without Repetitions

502 Definition We define the symbol ! (factorial), as follows: $0! = 1$, and for integer $n \geq 1$,

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

$n!$ is read *n factorial*.

503 Example We have

$$\begin{aligned} 1! &= 1, \\ 2! &= 1 \cdot 2 = 2, \\ 3! &= 1 \cdot 2 \cdot 3 = 6, \\ 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24, \\ 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120. \end{aligned}$$

504 Example We have

$$\begin{aligned} \frac{7!}{4!} &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 210, \\ \frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1), \\ \frac{(n-2)!}{(n+1)!} &= \frac{(n-2)!}{(n+1)(n)(n-1)(n-2)!} = \frac{1}{(n+1)(n)(n-1)}. \end{aligned}$$

505 Definition Let $x_1, x_2, \ldots, x_n$ be n distinct objects. A *permutation* of these objects is simply a rearrangement of them.

506 Example There are 24 permutations of the letters in $MATH$, namely

$MATH$	$MAHT$	$MTAH$	$MTHA$	$MHTA$	$MHAT$
$AMTH$	$AMHT$	$ATMH$	$ATHM$	$AHTM$	$AHMT$
$TAMH$	$TAHM$	$TMAH$	$TMHA$	$THMA$	$THAM$
$HATM$	$HAMT$	$HTAM$	$HTMA$	$HMTA$	$HMAT$

507 Theorem Let $x_1, x_2, \ldots, x_n$ be n distinct objects. Then there are $n!$ permutations of them.

Proof: *The first position can be chosen in n ways, the second object in $n-1$ ways, the third in $n-2$, etc. This gives*

$$n(n-1)(n-2)\cdots 2 \cdot 1 = n!.$$

❑

508 Example The number of permutations of the letters of the word $RETICULA$ is $8! = 40320$.

509 Example A bookshelf contains 5 German books, 7 Spanish books and 8 French books. Each book is different from one another.

1. How many different arrangements can be done of these books?
2. How many different arrangements can be done of these books if books of each language must be next to each other?
3. How many different arrangements can be done of these books if all the French books must be next to each other?
4. How many different arrangements can be done of these books if no two French books must be next to each other?

Solution:

1. We are permuting $5+7+8=20$ objects. Thus the number of arrangements sought is $20! = 2432902008176640000$.
2. "Glue" the books by language, this will assure that books of the same language are together. We permute the 3 languages in $3!$ ways. We permute the German books in $5!$ ways, the Spanish books in $7!$ ways and the French books in $8!$ ways. Hence the total number of ways is $3!5!7!8! = 146313216000$.
3. Align the German books and the Spanish books first. Putting these $5+7=12$ books creates $12+1=13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that all the French books are next each other, we "glue" them together and put them in one of these spaces. Now, the French books can be permuted in $8!$ ways and the non-French books can be permuted in $12!$ ways. Thus the total number of permutations is
$$(13)8!12! = 251073478656000.$$
4. Align the German books and the Spanish books first. Putting these $5+7=12$ books creates $12+1=13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that no two French books are next to each other, we put them into these spaces. The first French book can be put into any of 13 spaces, the second into any of 12, etc., the eighth French book can be put into any 6 spaces. Now, the non-French books can be permuted in $12!$ ways. Thus the total number of permutations is
$$(13)(12)(11)(10)(9)(8)(7)(6)12!,$$
which is 24856274386944000.

Homework

510 Problem How many changes can be rung with a peal of five bells?

511 Problem A bookshelf contains 3 Russian novels, 4 German novels, and 5 Spanish novels. In how many ways may we align them if

1. there are no constraints as to grouping?
2. all the Spanish novels must be together?
3. no two Spanish novels are next to one another?

512 Problem How many permutations of the word **IMPURE** are there? How many permutations start with **P** and end in **U**? How many permutations are there if the **P** and the **U** must always be together in the order **PU**? How many permutations are there in which no two vowels (**I, U, E**) are adjacent?

513 Problem How many arrangements can be made of out of the letters of the word **DRAUGHT**, the vowels never separated?

514 Problem (AIME 1991) Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between 0 and 1 will $20!$ be the resulting product?

515 Problem (AMC12 2001) A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?

516 Problem How many trailing 0's are there when $1000!$ is multiplied out?

517 Problem In how many ways can 8 people be seated in a row if

1. there are no constraints as to their seating arrangement?
2. persons X and Y must sit next to one another?
3. there are 4 women and 4 men and no 2 men or 2 women can sit next to each other?
4. there are 4 married couples and each couple must sit together?
5. there are 4 men and they must sit next to each other?

5.5 Permutations with Repetitions

We now consider permutations with repeated objects.

518 Example In how many ways may the letters of the word

$$MASSACHUSETTS$$

be permuted?

Solution: We put subscripts on the repeats forming

$$MA_1S_1S_2A_2CHUS_3ET_1T_2S_4.$$

There are now 13 distinguishable objects, which can be permuted in 13! different ways by Theorem 507. For each of these 13! permutations, A_1A_2 can be permuted in 2! ways, $S_1S_2S_3S_4$ can be permuted in 4! ways, and T_1T_2 can be permuted in 2! ways. Thus the over count 13! is corrected by the total actual count

$$\frac{13!}{2!4!2!} = 64864800.$$

A reasoning analogous to the one of example 518, we may prove

519 Theorem Let there be k types of objects: n_1 of type 1; n_2 of type 2; etc. Then the number of ways in which these $n_1 + n_2 + \cdots + n_k$ objects can be rearranged is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!n_2!\cdots n_k!}.$$

520 ExampleT In how many ways may we permute the letters of the word *56 Td [(4)-4.58984]TJ /R47 9.96264 Tf 4.08 1.56 Td [(:)0.74105]T* and we find the permutations of each triplet. We have

(a,b,c)	Number of permutations
$(1,1,7)$	$\frac{3!}{2!}=3$
$(1,2,6)$	$3!=6$
$(1,3,5)$	$3!=6$
$(1,4,4)$	$\frac{3!}{2!}=3$
$(2,2,5)$	$\frac{3!}{2!}=3$
$(2,3,4)$	$3!=6$
$(3,3,3)$	$\frac{3!}{3!}=1$

Thus the number desired is

$$3+6+6+3+3+6+1=28.$$

522 Example In how many ways can the letters of the word **MURMUR** be arranged without letting two letters which are alike come together?

Solution: If we started with, say , **MU** then the **R** could be arranged as follows:

M	**U**	**R**		**R**	

,

M	**U**	**R**			**R**

,

M	**U**		**R**		**R**

.

In the first case there are $2!=2$ of putting the remaining **M** and **U**, in the second there are $2!=2$ and in the third there is only $1!$. Thus starting the word with **MU** gives $2+2+1=5$ possible arrangements. In the general case, we can choose the first letter of the word in 3 ways, and the second in 2 ways. Thus the number of ways sought is $3\cdot 2\cdot 5=30$.

523 Example In how many ways can the letters of the word **AFFECTION** be arranged, keeping the vowels in their natural order and not letting the two **F**'s come together?

Solution: There are $\frac{9!}{2!}$ ways of permuting the letters of **AFFECTION**. The 4 vowels can be permuted in $4!$ ways, and in only one of these will they be in their natural order. Thus there are $\frac{9!}{2!4!}$ ways of permuting the letters of **AFFECTION** in which their vowels keep their natural order.

Now, put the 7 letters of **AFFECTION** which are not the two **F**'s. This creates 8 spaces in between them where we put the two **F**'s. This means that there are $8\cdot 7!$ permutations of **AFFECTION** that keep the two **F**'s together. Hence there are $\frac{8\cdot 7!}{4!}$ permutations of **AFFECTION** where the vowels occur in their natural order.

In conclusion, the number of permutations sought is

$$\frac{9!}{2!4!}-\frac{8\cdot 7!}{4!}=\frac{8!}{4!}\left(\frac{9}{2}-1\right)=\frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!}\cdot\frac{7}{2}=5880$$

524 Example How many arrangements of five letters can be made of the letters of the word **PALLMALL**?

Solution: We consider the following cases:

❶ there are four **L**'s and a different letter. The different letter can be chosen in 3 ways, so there are $\frac{3\cdot 5!}{4!} = 15$ permutations in this case.

❷ there are three **L**'s and two **A**'s. There are $\frac{5!}{3!2!} = 10$ permutations in this case.

❸ there are three **L**'s and two different letters. The different letters can be chosen in 3 ways (either **P** and **A**; or **P** and **M**; or **A** and **M**), so there are $\frac{3\cdot 5!}{3!} = 60$ permutations in this case.

❹ there are two **L**'s, two **A**'s and a different letter from these two. The different letter can be chosen in 2 ways. There are $\frac{2\cdot 5!}{2!2!} = 60$ permutations in this case.

❺ there are two **L**'s and three different letters. The different letters can be chosen in 1 way. There are $\frac{1\cdot 5!}{2!} = 60$ permutations in this case.

❻ there is one **L**. This forces having two **A**'s and two other different letters. The different letters can be chosen in 1 way. There are $\frac{1\cdot 5!}{2!} = 60$ permutations in this case.

The total number of permutations is thus seen to be

$$15+10+60+60+60+60 = 265.$$

Homework

525 Problem In how many ways may one permute the letters of the word **MEPHISTOPHELES**?

526 Problem How many arrangements of four letters can be made out of the letters of **KAFFEEKANNE** without letting the three **E**'s come together?

527 Problem How many numbers can be formed with the digits

$$1,2,3,4,3,2,1$$

so that the odd digits occupy the odd places?

528 Problem In this problem you will determine how many different signals, each consisting of 10 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, 2 blue flags, and 1 orange flag, if flags of the same colour are identical.

❶ How many are there if there are no constraints on the order?

❷ How many are there if the orange flag must always be first?

❸ How many are there if there must be a white flag at the beginning and another white flag at the end?

529 Problem In how many ways may we write the number 10 as the sum of three positive integer summands? Here order counts, so, for example, $1+8+1$ is to be regarded different from $8+1+1$.

530 Problem Three distinguishable dice are thrown. In how many ways can they land and give a sum of 9?

531 Problem In how many ways can 15 different recruits be divided into three equal groups? In how many ways can they be drafted into three different regiments?

5.6 Combinations without Repetitions

532 Definition Let n,k be non-negative integers with $0 \leq k \leq n$. The symbol $\binom{n}{k}$ (read "*n choose k*") is defined and denoted by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}.$$

☞ *Observe that in the last fraction, there are k factors in both the numerator and denominator. Also, observe the boundary conditions*

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

533 Example We have

$$\begin{aligned}\binom{6}{3} &= \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20,\\ \binom{11}{2} &= \frac{11\cdot 10}{1\cdot 2} = 55,\\ \binom{12}{7} &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = 792,\\ \binom{110}{109} &= 110,\\ \binom{110}{0} &= 1.\end{aligned}$$

☞ *Since $n-(n-k)=k$, we have for integer n,k, $0 \leq k \leq n$, the symmetry identity*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

This can be interpreted as follows: if there are n different tickets in a hat, choosing k of them out of the hat is the same as choosing $n-k$ of them to remain in the hat.

534 Example

$$\binom{11}{9} = \binom{11}{2} = 55,$$
$$\binom{12}{5} = \binom{12}{7} = 792.$$

535 Definition Let there be n distinguishable objects. A k-*combination* is a selection of k, $(0 \leq k \leq n)$ objects from the n made without regards to order.

536 Example The 2-combinations from the list $\{X,Y,Z,W\}$ are

$$XY, XZ, XW, YZ, YW, WZ.$$

537 Example The 3-combinations from the list $\{X,Y,Z,W\}$ are

$$XYZ, XYW, XZW, YWZ.$$

538 Theorem Let there be n distinguishable objects, and let k, $0 \leq k \leq n$. Then the numbers of k-combinations of these n objects is $\binom{n}{k}$.

Proof: *Pick any of the k objects. They can be ordered in $n(n-1)(n-2)\cdots(n-k+1)$, since there are n ways of choosing the* first, *$n-1$ ways of choosing the* second, *etc. This particular choice of k objects can be permuted in $k!$ ways. Hence the total number of k-combinations is*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k}.$$

❑

539 Example From a group of 10 people, we may choose a committee of 4 in $\binom{10}{4} = 210$ ways.

540 Example In a group of 2 camels, 3 goats, and 10 sheep in how many ways may one choose 6 animals if

❶ there are no constraints in species?

❷ the two camels must be included?

❸ the two camels must be excluded?

❹ there must be at least 3 sheep?

❺ there must be at most 2 sheep?

❻ Joe Camel, Billy Goat and Samuel Sheep hate each other and they will not work in the same group. How many compatible committees are there?

Solution:

❶ There are $2+3+10 = 15$ animals and we must choose 6, whence $\binom{15}{6} = 5005$

❷ Since the 2 camels are included, we must choose $6-2 = 4$ more animals from a list of $15-2 = 13$ animals, so $\binom{13}{4} = 715$

❸ Since the 2 camels must be excluded, we must choose 6 animals from a list of $15-2 = 13$, so $\binom{13}{6} = 1716$

❹ If k sheep are chosen from the 10 sheep, $6-k$ animals must be chosen from the remaining 5 animals, hence

$$\binom{10}{3}\binom{5}{3} + \binom{10}{4}\binom{5}{2} + \binom{10}{5}\binom{5}{1} + \binom{10}{6}\binom{5}{0},$$

which simplifies to 4770.

❺ First observe that there cannot be 0 sheep, since that would mean choosing 6 other animals. Hence, there must be either 1 or 2 sheep, and so 3 or 4 of the other animals. The total number is thus

$$\binom{10}{2}\binom{5}{4} + \binom{10}{1}\binom{5}{5} = 235.$$

❻ A compatible group will either exclude all these three animals or include exactly one of them. This can be done in

$$\binom{12}{6} + \binom{3}{1}\binom{12}{5} = 3300$$

ways.

541 Example To count the number of shortest routes from A to B in figure 5.8 observe that any shortest path must consist of 6 horizontal moves and 3 vertical ones for a total of $6+3 = 9$ moves. Of these 9 moves once we choose the 6 horizontal ones the 3 vertical ones are determined. Thus there are $\binom{9}{6} = 84$ paths.

542 Example To count the number of shortest routes from A to B in figure 5.9 that pass through point O we count the number of paths from A to O (of which there are $\binom{5}{3} = 20$) and the number of paths from O to B (of which there are $\binom{4}{3} = 4$). Thus the desired number of paths is $\binom{5}{3}\binom{4}{3} = (20)(4) = 80$.

543 Example Consider the set of 5-digit positive integers written in decimal notation.

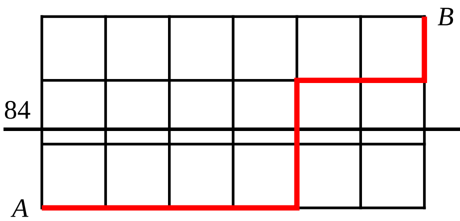

Figure 5.8: Example 541.

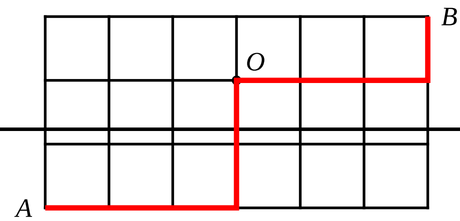

Figure 5.9: Example 542.

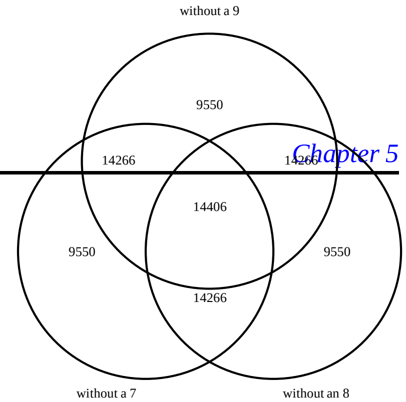

Figure 5.10: Example 543.

1. How many are there?
2. How many do not have a 9 in their decimal representation?
3. How many have at least one 9 in their decimal representation?
4. How many have exactly one 9?
5. How many have exactly two 9's?
6. How many have exactly three 9's?
7. How many have exactly four 9's?
8. How many have exactly five 9's?
9. How many have neither an 8 nor a 9 in their decimal representation?
10. How many have neither a 7, nor an 8, nor a 9 in their decimal representation?
11. How many have either a 7, an 8, or a 9 in their decimal representation?

Solution:

1. There are 9 possible choices for the first digit and 10 possible choices for the remaining digits. The number of choices is thus $9 \cdot 10^4 = 90000$.

2. There are 8 possible choices for the first digit and 9 possible choices for the remaining digits. The number of choices is thus $8 \cdot 9^4 = 52488$.

3. The difference $90000 - 52488 = 37512$.

4. We condition on the first digit. If the first digit is a 9 then the other four remaining digits must be different from 9, giving $9^4 = 6561$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{1} = 4$ ways of choosing were the 9 will be, and we have 9^3 ways of filling the 3 remaining spots. Thus in this case there are $8 \cdot 4 \cdot 9^3 = 23328$ such numbers. In total there are $6561 + 23328 = 29889$ five-digit positive integers with exactly one 9 in their decimal representation.

5. We condition on the first digit. If the first digit is a 9 then one of the remaining four must be a 9, and the choice of place can be accomplished in $\binom{4}{1} = 4$ ways. The other three remaining digits must be different from 9, giving $4 \cdot 9^3 = 2916$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{2} = 6$ ways of choosing were the two 9's will be, and we have 9^2 ways of filling the two remaining spots. Thus in this case there are $8 \cdot 6 \cdot 9^2 = 3888$ such numbers. Altogether there are $2916 + 3888 = 6804$ five-digit positive integers with exactly two 9's in their decimal representation.

6. Again we condition on the first digit. If the first digit is a 9 then two of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{2} = 6$ ways. The other two remaining digits must be different from 9, giving $6 \cdot 9^2 = 486$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{3} = 4$ ways of choosing were the three 9's will be, and we have 9 ways of filling the remaining spot. Thus in this case there are $8 \cdot 4 \cdot 9 = 288$ such numbers. Altogether there are $486 + 288 = 774$ five-digit positive integers with exactly three 9's in their decimal representation.

7. If the first digit is a 9 then three of the remaining four must be 9's, and the choice of place can be accomplished in $\binom{4}{3} = 4$ ways. The other remaining digit must be different from 9, giving $4 \cdot 9 = 36$ such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have $\binom{4}{4} = 4$ ways of choosing were the four 9's will be, thus filling all the spots. Thus in this case there are $8 \cdot 1 = 8$ such numbers. Altogether there are $36 + 8 = 44$ five-digit

positive integers with exactly three 9's in their decimal representation.

8. There is obviously only 1 such positive integer.

   Observe that
   $37512 = 29889 + 6804 + 774 + 44 + 1.$

9. We have 7 choices for the first digit and 8 choices for the remaining 4 digits, giving $7 \cdot 8^4 = 28672$ such integers.

10. We have 6 choices for the first digit and 7 choices for the remaining 4 digits, giving $6 \cdot 7^4 = 14406$ such integers.

11. We use inclusion-exclusion. From figure 5.10, the numbers inside the circles add up to 85854. Thus the desired number is $90000 - 85854 = 4146$.

Homework

544 Problem Verify the following.

❶ $\binom{20}{3} = 1140$

❷ $\binom{12}{4}\binom{12}{6} = 457380$

❸ $\dfrac{\binom{n}{1}}{\binom{n}{n-1}} = 1$

❹ $\binom{n}{2} = \dfrac{n(n-1)}{2}$

❺ $\binom{6}{1} + \binom{6}{3} + \binom{6}{6} = 2^5.$

❻ $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} = 2^6 - \binom{7}{6}$

545 Problem A publisher proposes to issue a set of dictionaries to translate from any one language to any other. If he confines his system to ten languages, how many dictionaries must be published?

546 Problem From a group of 12 people—7 of which are men and 5 women—in how many ways may choose a committee of 4 with 1 man and 3 women?

547 Problem N friends meet and shake hands with one another. How many handshakes?

548 Problem How many 4-letter words can be made by taking 4 letters of the word **RETICULA** and permuting them?

549 Problem (AHSME 1989) Mr. and Mrs. Zeta want to name baby Zeta so that its monogram (first, middle and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?

550 Problem In how many ways can $\{1,2,3,4\}$ be written as the union of two non-empty, disjoint subsets?

551 Problem How many lists of 3 elements taken from the set $\{1,2,3,4,5,6\}$ list the elements in increasing order?

552 Problem How many times is the digit 3 listed in the numbers 1 to 1000?

553 Problem How many subsets of the set $\{a,b,c,d,e\}$ have exactly 3 elements?

554 Problem How many subsets of the set $\{a,b,c,d,e\}$ have an odd number of elements?

555 Problem (AHSME 1994) Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?

556 Problem There are E (different) English novels, F (different) French novels, S (different) Spanish novels, and I (different) Italian novels on a shelf. How many different permutations are there if

❶ if there are no restrictions?

❷ if all books of the same language must be together?

❸ if all the Spanish novels must be together?

❹ if no two Spanish novels are adjacent?

❺ if all the Spanish novels must be together, and all the English novels must be together, but no Spanish novel is next to an English novel?

557 Problem How many committees of seven with a given chairman can be selected from twenty people?

558 Problem How many committees of seven with a given chairman and a given secretary can be selected from twenty people? Assume the chairman and the secretary are different persons.

559 Problem (AHSME 1990) How many of the numbers

$$100, 101, \ldots, 999,$$

have three different digits in increasing order or in decreasing order?

560 Problem There are twenty students in a class. In how many ways can the twenty students take five different tests if four of the students are to take each test?

561 Problem In how many ways can a deck of playing cards be arranged if no two hearts are adjacent?

562 Problem Given a positive integer n, find the number of quadruples $(a,b,c,d,)$ such that $0 \leq a \leq b \leq c \leq d \leq n$.

563 Problem There are T books on Theology, L books on Law and W books on Witchcraft on Dr. Faustus' shelf. In how many ways may one order the books

❶ there are no constraints in their order?

❷ all books of a subject must be together?

❸ no two books on Witchcraft are juxtaposed?

4. all the books on Witchcraft must be together?

564 Problem From a group of 20 students, in how many ways may a professor choose at least one in order to work on a project?

565 Problem From a group of 20 students, in how many ways may a professor choose an even number number of them, but at least four in order to work on a project?

566 Problem How many permutations of the word

CHICHICUILOTE

are there

1. if there are no restrictions?
2. if the word must start in an **I** and end also in an **I**?
3. if the word must start in an **I** and end in a **C**?
4. if the two **H**'s are adjacent?
5. if the two **H**'s are not adjacent?
6. if the particle **LOTE** must appear, with the letters in this order?

567 Problem There are M men and W women in a group. A committee of C people will be chosen. In how many ways may one do this if

1. there are no constraints on the sex of the committee members?
2. there must be exactly T women?
3. A committee must always include George and Barbara?
4. A committee must always exclude George and Barbara?

Assume George and Barbara form part of the original set of people.

568 Problem There are M men and W women in a group. A committee of C people will be chosen. In how many ways may one do this if George and Barbara are feuding and will not work together in a committee? Assume George and Barbara form part of the original set of people.

569 Problem Out of 30 consecutive integers, in how many ways can three be selected so that their sum be even?

570 Problem In how many ways may we choose three distinct integers from $\{1,2,\ldots,100\}$ so that one of them is the average of the other two?

571 Problem How many vectors $(a_1,a_2,\ldots,a_k)$ with integral

$$a_i \in \{1,2,\ldots,n\}$$

are there satisfying

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n?$$

572 Problem A square chessboard has 16 squares (4 rows and 4 columns). One puts 4 checkers in such a way that only one checker can be put in a square. Determine the number of ways of putting these checkers if

1. there must be exactly one checker per row and column.
2. there must be exactly one column without a checker.
3. there must be at least one column without a checker.

573 Problem A box contains 4 red, 5 white, 6 blue, and 7 magenta balls. In how many of all possible samples of size 5, chosen without replacement, will every colour be represented?

574 Problem In how many ways can eight students be divided into four indistinguishable teams of two each?

575 Problem How many ways can three boys share fifteen different sized pears if the youngest gets seven pears and the other two boys get four each?those in which the digit 1 occurs or those in which it does not occur?

576 Problem Four writers must write a book containing seventeen chapters. The first and third writers must each write five chapters, the second must write four chapters, and the fourth must write three chapters. How many ways can the book be divided between the authors? What if the first and third had to write ten chapters combined, but it did not matter which of them wrote how many (i.e. the first could write ten and the third none, the first could write none and the third one, etc.)?

577 Problem In how many ways can a woman choose three lovers or more from seven eligible suitors? may be opened by depressing—in any order—the correct five buttons. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?

1. how many straight lines are determined?
2. how many straight lines pass through a particular point?
3. how many triangles are determined?
4. how many triangles have a particular point as a vertex?

578 Problem In how many ways can you pack twelve books into four parcels if one parcel has one book, another has five books, and another has two books, and another has four books?

579 Problem In how many ways can a person invite three of his six friends to lunch every day for twenty days if he has the option of inviting the same or different friends from previous days?

580 Problem A committee is to be chosen from a set of nine women and five men. How many ways are there to form the committee if the committee has three men and three women?

581 Problem At a dance there are b boys and g girls. In how many ways can they form c couples consisting of different sexes?

582 Problem From three Russians, four Americans, and two Spaniards, how many selections of people can be made, taking at least one of each kind?

583 Problem The positive integer r satisfies

$$\frac{1}{\binom{9}{r}} - \frac{1}{\binom{10}{r}} = \frac{11}{6\binom{11}{r}}.$$

Find r.

584 Problem If $11\binom{28}{2r} = 225\binom{24}{2r-4}$, find r.

585 Problem Compute the number of ten-digit numbers which contain only the digits $1,2$, and 3 with the digit 2 appearing in each number exactly twice.

586 Problem Prove *Pascal's Identity*:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

for integers $1 \leq k \leq n$.

587 Problem Give a combinatorial interpretation of **Newton's Identity**:

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \quad (5.1)$$

for $0 \leq k \leq r \leq n$.

588 Problem Give a combinatorial proof that for integer $n \geq 1$,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

589 Problem In each of the 6-digit numbers

$$333333, 225522, 118818, 707099,$$

each digit in the number appears at least twice. Find the number of such 6-digit natural numbers.

590 Problem In each of the 7-digit numbers

$$1001011, 5550000, 3838383, 7777777,$$

each digit in the number appears at least thrice. Find the number of such 7-digit natural numbers.

591 Problem (AIME 1983) The numbers $1447, 1005$ and 1231 have something in common: each is a four-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

592 Problem If there are fifteen players on a baseball team, how many ways can the coach choose nine players for the starting lineup if it does not matter which position the players play (i.e., no distinction is made between player A playing shortstop, left field, or any other positions as long as he is on the field)? How many ways are there if it does matter which position the players play?

593 Problem (AHSME 1989) A child has a set of 96 distinct blocks. Each block is one of two materials (*plastic*, *wood*), three sizes (*small*, *medium*, *large*), four colours (*blue*, *green*, *red*, *yellow*), and four shapes (*circle*, *hexagon*, *square*, *triangle*). How many blocks in the set are different from the "*plastic medium red circle*" in exactly two ways? (The "*wood medium red square*" is such a block.)

594 Problem (AHSME 1989) Suppose that k boys and $n-k$ girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row

$$GBBGGGBGBGGGBGBGGBG,$$

with $k = 7, n = 20$ we have $S = 12$. Shew that the average value of S is $\frac{2k(n-k)}{n}$.

595 Problem There are four different kinds of sweets at a sweets store. I want to buy up to four sweets (I'm not sure if I want none, one, two, three, or four sweets) and I refuse to buy more than one of any kind of sweet. How many ways can I do this?

596 Problem Suppose five people are in a lift. There are eight floors that the lift stops at. How many distinct ways can the people exit the lift if either one or zero people exit at each stop?

597 Problem If the natural numbers from 1 to 222222222 are written down in succession, how many 0's are written?

598 Problem In how many ways can we distribute k identical balls into n different boxes so that each box contains at most one ball and no two consecutive boxes are empty?

599 Problem In a row of n seats in the doctor's waiting-room k patients sit down in a particular order from left to right. They sit so that no two of them are in adjacent seats. In how many ways could a suitable set of k seats be chosen?

5.7 Combinations with Repetitions

600 Theorem (De Moivre) Let n be a positive integer. The number of positive integer solutions to

$$x_1 + x_2 + \cdots + x_r = n$$

is

$$\binom{n-1}{r-1}.$$

Proof: *Write n as*

$$n = 1 + 1 + \cdots + 1 + 1,$$

where there are n 1s and $n-1$ +s. To decompose n in r summands we only need to choose $r-1$ pluses from the $n-1$, which proves the theorem. ❑

601 Example In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, $1+7+1$ is to be regarded different from $7+1+1$.

Solution: Notice that this is example 521. We are seeking integral solutions to

$$a+b+c = 9, \quad a > 0, b > 0, c > 0.$$

By Theorem 600 this is

$$\binom{9-1}{3-1} = \binom{8}{2} = 28.$$

602 Example In how many ways can 100 be written as the sum of four positive integer summands?

Solution: We want the number of positive integer solutions to

$$a+b+c+d=100,$$

which by Theorem 600 is

$$\binom{99}{3}=156849.$$

603 Corollary Let n be a positive integer. The number of non-negative integer solutions to

$$y_1+y_2+\cdots+y_r=n$$

is

$$\binom{n+r-1}{r-1}.$$

Proof: *Put* $x_r-1=y_r$. *Then* $x_r\geq 1$. *The equation*

$$x_1-1+x_2-1+\cdots+x_r-1=n$$

is equivalent to

$$x_1+x_2+\cdots+x_r=n+r,$$

which from Theorem 600, has

$$\binom{n+r-1}{r-1}$$

solutions. ❑

604 Example Find the number of quadruples (a,b,c,d) of integers satisfying

$$a+b+c+d=100,\ a\geq 30, b>21, c\geq 1, d\geq 1.$$

Solution: Put $a'+29=a, b'+20=b$. Then we want the number of positive integer solutions to

$$a'+29+b'+21+c+d=100,$$

or

$$a'+b'+c+d=50.$$

By Theorem 600 this number is

$$\binom{49}{3}=18424.$$

605 Example There are five people in a lift of a building having eight floors. In how many ways can they choose their floor for exiting the lift?

Solution: Let x_i be the number of people that floor i receives. We are looking for non-negative solutions of the equation

$$x_1+x_2+\cdots+x_8=5.$$

Putting $y_i=x_i+1$, then

$$x_1+x_2+\cdots+x_8=5 \quad\Longrightarrow\quad (y_1-1)+(y_2-1)+\cdots+(y_8-1)=5$$

$$\Longrightarrow\quad y_1+y_2+\cdots+y_8=13,$$

whence the number sought is the number of positive solutions to

$$y_1 + y_2 + \cdots + y_8 = 13$$

which is $\binom{12}{7} = 792$.

606 Example Find the number of quadruples (a, b, c, d) of non-negative integers which satisfy the inequality

$$a + b + c + d \leq 2001.$$

Solution: The number of non-negative solutions to

$$a + b + c + d \leq 2001$$

equals the number of solutions to

$$a + b + c + d + f = 2001$$

where f is a non-negative integer. This number is the same as the number of positive integer solutions to

$$a_1 - 1 + b_1 - 1 + c_1 - 1 + d_1 - 1 + f_1 - 1 = 2001,$$

which is easily seen to be $\binom{2005}{4}$.

607 Example

How many integral solutions to the equation

$$a + b + c + d = 100,$$

are there given the following constraints:

$$1 \leq a \leq 10,\ b \geq 0,\ c \geq 2, 20 \leq d \leq 30?$$

Solution: We use Inclusion-Exclusion. There are $\binom{80}{3} = 82160$ integral solutions to

$$a + b + c + d = 100,\ \ a \geq 1, b \geq 0, c \geq 2, d \geq 20.$$

Let A be the set of solutions with

$$a \geq 11, b \geq 0, c \geq 2, d \geq 20$$

and B be the set of solutions with

$$a \geq 1, b \geq 0, c \geq 2, d \geq 31.$$

Then $\operatorname{card}(A) = \binom{70}{3}$, $\operatorname{card}(B) = \binom{69}{3}$, $\operatorname{card}(A \cap B) = \binom{59}{3}$ and so

$$\operatorname{card}(A \cup B) = \binom{70}{3} + \binom{69}{3} - \binom{59}{3} = 74625.$$

The total number of solutions to

$$a + b + c + d = 100$$

with

$$1 \leq a \leq 10,\ b \geq 0,\ c \geq 2, 20 \leq d \leq 30$$

is thus

$$\binom{80}{3} - \binom{70}{3} - \binom{69}{3} + \binom{59}{3} = 7535.$$

Homework

608 Problem How many positive integral solutions are there to

$$a+b+c=10?$$

609 Problem Three fair dice, one red, one white, and one blue are thrown. In how many ways can they land so that their sum be 10 ?

610 Problem Adena has twenty indistinguishable pieces of sweet-meats that she wants to divide amongst her five stepchildren. How many ways can she divide the sweet-meats so that each stepchild gets at least two pieces of sweet-meats?

611 Problem How many integral solutions are there to the equation

$$x_1+x_2+\cdots+x_{100}=n$$

subject to the constraints

$$x_1\geq 1, x_2\geq 2, x_3\geq 3, \ldots, x_{99}\geq 99, x_{100}\geq 100?$$

612 Problem (AIME 1998) Find the number of ordered quadruplets (a,b,c,d) of positive odd integers satisfying $a+b+c+d=98$.

5.8 The Binomial Theorem

We recall that the symbol

$$\binom{n}{k}=\frac{n!}{(n-k)!k!}, n,k\in\mathbb{N}, 0\leq k\leq n,$$

counts the number of ways of selecting k different objects from n different objects. Observe that we have the following *absorbtion identity:*

$$\binom{n}{k}=\frac{n}{k}\binom{n-1}{k-1}.$$

613 Example Prove *Pascal's Identity*:

$$\binom{n}{k}=\binom{n-1}{k-1}+\binom{n-1}{k},$$

for integers $1\leq k\leq n$.

Solution: We have

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\ &= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right)\\ &= \frac{(n-1)!}{(n-k-1)!(k-1)!}\frac{n}{(n-k)k}\\ &= \frac{n!}{(n-k)!k!}.\\ &= \binom{n}{k}\end{aligned}$$

614 Example Prove *Newton's Identity*:

$$\binom{n}{i}\binom{i}{j}=\binom{n}{j}\binom{n-j}{j-i},$$

for integers $0\leq j\leq i\leq n$.

Solution: We have

$$\binom{n}{i}\binom{i}{j}=\frac{n!i!}{i!(n-i)!j!(i-j)!}=\frac{n!(n-j)!}{(n-j)!j!(n-i)!(i-j)!}$$

which is the same as

$$\binom{n}{j}\binom{n-j}{i-j}.$$

Using Pascal's Identity we obtain *Pascal's Triangle.*

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

When the numerical values are substituted, the triangle then looks like this.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

We see from Pascal's Triangle that binomial coefficients are symmetric. This symmetry is easily justified by the identity $\binom{n}{k} = \binom{n}{n-k}$. We also notice that the binomial coefficients tend to increase until they reach the middle, and that they decrease symmetrically. That is, the $\binom{n}{k}$ satisfy

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{[n/2]-1} < \binom{n}{[n/2]} > \binom{n}{[n/2]+1} > \binom{n}{[n/2]+2} > \cdots > \binom{n}{n-1} > \binom{n}{n}$$

if n is even, and that

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{[n/2]-1} < \binom{n}{[n/2]} = \binom{n}{[n/2]+1} > \binom{n}{[n/2]+2} > \binom{n}{[n/2]+3} > \cdots > \binom{n}{n-1} > \binom{n}{n}$$

for odd n. We call this property the *unimodality* of the binomial coefficients. For example, without finding the exact numerical values we can see that $\binom{200}{17} < \binom{200}{69}$ and that $\binom{200}{131} = \binom{200}{69} < \binom{200}{99}$.
We now present some examples on the use of binomial coefficients.

615 Example The *Catalan number of order n* is defined as

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Prove that C_n is an integer for all natural numbers n.

Solution: Observe that

$$\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1},$$

the difference of two integers.

616 Example (Putnam 1972) Shew that no four consecutive binomial coefficients

$$\binom{n}{r},\binom{n}{r+1},\binom{n}{r+2},\binom{n}{r+3}$$

(n,r positive integers and $r+3\leq n$) are in arithmetic progression.

Solution: Assume that $a=\binom{n}{r}, a+d=\binom{n}{r+1}, a+2d=\binom{n}{r+2}, a+3d=\binom{n}{r+3}$. This yields

$$2\binom{n}{r+1}=\binom{n}{r}+\binom{n}{r+2},$$

or equivalently

$$2=\frac{r+1}{n-r}+\frac{n-r-1}{r+2} \qquad (*).$$

This is a quadratic equation in r, having r as one of its roots. The condition that the binomial coefficients are in arithmetic progression means that $r+1$ is also a root of $(*)$. Replacing r by $n-r-2$ we also obtain

$$2=\frac{n-r-1}{r+2}+\frac{r+1}{n-r},$$

which is the same as $(*)$. This means that $n-r-3$ and $n-r-2$ are also roots of $(*)$. Since a quadratic equation can only have two roots, we must have $r=n-r-3$. The four binomial coefficients must then be

$$\binom{2r+3}{r},\ \binom{2r+3}{r+1},\ \binom{2r+3}{r+2},\ \binom{2r+3}{r+3}.$$

But these cannot be in an arithmetic progression, since binomial coefficients are unimodal and symmetric.

617 Example Let $N(a)$ denote the number of solutions to the equation $a=\binom{n}{k}$ for nonnegative integers n,k. For example, $N(1)=\infty, N(3)=N(5)=2, N(6)=3$, etc. Prove that $N(a)\leq 2+2\log_2 a$.

Solution: Let b be the first time that $\binom{2b}{b}>a$. By the unimodality of the binomial coefficients, $\binom{i+j}{i}=\binom{i+j}{j}$ is monotonically increasing in i and j. Hence

$$\binom{b+i+b+j}{b+j}\geq\binom{b+b+j}{b}\geq\binom{2b}{b}>a$$

for all $i,j\geq 0$. Hence $\binom{i+j}{j}=a$ implies $i<b$, or $j<b$. Also, for each fixed value of i (or j), $\binom{i+j}{i}=a$ has at most one solution. It follows that $N(a)<2b$. Since

$$a\geq\binom{2(b-1)}{b-1}\geq 2^{b-1},$$

it follows that $b\leq\log_2 a+1$, and the statement is proven.
We now use Pascal's Triangle in order to expand the binomial

$$(a+b)^n.$$

The *Binomial Theorem* states that for $n\in\mathbb{Z}, n\geq 0$,

$$(1+x)^n=\sum_{k=0}^{n}\binom{n}{k}x^k.$$

As a way of proving this, we observe that expanding

$$\underbrace{(1+x)(1+x)\cdots(1+x)}_{n \text{ factors}}$$

consists of adding up all the terms obtained from multiplying either a 1 or a x from the first set of parentheses times either a 1 or an x from the second set of parentheses etc. To get x^k, x must be chosen from exactly k of the sets of parentheses. Thus the number of x^k terms is $\binom{n}{k}$. It follows that

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

618 Example Prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Solution: This follows from letting $x = 1$ in the expansion

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

619 Example Prove that for integer $n \geq 1$,

$$\sum_{j=i}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i}2^{n-i},\ i \leq n.$$

Solution: Recall that by Newton's Identity

$$\binom{n}{j}\binom{j}{i} = \binom{n}{i}\binom{n-i}{j-i}.$$

Thus

$$\sum_{j=0}^{n} \binom{n}{j}\binom{j}{i} = \binom{n}{i}\sum_{j=0}^{n}\binom{n-i}{j-i}.$$

But upon re-indexing

$$\sum_{j=0}^{n} \binom{n-i}{j-i} = \sum_{j=0}^{n-i}\binom{n-i}{j} = 2^{n-i},$$

by the preceding problem. Thus the assertion follows.

620 Example Prove that

$$\sum_{k \leq n} \binom{m+k}{k} = \binom{n+m+1}{n}.$$

Solution: Using Pascal's Identity

$$\begin{aligned}
\sum_{k=0}^{n}\binom{k+m}{k} &= \binom{0+m}{-1}+\binom{0+m}{0}+\binom{1+m}{1}+\binom{2+m}{2}+\binom{3+m}{3} \\
&\quad +\cdots+\binom{n-1+m}{n-1}+\binom{n+m}{n} \\
&= \binom{1+m}{0}+\binom{1+m}{1}+\binom{2+m}{2}+\binom{3+m}{3} \\
&\quad +\cdots+\binom{n-1+m}{n-1}+\binom{n+m}{n} \\
&= \binom{2+m}{1}+\binom{2+m}{2}+\binom{3+m}{3} \\
&\quad +\cdots+\binom{n-1+m}{n-1}+\binom{n+m}{n} \\
&= \binom{3+m}{2}+\binom{3+m}{3} \\
&\quad +\cdots+\binom{n-1+m}{n-1}+\binom{n+m}{n} \\
&\ \vdots \\
&= \binom{n+m}{n-1}+\binom{n+m}{n} \\
&= \binom{n+m+1}{n},
\end{aligned}$$

which is what we wanted.

621 Example Find a closed formula for

$$\sum_{0\le k\le m}\binom{m}{k}\binom{n}{k}^{-1} \quad n\ge m\ge 0.$$

Solution: Using Newton's Identity,

$$\sum_{0\le k\le m}\binom{m}{k}\binom{n}{k}^{-1}=\binom{n}{m}^{-1}\sum_{0\le k\le m}\binom{n-k}{m-k}.$$

Re-indexing,

$$\sum_{0\le k\le m}\binom{n-k}{m-k}=\sum_{k\le m}\binom{n-m+k}{k}=\binom{n+1}{m},$$

by the preceding problem. Thus

$$\sum_{0\le k\le m}\binom{m}{k}\binom{n}{k}^{-1}=\binom{n+1}{m}\Big/\binom{n}{m}=\frac{n+1}{n+1-m}.$$

622 Example Simplify

$$\sum_{0\le k\le 50}\binom{100}{2k}.$$

Solution: By the Binomial Theorem

$$\begin{aligned}
(1+1)^{100} &= \binom{100}{0}+\binom{100}{1}+\binom{100}{2}+\ldots+\binom{100}{99}+\binom{100}{100} \\
(1-1)^{100} &= \binom{100}{0}-\binom{100}{1}+\binom{100}{2}-\ldots-\binom{100}{99}+\binom{100}{100},
\end{aligned}$$

whence summing both columns

$$2^{100} = 2\binom{100}{0} + 2\binom{100}{2} + \ldots + 2\binom{100}{100}.$$

Dividing by 2, the required sum is thus 2^{99}.

623 Example Simplify

$$\sum_{k=1}^{50} \binom{100}{2k-1}.$$

Solution: We know that

$$\sum_{k=0}^{100} \binom{100}{k} = 2^{100}$$

and

$$\sum_{k=0}^{50} \binom{100}{2k} = 2^{99}.$$

The desired sum is the difference of these two values $2^{100} - 2^{99} = 2^{99}$.

624 Example Simplify

$$\sum_{k=1}^{10} 2^k \binom{11}{k}.$$

Solution: By the Binomial Theorem, the complete sum $\sum_{k=0}^{11} \binom{11}{k} 2^k = 3^{11}$. The required sum lacks the zeroth term, $\binom{11}{0} 2^0 = 1$, and the eleventh term, $\binom{11}{11} 2^{11}$ from this complete sum. The required sum is thus $3^{11} - 2^{11} - 1$.

625 Example Which coefficient of the expansion of

$$\left(\frac{1}{3} + \frac{2}{3}x\right)^{10}$$

has the greatest magnitude?

Solution: By the Binomial Theorem,

$$\left(\frac{1}{3} + \frac{2}{3}x\right)^{10} = \sum_{k=0}^{10} \binom{10}{k} (1/3)^k (2x/3)^{10-k} = \sum_{k=0}^{10} a_k x^k.$$

We consider the ratios $\frac{a_k}{a_{k-1}}, k = 1, 2, \ldots n$. This ratio is seen to be

$$\frac{a_k}{a_{k-1}} = \frac{2(10-k+1)}{k}.$$

This will be < 1 if $k < 22/3 < 8$. Thus $a_0 < a_1 < a_2 < \ldots < a_7$. If $k > 22/3$, the ratio above will be < 1. Thus $a_7 > a_8 > a_9 > a_{10}$. The largest term is that of $k = 7$, i.e. the eighth term.

626 Example At what positive integral value of x is the x^4 term in the expansion of $(2x+9)^{10}$ greater than the adjacent terms?

Solution: We want to find integral x such that

$$\binom{10}{4}(2x)^4(9)^6 \geq \binom{10}{3}(2x)^3(9)^7,$$

and

$$\binom{10}{4}(2x)^4(9)^6 \geq \binom{10}{5}(2x)^5(9)^5.$$

After simplifying the factorials, the two inequalities sought are

$$x \geq 18/7$$

and

$$15/4 \geq x.$$

The only integral x that satisfies this is $x = 3$.

627 Example Prove that for integer $n \geq 1$,

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Solution: Using the absorption identity

$$\sum_{k=0}^{n} n\binom{n-1}{k-1} = \sum_{k=0}^{n} k\binom{n}{k},$$

with the convention that $\binom{n-1}{-1} = 0$. But since

$$\sum_{k=0}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1},$$

we obtain the result once again.

628 Example Find a closed formula for

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}.$$

Solution: Using the absorption identity

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1} = \frac{1}{n+1}(2^{n+1}-1).$$

629 Example Prove that if m, n are nonnegative integers then

$$\binom{n+1}{m+1} = \sum_{k=m}^{n}\binom{k}{m}.$$

Solution: Using Pascal's Identity

$$\begin{aligned}
\sum_{k=m}^{n}\binom{k}{m} &= \binom{m}{m+1}+\binom{m}{m}+\binom{m+1}{m}+\cdots+\binom{n}{m}\\
&= \binom{m+1}{m+1}+\binom{m+1}{m}+\binom{m+2}{m}+\cdots+\binom{n}{m}\\
&= \binom{m+2}{m+1}+\binom{m+2}{m}+\binom{m+3}{m}+\cdots+\binom{n}{m}\\
&\vdots\\
&= \binom{n}{m+1}+\binom{n}{m}\\
&= \binom{n+1}{m+1}.
\end{aligned}$$

630 Example Find a closed form for

$$\sum_{k\leq n} k(k+1).$$

Solution: Let

$$S=\sum_{k\leq n} k(k+1).$$

Then

$$S/2! = \sum_{k\leq n}\frac{k(k+1)}{2!} = \sum_{k\leq n}\binom{k+1}{2}.$$

By the preceding problem

$$\sum_{k\leq n}\binom{k+1}{2} = \binom{n+2}{3}.$$

We gather that $S = 2\binom{n+2}{3} = n(n+1)(n+2)/3$.

Practice

631 Problem Prove that

$$\sum_{0\leq k\leq n/2}\binom{n}{2k+1} = 2^{n-1}.$$

632 Problem Expand

$$\frac{1}{2}(1+\sqrt{x})^{100}+\frac{1}{2}(1-\sqrt{x})^{100}.$$

633 Problem Four writers must write a book containing seventeen chapters. The first and third writers must each write five chapters, the second must write four chapters, and the fourth writer must write three chapters. How many ways can the book be written? What if the first and third writers had to write ten chapters combined, but it did not matter which of them wrote how many (e.g., the first could write ten and the third none, the first could write none and the third one, etc.)?

634 Problem Prove that

$$\sum_{j_n=1}^{m}\sum_{j_{n-1}=1}^{j_n}\cdots\sum_{k=1}^{j_1} 1 = \binom{n+m}{n+1}.$$

635 Problem The expansion of $(x+2y)^{20}$ contains two terms with the same coefficient, $Kx^a y^b$ and $Kx^{a+1}y^{b-1}$. Find a.

636 Problem Prove that for $n\in\mathbb{N}, n>1$ the identity

$$\sum_{k=1}^{n}(-1)^{k-1}k\binom{n}{k} = 0$$

holds true.

637 Problem If n is an even natural number, shew that

$$\begin{aligned}
&\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}\\
&\quad+\frac{1}{5!(n-5)!}+\cdots+\frac{1}{(n-1)!1!}\\
&\quad=\frac{2^{n-1}}{n!}.
\end{aligned}$$

638 Problem Find a closed formula for

$$\sum_{0\leq k\leq n}\binom{n-k}{k}(-1)^k.$$

639 Problem What is the exact numerical value of

$$\sum_{k\le 100} k3^k \binom{100}{k}?$$

640 Problem Find a closed formula for

$$\sum_{k=1}^{n} k^2 - k.$$

641 Problem Find a closed formula for

$$\sum_{0\le k\le n} k\binom{m-k-1}{n-k-1} \quad m > n \ge 0.$$

642 Problem What is the exact numerical value of

$$\sum_{k\le 100} \frac{5^k}{k+1}\binom{100}{k}?$$

643 Problem Find n if $\binom{10}{4}+\binom{10}{3}=\binom{n}{4}$.

644 Problem If

$$\binom{1991}{1}+\binom{1991}{3}+\binom{1991}{5}+\cdots+\binom{1991}{1991}=2^a,$$

find a.

645 Problem True or False: $\binom{20}{5}=\binom{20}{15}$.

646 Problem True or False:

$$49\binom{48}{9}=10\binom{49}{10}.$$

647 Problem What is the coefficient of $x^{24}y^{24}$ in the expansion

$$(2x^3+3y^2)^{20}?$$

648 Problem What is the coefficient of $x^{12}y^7$ in the expansion

$$(x^{3/2}+y)^{15}?$$

649 Problem What is the coefficient of x^4y^6 in

$$(x\sqrt{2}-y)^{10}?$$

650 Problem Shew that the binomial coefficients satisfy the following hexagonal property:

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n+1}{k+1}\binom{n}{k-1}.$$

651 Problem (AIME 1991) In the expansion

$$(1+0.2)^{1000}=\sum_{k=0}^{1000}\binom{1000}{k}(0.2)^k,$$

which one of the 1001 terms is the largest?

652 Problem (Putnam 1971) Shew that for $0<\varepsilon<1$ the expression

$$(x+y)^n(x^2-(2-\varepsilon)xy+y^2)$$

is a polynomial with positive coefficients for integral n sufficiently large. For $\varepsilon=.002$ find the smallest admissible value of n.

653 Problem Prove that for integer $n\ge 1$,

$$\sum_{k=1}^{n} k^3\binom{n}{k}=n^2(n+3)2^{n-3}.$$

654 Problem Expand and simplify

$$(\sqrt{1-x^2}+1)^7-(\sqrt{1-x^2}-1)^7.$$

655 Problem Simplify

$$\binom{5}{5}+\binom{6}{5}+\binom{7}{5}+\cdots+\binom{999}{5}$$

656 Problem Simplify

$$\binom{15}{1}-\binom{15}{2}+\binom{15}{3}-\binom{15}{4}\cdots+\binom{15}{13}-\binom{15}{14}$$

657 Problem What is the exact numerical value of

$$\sum_{k=0}^{1994}(-1)^{k-1}\binom{1994}{k}?$$

658 Problem True or False:

$$\binom{4}{4}+\binom{5}{4}+\cdots+\binom{199}{4}>\binom{16}{16}+\binom{17}{16}+\cdots+\binom{199}{16}.$$

659 Problem (AIME 1992) In which row of Pascal's triangle (we start with zeroth row, first row ,etc.) do three consecutive entries occur that are in the ratio $3:4:5$?

5.9 Multinomial Theorem

If $n, n_1, n_2, \ldots, n_k$ are nonnegative integers and $n = n_1 + n_2 + \cdots n_k$ we put

$$\binom{n}{n_1, n_2 \cdots n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}.$$

Using the De-Polignac Legendre Theorem, it is easy to see that this quantity is an integer. Proceeding in the same way we proved the Binomial Theorem, we may establish the *Multinomial Theorem:*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{n_1+n_2+\cdots+n_k=n \\ n_1,n_2,\ldots,n_k \geq 0}} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

We give a few examples on the use of the Multinomial Theorem.

660 Example Determine the coefficient of $x^2y^3z^3$ in

$$(x + 2y + z)^8$$

.

Solution: By the Multinomial Theorem

$$(x + 2y + z)^8 = \sum_{\substack{n_1,n_2,n_3 \geq 0 \\ n_1+n_2+n_3=8}} \binom{8}{n_1, n_2, n_3} x^{n_1} (2y)^{n_2} z^{n_3}.$$

This requires $n_1 = 2, n_2 = 3, n_3 = 3$. The coefficient sought is then $2^3 \binom{8}{2,3,3}$.

661 Example In $(1 + x^5 + x^9)^{23}$, find the coefficient of x^{23}.

Solution: By the Multinomial Theorem

$$\sum_{\substack{n_1,n_2,n_3 \geq 0 \\ n_1+n_2+n_3=23}} \binom{23}{n_1, n_2, n_3} x^{5n_2+9n_3}.$$

Since $5n_2 + 9n_3 = 23$ and $n_1 + n_2 + n_3 = 23$, we must have $n_1 = 20, n_2 = 1, n_3 = 2$. The coefficient sought is thus $\binom{23}{20,1,2}$.

662 Example How many different terms are there in the expansion of

$$(x + y + z + w + s + t)^{20}?$$

Solution: There as many terms as nonnegative integral solutions of

$$n_1 + n_2 + \cdots + n_6 = 20.$$

But we know that there are $\binom{25}{5}$ of these.

Practice

663 Problem How many terms are in the expansion $(x+y+z)^{10}$?

664 Problem Find the coefficient of x^4 in the expansion of

$$(1+3x+2x^3)^{10}?$$

665 Problem Find the coefficient of $x^2y^3z^5$ in the expansion of

$$(x+y+z)^{10}?$$

Chapter 6

Equations

6.1 Equations in One Variable

Let us start with the following example.

666 Example Solve the equation $2^{|x|} = \sin x^2$.

Solution: Clearly $x = 0$ is not a solution. Since $2^y > 1$ for $y > 0$, the equation does not have a solution.

667 Example Solve the equation $|x-3|^{(x^2-8x+15)/(x-2)} = 1$.

Solution: We want either the exponent to be zero, or the base to be 1. We cannot have, however, 0^0 as this is undefined. So, $|x-3| = 1$ implies $x = 4$ or $x = 2$. We discard $x = 2$ as the exponent is undefined at this value. For the exponent we want $x^2 - 8x + 15 = 0$ or $x = 5$ or $x = 3$. We cannot have $x = 3$ since this would give 0^0. So the only solutions are $x = 4$ and $x = 5$.

668 Example What would be the appropriate value of $x$if

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$$

made sense?

Solution: Since $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$, we have $x^2 = 2$ (the chain is infinite, so cutting it at one step does not change the value). Since we want a positive value we must have $x = \sqrt{2}$.

669 Example Solve $9 + x^{-4} = 10x^{-2}$.

Solution: Observe that

$$x^{-4} - 10x^{-2} + 9 = (x^{-2} - 9)(x^{-2} - 1).$$

Then $x = \pm\frac{1}{3}$ and $x = \pm 1$.

670 Example Solve $9^x - 3^{x+1} - 4 = 0$.

Solution: Observe that $9^x - 3^{x+1} - 4 = (3^x - 4)(3^x + 1)$. As no real number x satisfies $3^x + 1 = 0$, we discard this factor. So $3^x - 4 = 0$ yields $x = \log_3 4$.

671 Example Solve

$$(x-5)(x-7)(x+6)(x+4)=504.$$

Solution: Reorder the factors and multiply in order to obtain

$$(x-5)(x-7)(x+6)(x+4)=(x-5)(x+4)(x-7)(x+6)=(x^2-x-20)(x^2-x-42).$$

Put $y=x^2-x$. Then $(y-20)(y-42)=504$, which is to say, $y^2-62y+336=(y-6)(y-56)=0$. Now, $y=6,\ 56$, implies

$$x^2-x=6$$

and

$$x^2-x=56.$$

Solving both quadratics, $x=-2,4,-7,8$.

672 Example Solve $12x^4-56x^3+89x^2-56x+12=0$.

Solution: Reordering

$$12x^4+12-56(x^3+x)+89x^2=0. \tag{6.1}$$

Dividing by x^2,

$$12(x^2+\frac{1}{x^2})-56(x+\frac{1}{x})+89=0.$$

Put $u=x+1/x$. Then $u^2-2=x^2+1/x^2$. Using this, (6) becomes $12(u^2-2)-56u+89=0$, whence $u=5/2,\ 13/6$. From this

$$x+\frac{1}{x}=\frac{5}{2}$$

and

$$x+\frac{1}{x}=\frac{13}{6}.$$

Solving both quadratics we conclude that $x=1/2,2,2/3,3/2$.

673 Example Find the real solutions to

$$x^2-5x+2\sqrt{x^2-5x+3}=12.$$

Solution: Observe that

$$x^2-5x+3+2\sqrt{x^2-5x+3}-15=0.$$

Let $u=x^2-5x+3$ and so $u+2u^{1/2}-15=(u^{1/2}+5)(u^{1/2}-3)=0$. This means that $u=9$ (we discard $u^{1/2}+5=0$, why?). Therefore $x^2-5x+3=9$ or $x=-1,\ 6$.

674 Example Solve

$$\sqrt{3x^2-4x+34}-\sqrt{3x^2-4x-11}=9. \tag{6.2}$$

Solution: Notice the trivial identity

$$(3x^2-4x+34)-(3x^2-4x-11)=45. \tag{6.3}$$

Dividing each member of (8) by the corresponding members of (7), we obtain

$$\sqrt{3x^2-4x+34}+\sqrt{3x^2-4x-11}=5. \tag{6.4}$$

Adding (7) and (9)

$$\sqrt{3x^2-4x+34}=7,$$

from where $x=-\frac{5}{3},3$.

675 Example Solve

$$\sqrt[3]{14+x}+\sqrt[3]{14-x}=4.$$

Solution: Let$u=\sqrt[3]{14+x}, v=\sqrt[3]{14-x}$. Then

$$64=(u+v)^3=u^3+v^3+3uv(u+v)=14+x+14-x+12(196-x^2)^{1/3},$$

whence

$$3=(196-x^2)^{1/3},$$

which upon solving yields $x=\pm 13$.

676 Example Find the exact value of $\cos 2\pi/5$.

Solution: Using the identity

$$\cos(u\pm v)=\cos u\cos v\mp\sin u\sin v$$

twice, we obtain

$$\cos 2\theta = 2\cos^2\theta - 1 \qquad (6.5)$$

and

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta. \qquad (6.6)$$

Let $x=\cos 2\pi/5$. As $\cos 6\pi/5=\cos 4\pi/5$, thanks to (5) and (6), we see that x satisfies the equation

$$4x^3-2x^2-3x+1=0,$$

which is to say

$$(x-1)(4x^2+2x-1)=0.$$

As $x=\cos 2\pi/5\neq 1$, and $\cos 2\pi/5>0$, x positive root of the quadratic equation $4x^2+2x-1=0$, which is to say

$$\cos\frac{2\pi}{5}=\frac{\sqrt{5}-1}{4}.$$

677 Example How many real numbers x satisfy

$$\sin x=\frac{x}{100}?$$

Solution: Plainly $x=0$ is a solution. Also, if $x>0$ is a solution, so is $-x<0$. So, we can restrict ourselves to positive solutions.
If x is a solution then $|x|=100|\sin x|\leq 100$. So we can further restrict x to the interval $]0;100]$. Decompose $]0;100]$ into 2π-long intervals (the last interval is shorter):

$$]0;100]=]0;2\pi]\cup]2\pi;4\pi]\cup]4\pi;6\pi]\cup\cdots\cup]28\pi;30\pi]\cup]30\pi;100].$$

From the graphs of $y=\sin x, y=x/100$ we that the interval $]0;2\pi]$ contains only one solution. Each interval of the form $]2\pi k;\ 2(k+1)\pi], k=1,2,\ldots,14$ contains two solutions. As $31\pi<100$, the interval $]30\pi;100]$ contains a full wave, hence it contains two solutions. Consequently, there are $1+2\cdot 14+2=31$ positive solutions, and hence, 31 negative solutions. Therefore, there is a total of $31+31+1=63$ solutions.

Practice

678 Problem Solve for x

$$2\sqrt{\frac{x}{a}}+3\sqrt{\frac{a}{x}}=\frac{b}{a}+\frac{6a}{b}.$$

679 Problem Solve

$$(x-7)(x-3)(x+5)(x+1)=1680.$$

680 Problem Solve

$$x^4+x^3-4x^2+x+1=0.$$

681 Problem Solve the equation

$$2^{\sin^2 x}+5\cdot 2^{\cos^2 x}=7.$$

682 Problem If the equation $\sqrt{x+\sqrt{x+\sqrt{x+\sqrt{\cdots}}}}=2$ made sense, what would be the value of x?

683 Problem How many real solutions are there to

$$\sin x=\log_e x?$$

684 Problem Solve the equation

$$|x+1|-|x|+3|x-1|-2|x-2|=x+2.$$

685 Problem Find the real roots of

$$\sqrt{x+3-4\sqrt{x-1}}+\sqrt{x+8-6\sqrt{x-1}}=1.$$

686 Problem Solve the equation

$$6x^4-25x^3+12x^2+25x+6=0.$$

687 Problem Solve the equation

$$x(2x+1)(x-2)(2x-3)=63.$$

688 Problem Find the value of

$$\sqrt{30\cdot 31\cdot 32\cdot 33+1}.$$

689 Problem Solve

$$\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}+\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}=98.$$

690 Problem Find a real solution to

$$(x^2-9x-1)^{10}+99x^{10}=10x^9(x^2-1).$$

Hint: Write this equation as

$$(x^2-9x-1)^{10}-10x^9(x^2-9x-1)+9x^{10}=0.$$

691 Problem Find the real solutions to

$$\underbrace{\sqrt{x+2\sqrt{x+2\sqrt{x+\cdots+2\sqrt{x+2\sqrt{3x}}}}}}_{n \text{ radicals}}=x.$$

692 Problem Solve the equation

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots\, 1+\cfrac{1}{1+\cfrac{1}{x}}}}}=x.$$

where the fraction is repeated n times.

693 Problem Solve for x

$$\sqrt{x+\sqrt{x+11}}+\sqrt{x+\sqrt{x-11}}=4.$$

6.2 Systems of Equations

694 Example Solve the system of equations

$$\begin{aligned} x+y+u &= 4,\\ y+u+v &= -5,\\ u+v+x &= 0,\\ v+x+y &= -8. \end{aligned}$$

Solution: Adding all the equations and dividing by 3,

$$x+y+u+v=-3.$$

This implies

$$\begin{aligned} 4+v &= -3,\\ -5+x &= -3,\\ 0+y &= -3,\\ -8+u &= -3, \end{aligned}$$

whence $x=2, y=-3, u=5, v=-7$.

695 Example Solve the system

$$(x+y)(x+z)=30,$$
$$(y+z)(y+x)=15,$$
$$(z+x)(z+y)=18.$$

Solution: Put $u=y+z, v=z+x, w=x+y$. The system becomes

$$vw=30,\ wu=15,\ uv=18. \tag{6.7}$$

Multiplying all of these equations we obtain $u^2v^2w^2=8100$, that is, $uvw=\pm 90$. Dividing each of the equations in (7), we gather $u=3, v=6, w=5$, or $u=-3, v=-6, w=-5$. This yields

$$\begin{array}{lllllll} y+z & = & 3, & \text{or} & y+z & = & -3, \\ z+x & = & 6, & \text{or} & z+x & = & -6, \\ x+y & = & 5, & \text{or} & x+y & = & -5, \end{array}$$

whence $x=4,\ y=1,\ z=2$ or $x=-4,\ y=-1,\ z=-2$..

Practice

696 Problem Let a,b, c be real constants, $abc \neq 0$. Solve

$$x^2-(y-z)^2=a^2,$$
$$y^2-(z-x)^2=b^2,$$
$$z^2-(x-y)^2=c^2.$$

697 Problem Solve

$$x^3+3x^2y+y^3=8,$$
$$2x^3-2x^2y+xy^2=1.$$

698 Problem Solve the system

$$x+2+y+3+\sqrt{(x+2)(y+3)}=39,$$
$$(x+2)^2+(y+3)^2+(x+2)(y+3)=741.$$

699 Problem Solve the system

$$x^4+y^4=82,$$
$$x-y=2.$$

700 Problem Solve the system

$$x_1x_2=1,\ x_2x_3=2,\ \ldots,\ x_{100}x_{101}=100,\ x_{101}x_1=101.$$

701 Problem Solve the system

$$x^2-yz=3,$$
$$y^2-zx=4,$$
$$z^2-xy=5.$$

702 Problem Solve the system

$$2x+y+z+u=-1$$
$$x+2y+z+u=12$$
$$x+y+2z+u=5$$
$$x+y+z+2u=-1$$

703 Problem Solve the system

$$x^2+x+y=8,$$
$$y^2+2xy+z=168,$$
$$z^2+2yz+2xz=12480.$$

6.3 Remainder and Factor Theorems

The *Division Algorithm* for polynomials states that if the polynomial $p(x)$ is divided by $a(x)$ then there exist polynomials $q(x), r(x)$ with

$$p(x)=a(x)q(x)+r(x) \tag{6.8}$$

and $0 \leq$ degree $r(x) <$ degree $a(x)$. For example, if x^5+x^4+1 is divided by x^2+1 we obtain

$$x^5+x^4+1=(x^3+x^2-x-1)(x^2+1)+x+2,$$

and so the quotient is $q(x)=x^3+x^2-x-1$ and the remainder is $r(x)=x+2$.

704 Example Find the remainder when $(x+3)^5+(x+2)^8+(5x+9)^{1997}$ is divided by $x+2$.

Solution: As we are dividing by a polynomial of degree 1, the remainder is a polynomial of degree 0, that is, a constant. Therefore, there is a polynomial $q(x)$ and a constant r with

$$(x+3)^5+(x+2)^8+(5x+9)^{1997}=q(x)(x+2)+r$$

Letting $x=-2$ we obtain

$$(-2+3)^5+(-2+2)^8+(5(-2)+9)^{1997}=q(-2)(-2+2)+r=r.$$

As the sinistral side is 0 we deduce that the remainder $r=0$.

705 Example A polynomial leaves remainder -2 upon division by $x-1$ and remainder -4 upon division by $x+2$. Find the remainder when this polynomial is divided by x^2+x-2.

Solution: From the given information, there exist polynomials $q_1(x), q_2(x)$ with $p(x)=q_1(x)(x-1)-2$ and $p(x)=q_2(x)(x+2)-4$. Thus $p(1)=-2$ and $p(-2)=-4$. As $x^2+x-2=(x-1)(x+2)$ is a polynomial of degree 2 the remainder $r(x)$ upon dividing $p(x)$ by x^2+x-1 is of degree 1 or less, that is $r(x)=ax+b$ for some constants a,b which we must determine. By the Division Algorithm,

$$p(x)=q(x)(x^2+x-1)+ax+b.$$

Hence

$$-2=p(1)=a+b$$

and

$$-4=p(-2)=-2a+b.$$

From these equations we deduce that $a=2/3, b=-8/3$. The remainder sought is $r(x)=2x/3-8/3$.

706 Example Let $f(x)=x^4+x^3+x^2+x+1$. Find the remainder when $f(x^5)$ is divided by $f(x)$.

Solution: Observe that $f(x)(x-1)=x^5-1$ and

$$f(x^5)=x^{20}+x^{15}+x^{10}+x^5+1=(x^{20}-1)+(x^{15}-1)+(x^{10}-1)+(x^5-1)+5.$$

Each of the summands in parentheses is divisible by x^5-1 and, a fortiori, by $f(x)$. The remainder sought is thus 5. Using the Division Algorithm we may derive the following theorem.

707 Theorem Factor Theorem The polynomial $p(x)$ is divisible by $x-a$ if and only if $p(a)=0$.

Proof As $x-a$ is a polynomial of degree 1, the remainder after diving $p(x)$ by $x-a$ is a polynomial of degree 0, es that is, a constant. Therefore

$$p(x)=q(x)(x-a)+r.$$

From this we gather that $p(a)=q(a)(a-a)+r=r$, from where the theorem easily follows.

708 Example If $p(x)$ is a cubic polynomial with $p(1)=1, p(2)=2, p(3)=3, p(4)=5$, find $p(6)$.

Solution: Put $g(x)=p(x)-x$. Observe that $g(x)$ is a polynomial of degree 3 and that $g(1)=g(2)=g(3)=0$. Thus $g(x)=c(x-1)(x-2)(x-3)$ for some constant c that we must determine. Now, $g(4)=c(4-1)(4-2)(4-3)=6c$ and $g(4)=p(4)-4=1$, whence $c=1/6$. Finally

$$p(6)=g(6)+6=\frac{(6-1)(6-2)(6-3)}{6}+6=16.$$

709 Example The polynomial $p(x)$ has integral coefficients and $p(x) = 7$ for four different values of x. Shew that $p(x)$ never equals 14.

Solution: The polynomial $g(x) = p(x) - 7$ vanishes at the 4 different integer values a, b, c, d. In virtue of the Factor Theorem,

$$g(x) = (x-a)(x-b)(x-c)(x-d)q(x),$$

where $q(x)$ is a polynomial with integral coefficients. Suppose that $p(t) = 14$ for some integer t. Then $g(t) = p(t) - 7 = 14 - 7 = 7$. It follows that

$$7 = g(t) = (t-a)(t-b)(t-c)(t-d)q(t),$$

that is, we have factorised 7 as the product of at least 4 different factors, which is impossible since 7 can be factorised as $7(-1)1$, the product of at most 3 distinct integral factors. From this contradiction we deduce that such an integer t does not exist.

Practice

710 Problem If $p(x)$ is a polynomial of degree n such that $p(k) = 1/k, k = 1, 2, \ldots, n+1$, find $p(n+2)$.

711 Problem The polynomial $p(x)$ satisfies $p(-x) = -p(x)$. When $p(x)$ is divided by $x - 3$ the remainder is 6. Find the remainder when $p(x)$ is divided by $x^2 - 9$.

6.4 Viète's Formulae

Let us consider first the following example.

712 Example Expand the product

$$(x+1)(x-2)(x+4)(x-5)(x+6).$$

Solution: The product is a polynomial of degree 5. To obtain the coefficient of x^5 we take an x from each of the five binomials. Therefore, the coefficient of x^5 is 1. To form the x^4 term, we take an x from 4 of the binomials and a constant from the remaining binomial. Thus the coefficient of x^4 is

$$1 - 2 + 4 - 5 + 6 = 4.$$

To form the coefficient of x^3 we take three x from 3 of the binomials and two constants from the remaining binomials. Thus the coefficient of x^3 is

$$(1)(-2)+(1)(4)+(1)(-5)+(1)(6)+(-2)(4)+(-2)(-5)+(-2)(6)$$
$$+(4)(-5)+(4)(6)+(-5)(6) = -33.$$

Similarly, the coefficient of x^2 is

$$(1)(-2)(4)+(1)(-2)(-5)+(1)(-2)(6)+(1)(4)(-5)+(1)(4)(6)+(-2)(4)(-5)$$
$$+(-2)(4)(6)+(4)(-5)(6) = -134$$

and the coefficient of x is

$$(1)(-2)(4)(-5)+(1)(-2)(4)(6)+(1)(-2)(-5)(6)+(1)(4)(-5)(6)+(-2)(4)(-5)(6) = 172.$$

Finally, the constant term is $(1)(-2)(4)(-5)(6) = 240$. The product sought is thus

$$x^5 + 4x^4 - 33x^3 - 134x^2 + 172x + 240.$$

From the preceding example, we see that each summand of the expanded product has "weight" 5, because of the five given binomials we either take the x or take the constant.

If $a_0 \neq 0$ and
$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$
is a polynomial with roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ then we may write
$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_{n-1})(x-\alpha_n).$$
From this we deduce the *Viète Formulæ:*
$$-\frac{a_1}{a_0} = \sum_{k=1}^{n} \alpha_k,$$
$$\frac{a_2}{a_0} = \sum_{1\leq j<k\leq n} \alpha_j\alpha_k,$$
$$-\frac{a_3}{a_0} = \sum_{1\leq j<k<l\leq n} \alpha_j\alpha_k\alpha_l,$$
$$\frac{a_4}{a_0} = \sum_{1\leq j<k<l<s\leq n} \alpha_j\alpha_k\alpha_l\alpha_s,$$
....

....

....
$$(-1)^n\frac{a_n}{a_0} = \alpha_1\alpha_2\cdots\alpha_n.$$

713 Example Find the sum of the roots, the sum of the roots taken two at a time, the sum of the square of the roots and the sum of the reciprocals of the roots of
$$2x^3 - x + 2 = 0.$$

Solution: Let a, b, c be the roots of $2x^3 - x + 2 = 0$. From the Viète Formulæ the sum of the roots is
$$a+b+c = -\frac{0}{2} = 0$$
and the sum of the roots taken two at a time is
$$ab+ac+bc = \frac{-1}{2}.$$
To find $a^2+b^2+c^2$ we observe that
$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+ac+bc).$$
Hence
$$a^2+b^2+c^2 = 0^2 - 2(-1/2) = 1.$$
Finally, as $abc = -2/2 = -1$, we gather that
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{ab+ac+bc}{abc} = \frac{-1/2}{-1} = 1/2.$$

714 Example Let α, β, γ be the roots of $x^3 - x^2 + 1 = 0$. Find
$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}.$$

Solution: From $x^3-x^2+1=0$ we deduce that $1/x^2=1-x$. Hence

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=(1-\alpha)+(1-\beta)+(1-\gamma)=3-(\alpha+\beta+\gamma)=3-1=2.$$

Together with the Viète Formulæ we also have the *Newton-Girard Identities* for the sum of the powers $s_k=\alpha_1^k+\alpha_2^k+\cdots+\alpha_n^k$ of the roots:

$$a_0s_1+a_1=0,$$

$$a_0s_2+a_1s_1+2a_2=0,$$

$$a_0s_3+a_1s_2+a_2s_1+3a_3=0,$$

etc..

715 Example If a,b,c are the roots of $x^3-x^2+2=0$, find

$$a^2+b^2+c^2$$

$$a^3+b^3+c^3$$

and

$$a^4+b^4+c^4.$$

Solution: First observe that

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+ac+bc)=1^2-2(0)=1.$$

As $x^3=x^2-2$, we gather

$$a^3+b^3+c^3=a^2-2+b^2-2+c^2-2=a^2+b^2+c^2-6=1-6=-5.$$

Finally, from $x^3=x^2-2$ we obtain $x^4=x^3-2x$, whence

$$a^4+b^4+c^4=a^3-2a+b^3-2b+c^3-2c=a^3+b^3+c^3-2(a+b+c)=-5-2(1)=-7.$$

716 Example (USAMO 1973) Find all solutions (real or complex) of the system

$$x+y+z=3,$$

$$x^2+y^2+z^2=3,$$

$$x^3+y^3+z^3=3.$$

Solution: Let x,y,z be the roots of

$$p(t)=(t-x)(t-y)(t-z)=t^3-(x+y+z)t^2+(xy+yz+zx)t-xyz.$$

Now $xy+yz+zx=(x+y+z)^2/2-(x^2+y^2+z^2)/2=9/2-3/2=3$ and from

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

we gather that $xyz=1$. Hence

$$p(t)=t^3-3t^2+3t-1=(t-1)^3.$$

Thus $x=y=z=1$ is the only solution of the given system.

Practice

717 Problem Suppose that

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = (x+r_1)(x+r_2)\cdots(x+r_n)$$

where $r_1, r_2, \ldots, r_n$ are real numbers. Shew that

$$(n-1)a_1^2 \geq 2na_2.$$

718 Problem (USAMO 1984) The product of the roots of

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Determine k.

719 Problem The equation $x^4 - 16x^3 + 94x^2 + px + q = 0$ has two double roots. Find $p+q$.

720 Problem If $\alpha_1, \alpha_2, \ldots, \alpha_{100}$ are the roots of

$$x^{100} - 10x + 10 = 0,$$

find the sum

$$\alpha_1^{100} + \alpha_2^{100} + \cdots + \alpha_{100}^{100}.$$

721 Problem Let α, β, γ be the roots of $x^3 - x - 1 = 0$. Find

$$\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3}$$

y

$$\alpha^5 + \beta^5 + \gamma^5.$$

722 Problem The real numbers α, β satisfy

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0,$$

$$\beta^3 - 3\beta^2 + 5\beta + 11 = 0.$$

Find $\alpha + \beta$.

6.5 Lagrange's Interpolation

723 Example Find a cubic polynomial $p(x)$ vanishing at $x = 1, 2, 3$ and satisfying $p(4) = 666$.

Solution: The polynomial must be of the form $p(x) = a(x-1)(x-2)(x-3)$, where a is a constant. As $666 = p(4) = a(4-1)(4-2)(4-3) = 6a$, $a = 111$. The desired polynomial is therefore $p(x) = 111(x-1)(x-2)(x-3)$.

724 Example Find a cubic polynomial $p(x)$ satisfying $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5$.

Solution: We shall use the following method due to Lagrange. Let

$$p(x) = a(x) + 2b(x) + 3c(x) + 5d(x),$$

where $a(x), b(x), c(x), d(x)$ are cubic polynomials with the following properties: $a(1) = 1$ and $a(x)$ vanishes when $x = 2, 3, 4; b(2) = 1$ and $b(x)$ vanishes when $x = 1, 3, 4$; $c(3) = 1$ and $c(3) = 1$ vanishes when $x = 1, 2, 4$, and finally, $d(4) = 1$, $d(x)$ vanishing at $x = 1, 2, 3$.
Using the technique of the preceding example, we find

$$a(x) = -\frac{(x-2)(x-3)(x-4)}{6},$$

$$b(x) = \frac{(x-1)(x-3)(x-4)}{2},$$

$$c(x) = -\frac{(x-1)(x-2)(x-4)}{2}$$

y

$$d(x) = \frac{(x-1)(x-2)(x-3)}{6}.$$

Thus

$$p(x) = -\frac{1}{6}\cdot(x-2)(x-3)(x-4) + (x-1)(x-3)(x-4) - \frac{3}{2}\cdot(x-1)(x-2)(x-4) + \frac{5}{6}(x-1)(x-2)(x-3).$$

It is left to the reader to verify that the polynomial satisfies the required properties.

Practice

725 Problem Find a polynomial $p(x)$ of degree 4 with $p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 4, p(5) = 5$.

726 Problem Find a polynomial $p(x)$ of degree 4 with $p(1) = -1, p(2) = 2, p(-3) = 4, p(4) = 5, p(5) = 8$.

Chapter 7

Inequalities

7.1 Absolute Value

727 Definition (The Signum (Sign) Function) Let x be a real number. We define $\text{signum}\,(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ +1 & \text{if } x > 0. \end{cases}$

728 Lemma The signum function is multiplicative, that is, if $(x,y) \in \mathbb{R}^2$ then $\text{signum}\,(x \cdot y) = \text{signum}\,(x)\,\text{signum}\,(y)$.

Proof: *Immediate from the definition of signum.* ❑

729 Definition (Absolute Value) Let $x \in \mathbb{R}$. The *absolute value* of x is defined and denoted by

$$|x| = \text{signum}\,(x)\,x.$$

730 Theorem Let $x \in \mathbb{R}$. Then

1. $|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$
2. $|x| \geq 0$,
3. $|x| = \max(x, -x)$,
4. $|-x| = |x|$,
5. $-|x| \leq x \leq |x|$.
6. $\sqrt{x^2} = |x|$
7. $|x|^2 = |x^2| = x^2$
8. $x = \text{signum}\,(x)\,|x|$

Proof: *These are immediate from the definition of* $|x|$. ❑

731 Theorem $(\forall (x,y) \in \mathbb{R}^2)$,

$$|xy| = |x|\,|y|\,.$$

Proof: *We have*

$$|xy| = \text{signum}(xy)\,xy = (\text{signum}(x)\,x)\,(\text{signum}(y)\,y) = |x|\,|y|,$$

where we have used Lemma 728. ❑

732 Theorem Let $t \geq 0$. Then

$$|x| \leq t \iff -t \leq x \leq t.$$

Proof: *Either* $|x| = x$ *or* $|x| = -x$. *If* $|x| = x$,

$$|x| \leq t \iff x \leq t \iff -t \leq 0 \leq x \leq t.$$

If $|x| = -x$,

$$|x| \leq t \iff -x \leq t \iff -t \leq x \leq 0 \leq t.$$

❑

733 Theorem If $(x,y) \in \mathbb{R}^2$, $\max(x,y) = \dfrac{x+y+|x-y|}{2}$ and $\min(x,y) = \dfrac{x+y-|x-y|}{2}$.

Proof: *Observe that* $\max(x,y) + \min(x,y) = x+y$, *since one of these quantities must be the maximum and the other the minimum, or else, they are both equal.*

Now, either $|x-y| = x-y$, *and so* $x \geq y$, *meaning that* $\max(x,y) - \min(x,y) = x-y$, *or* $|x-y| = -(x-y) = y-x$, *which means that* $y \geq x$ *and so* $\max(x,y) - \min(x,y) = y-x$. *In either case we get* $\max(x,y) - \min(x,y) = |x-y|$. *Solving now the system of equations*

$$\begin{aligned} \max(x,y) + \min(x,y) &= x+y \\ \max(x,y) - \min(x,y) &= |x-y|, \end{aligned}$$

for $\max(x,y)$ *and* $\min(x,y)$ *gives the result.* ❑

7.2 Triangle Inequality

734 Theorem (Triangle Inequality) Let $(a,b) \in \mathbb{R}^2$. Then

$$\boxed{|a+b| \leq |a| + |b|.} \tag{7.1}$$

Proof: *From 5 in Theorem 730, by addition,*

$$-|a| \leq a \leq |a|$$

to

$$-|b| \leq b \leq |b|$$

we obtain

$$-(|a|+|b|) \leq a+b \leq (|a|+|b|),$$

whence the theorem follows by applying Theorem 732. ❑

By induction, we obtain the following generalisation to n terms.

735 Corollary Let $x_1, x_2, \ldots, x_n$ be real numbers. Then

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Proof: *We apply Theorem 734 $n-1$ times*

$$\begin{array}{lll} |x_1+x_2+\cdots+x_n| & \leq & |x_1|+|x_2+\cdots x_{n-1}+x_n| \\ & \leq & |x_1|+|x_2|+|x_3+\cdots x_{n-1}+x_n| \\ & \vdots & \\ & \leq & |x_1|+|x_2|+\cdots+|x_{n-1}+x_n| \\ & \leq & |x_1|+|x_2|+\cdots+|x_{n-1}|+|x_n|. \end{array}$$

❑

736 Corollary Let $(a,b)\in\mathbb{R}^2$. Then

$$\boxed{||a|-|b||\leq|a-b|}. \tag{7.2}$$

Proof: *We have*

$$|a|=|a-b+b|\leq|a-b|+|b|,$$

giving

$$|a|-|b|\leq|a-b|.$$

Similarly,

$$|b|=|b-a+a|\leq|b-a|+|a|=|a-b|+|a|,$$

gives

$$|b|-|a|\leq|a-b| \implies -|a-b|\leq|a|-|b|.$$

Thus

$$-|a-b|\leq|a|-|b|\leq|a-b|,$$

and we now apply Theorem 732. ❑

7.3 Rearrangement Inequality

737 Definition Given a set of real numbers $\{x_1,x_2,\ldots,x_n\}$ denote by

$$\check{x}_1\geq\check{x}_2\geq\cdots\geq\check{x}_n$$

the decreasing rearrangement of the x_i and denote by

$$\hat{x}_1\leq\hat{x}_2\leq\cdots\leq\hat{x}_n$$

the increasing rearrangement of the x_i.

738 Definition Given two sequences of real numbers $\{x_1,x_2,\ldots,x_n\}$ and $\{y_1,y_2,\ldots,y_n\}$ of the same length n, we say that they are *similarly sorted* if they are both increasing or both decreasing, and *differently sorted* if one is increasing and the other decreasing..

739 Example The sequences $1\leq 2\leq\cdots\leq n$ and $1^2\leq 2^2\leq\cdots\leq n^2$ are similarly sorted, and the sequences $\frac{1}{1^2}\geq\frac{1}{2^2}\geq\cdots\geq\frac{1}{n^2}$ and $1^3\leq 2^3\leq\cdots\leq n^3$ are differently sorted.

740 Theorem (Rearrangement Inequality) Given sets of real numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ we have

$$\sum_{1\leq k\leq n} \breve{a}_k\hat{b}_k \leq \sum_{1\leq k\leq n} a_kb_k \leq \sum_{1\leq k\leq n} \hat{a}_k\hat{b}_k.$$

Thus the sum $\sum_{1\leq k\leq n} a_kb_k$ is minimised when the sequences are differently sorted, and maximised when the sequences are similarly sorted.

Observe that

$$\sum_{1\leq k\leq n} \hat{a}_k\hat{b}_k = \sum_{1\leq k\leq n} \breve{a}_k\breve{b}_k.$$

Proof: *Let $\{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$ be a reordering of $\{1, 2, \ldots, n\}$. If there are two sub-indices i, j, such that the sequences pull in opposite directions, say, $a_i > a_j$ and $b_{\sigma(i)} < b_{\sigma(j)}$, then consider the sums*

$$\begin{aligned} S &= a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_ib_{\sigma(i)} + \cdots + a_jb_{\sigma(j)} + \cdots + a_nb_{\sigma(n)} \\ S' &= a_1b_{\sigma(1)} + a_2b_{\sigma(2)} + \cdots + a_ib_{\sigma(j)} + \cdots + a_jb_{\sigma(i)} + \cdots + a_nb_{\sigma(n)} \end{aligned}$$

Then

$$S' - S = (a_i - a_j)(b_{\sigma(j)} - b_{\sigma(i)}) > 0.$$

This last inequality shews that the closer the a's and the b's are to pulling in the same direction the larger the sum becomes. This proves the result. ❑

7.4 Mean Inequality

741 Theorem (Arithmetic Mean-Geometric Mean Inequality) Let $a_1, \ldots, a_n$ be positive real numbers. Then their geometric mean is at most their arithmetic mean, that is,

$$\sqrt[n]{a_1\cdots a_n} \leq \frac{a_1 + \cdots + a_n}{n},$$

with equality if and only if $a_1 = \cdots = a_n$.

We will provide multiple proofs of this important inequality. Some other proofs will be found in latter chapters.

First Proof: *Our first proof uses the Rearrangement Inequality (Theorem 740) in a rather clever way. We may assume that the a_k are strictly positive. Put*

$$x_1 = \frac{a_1}{(a_1a_2\cdots a_n)^{1/n}}, \quad x_2 = \frac{a_1a_2}{(a_1a_2\cdots a_n)^{2/n}}, \quad \ldots, \quad x_n = \frac{a_1a_2\cdots a_n}{(a_1a_2\cdots a_n)^{n/n}} = 1,$$

and

$$y_1 = \frac{1}{x_1}, \quad y_2 = \frac{1}{x_2}, \quad \ldots, \quad y_n = \frac{1}{x_n} = 1.$$

Observe that for $2 \leq k \leq n$,

$$x_ky_{k-1} = \frac{a_1a_2\cdots a_k}{(a_1a_2\cdots a_n)^{k/n}} \cdot \frac{(a_1a_2\cdots a_n)^{(k-1)/n}}{a_1a_2\cdots a_{k-1}} = \frac{a_k}{(a_1a_2\cdots a_n)^{1/n}}.$$

The x_k and y_k are differently sorted, so by virtue of the Rearrangement Inequality we gather

$$\begin{aligned} 1 + 1 + \cdots + 1 &= x_1y_1 + x_2y_2 + \cdots + x_ny_n \\ &\leq x_1y_n + x_2y_1 + \cdots + x_ny_{n-1} \\ &= \frac{a_1}{(a_1a_2\cdots a_n)^{1/n}} + \frac{a_2}{(a_1a_2\cdots a_n)^{1/n}} + \cdots + \frac{a_n}{(a_1a_2\cdots a_n)^{1/n}}, \end{aligned}$$

or

$$n \le \frac{a_1 + a_2 + \cdots + a_n}{(a_1 a_2 \cdots a_n)^{1/n}},$$

from where we obtain the result. ❑

Second Proof: *This second proof is a clever induction argument due to Cauchy. It proves the inequality first for powers of* 2 *and then interpolates for numbers between consecutive powers of* 2.

Since the square of a real number is always positive, we have, for positive real numbers a, b

$$(\sqrt{a} - \sqrt{b})^2 \ge 0 \implies \sqrt{ab} \le \frac{a+b}{2},$$

proving the inequality for $k = 2$. *Observe that equality happens if and only if* $a = b$. *Assume now that the inequality is valid for* $k = 2^{n-1} > 2$. *This means that for any positive real numbers* $x_1, x_2, \ldots, x_{2^{n-1}}$ *we have*

$$(x_1 x_2 \cdots x_{2^{n-1}})^{1/2^{n-1}} \le \frac{x_1 + x_2 + \cdots + x_{2^{n-1}}}{2^{n-1}}. \tag{7.3}$$

Let us prove the inequality for $2k = 2^n$. *Consider any any positive real numbers* $y_1, y_2, \ldots, y_{2^n}$. *Notice that there are* $2^n - 2^{n-1} = 2^{n-1}(2-1) = 2^{n-1}$ *integers in the interval* $\left[2^{n-1}+1\,; 2^n\right]$. *We have*

$$\begin{aligned}
(y_1 y_2 \cdots y_{2^n})^{1/2^n} &= \sqrt{(y_1 y_2 \cdots y_{2^{n-1}})^{1/2^{n-1}} \left(y_{2^{n-1}+1} \cdots y_{2^n}\right)^{1/2^{n-1}}} \\
&\le \frac{(y_1 y_2 \cdots y_{2^{n-1}})^{1/2^{n-1}} + \left(y_{2^{n-1}+1} \cdots y_{2^n}\right)^{1/2^{n-1}}}{2} \\
&\le \frac{\dfrac{y_1 + y_2 + \cdots + y_{2^{n-1}}}{2^{n-1}} + \dfrac{y_{2^{n-1}+1} + \cdots + y_{2^n}}{2^{n-1}}}{2} \\
&= \frac{y_1 + \cdots + y_{2^n}}{2^n},
\end{aligned}$$

where the first inequality follows by the Case $n = 2$ *and the second by the induction hypothesis (7.3). The theorem is thus proved for powers of* 2.

Assume now that $2^{n-1} < k < 2^n$, *and consider the* k *positive real numbers* $a_1, a_2, \ldots, a_k$. *The trick is to pad this collection of real numbers up to the next highest power of* 2, *the added real numbers being the average of the existing ones. Hence consider the* 2^n *real numbers*

$$a_1, a_2, \ldots, a_k, a_{k+1}, \ldots, a_{2^n}$$

with $a_{k+1} = \ldots = a_{2^n} = \dfrac{a_1 + a_2 + \cdots + a_k}{k}$. *Since we have already proved the theorem for* 2^n *we have*

$$\left(a_1 a_2 \cdots a_k \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^{2^n - k}\right)^{1/2^n} \le \frac{a_1 + a_2 + \cdots + a_k + (2^n - k)\left(\dfrac{a_1 + a_2 + \cdots + a_k}{k}\right)}{2^n},$$

whence

$$(a_1 a_2 \cdots a_k)^{1/2^n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^{1 - k/2^n} \le \frac{k\dfrac{a_1 + a_2 + \cdots + a_k}{k} + (2^n - k)\left(\dfrac{a_1 + a_2 + \cdots + a_k}{k}\right)}{2^n},$$

which implies

$$(a_1 a_2 \cdots a_k)^{1/2^n} \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right)^{1 - k/2^n} \le \left(\frac{a_1 + a_2 + \cdots + a_k}{k}\right),$$

Solving for $\dfrac{a_1 + a_2 + \cdots + a_k}{k}$ *gives the desired inequality.* ❑

Third Proof: *As in the second proof, the Case $k = 2$ is easily established. Put*

$$A_k = \frac{a_1 + a_2 + \cdots + a_k}{k}, \qquad G_k = (a_1 a_2 \cdots a_k)^{1/k}.$$

Observe that

$$a_{k+1} = (k+1)A_{k+1} - kA_k.$$

The inductive hypothesis is that $A_k \geq G_k$ and we must shew that $A_{k+1} \geq G_{k+1}$. Put

$$A = \frac{a_{k+1} + (k-1)A_{k+1}}{k}, \qquad G = \left(a_{k+1} A_{k+1}^{k-1}\right)^{1/k}.$$

By the inductive hypothesis $A \geq G$. Now,

$$\frac{A + A_k}{2} = \frac{\frac{(k+1)A_{k+1} - kA_k + (k-1)A_{k+1}}{k} + A_k}{2} = A_{k+1}.$$

Hence

$$\begin{aligned} A_{k+1} &= \frac{A + A_k}{2} \\ &\geq (AA_k)^{1/2} \\ &\geq (GG_k)^{1/2}. \\ &= \left(G_{k+1}^{k+1} A_{k+1}^{k-1}\right)^{1/2k} \end{aligned}$$

We have established that

$$A_{k+1} \geq \left(G_{k+1}^{k+1} A_{k+1}^{k-1}\right)^{1/2k} \implies A_{k+1} \geq G_{k+1},$$

completing the induction. ❑

Fourth Proof: *We will make a series of substitutions that preserve the sum*

$$a_1 + a_2 + \cdots + a_n$$

while strictly increasing the product

$$a_1 a_2 \cdots a_n.$$

At the end, the a_i will all be equal and the arithmetic mean A of the numbers will be equal to their geometric mean G. If the a_i where all $> A$ then $\frac{a_1 + a_2 + \cdots + a_n}{n} > \frac{nA}{n} = A$, impossible. Similarly, the a_i cannot be all $< A$. Hence there must exist two indices say i, j, such that $a_i < A < a_j$. Put $a_i' = A$, $a_j' = a_i + a_j - A$. Observe that $a_i + a_j = a_i' + a_j'$, so replacing the original a's with the primed a's does not alter the arithmetic mean. On the other hand,

$$a_i' a_j' = A(a_i + a_j - A) = a_i a_j + (a_j - A)(A - a_i) > a_i a_j$$

since $a_j - A > 0$ and $A - a_i > 0$.

This change has replaced one of the a's by a quantity equal to the arithmetic mean, has not changed the arithmetic mean, and made the geometric mean larger. Since there at most n a's to be replaced, the procedure must eventually terminate when all the a's are equal (to their arithmetic mean). Strict inequality hence holds if when at least two of the a's are unequal. ❑

742 Theorem (Cauchy-Bunyakovsky-Schwarz Inequality) Let x_k, y_k be real numbers, $1 \leq k \leq n$. Then

$$\left|\sum_{k=1}^{n} x_k y_k\right| \leq \left(\sum_{k=1}^{n} x_k^2\right)^{1/2} \left(\sum_{k=1}^{n} y_k^2\right)^{1/2},$$

with equality if and only if

$$(a_1, a_2, \ldots, a_n) = t(b_1, b_2, \ldots, b_n)$$

for some real constant t.

First Proof: *The inequality follows at once from Lagrange's Identity*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 = \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right) - \sum_{1\leq k<j\leq n} (x_k y_j - x_j y_k)^2$$

*(Theorem **??**), since* $\sum_{1\leq k<j\leq n} (x_k y_j - x_j y_k)^2 \geq 0$. ❑

Second Proof: *Put* $a = \sum_{k=1}^{n} x_k^2$, $b = \sum_{k=1}^{n} x_k y_k$, *and* $c = \sum_{k=1}^{n} y_k^2$. *Consider the quadratic polynomial*

$$at^2 + bt + c = t^2 \sum_{k=1}^{n} x_k^2 - 2t \sum_{k=1}^{n} x_k y_k + \sum_{k=1}^{n} y_k^2 = \sum_{k=1}^{n} (tx_k - y_k)^2 \geq 0,$$

where the inequality follows because a sum of squares of real numbers is being summed. Thus this quadratic polynomial is positive for all real t, so it must have complex roots. Its discriminant $b^2 - 4ac$ must be non-positive, from where we gather

$$4\left(\sum_{k=1}^{n} x_k y_k\right)^2 \leq 4\left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right),$$

which gives the inequality ❑

For our third proof of the CBS Inequality we need the following lemma.

743 Lemma For $(a,b,x,y) \in \mathbb{R}^4$ with $x > 0$ and $y > 0$ the following inequality holds:

$$\frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}.$$

Equality holds if and only if $\frac{a}{x} = \frac{b}{y}$.

Proof: *Since the square of a real number is always positive, we have*

$$\begin{aligned} (ay - bx)^2 \geq 0 \quad & \implies \quad a^2y^2 - 2abxy + b^2x^2 \geq 0 \\ & \implies \quad a^2y(x+y) + b^2x(x+y) \geq (a+b)^2xy \\ & \implies \quad \frac{a^2}{x} + \frac{b^2}{y} \geq \frac{(a+b)^2}{x+y}. \end{aligned}$$

Equality holds if and only if the first inequality is 0.❑

☞ *Iterating the result on Lemma 743,*

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{b_1 + b_2 + \cdots + b_n},$$

with equality if and only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

Third Proof: *By the preceding remark, we have*

$$\begin{aligned} x_1^2 + x_2^2 + \cdots + x_n^2 & = \frac{x_1^2y_1^2}{y_1^2} + \frac{x_2^2y_2^2}{y_2^2} + \cdots + \frac{x_n^2y_n^2}{y_n^2} \\ & \geq \frac{(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2}{y_1^2 + y_2^2 + \cdots + y_n^2}, \end{aligned}$$

and upon rearranging, CBS is once again obtained.❑

744 Theorem (Minkowski's Inequality) Let x_k, y_k be any real numbers. Then

$$\left(\sum_{k=1}^{n}(x_k+y_k)^2\right)^{1/2} \leq \left(\sum_{k=1}^{n}x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n}y_k^2\right)^{1/2}.$$

Proof: *We have*

$$\begin{aligned}\sum_{k=1}^{n}(x_k+y_k)^2 &= \sum_{k=1}^{n}x_k^2 + 2\sum_{k=1}^{n}x_ky_k + \sum_{k=1}^{n}y_k^2 \\ &\leq \sum_{k=1}^{n}x_k^2 + 2\left(\sum_{k=1}^{n}x_k^2\right)^{1/2}\left(\sum_{k=1}^{n}y_k^2\right)^{1/2} + \sum_{k=1}^{n}y_k^2 \\ &= \left(\left(\sum_{k=1}^{n}x_k^2\right)^{1/2} + \left(\sum_{k=1}^{n}y_k^2\right)^{1/2}\right)^2,\end{aligned}$$

where the inequality follows from the CBS Inequality.❑

Practice

745 Problem Let x, y be real numbers. Then

$$0 \leq x < y \iff x^2 < y^2.$$

746 Problem Let $t \geq 0$. Prove that

$$|x| \geq t \iff (x \geq t) \quad \text{or} \quad (x \leq -t).$$

747 Problem Let $(x,y) \in \mathbb{R}^2$. Prove that $\max(x,y) = -\min(-x,-y)$.

748 Problem Let x, y, z be real numbers. Prove that

$$\max(x,y,z) = x+y+z-\min(x,y)-\min(y,z)-\min(z,x)+\min(x,y,z).$$

749 Problem Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be such that

$$x_1^2+x_2^2+\cdots+x_n^2 = x_1^3+x_2^3+\cdots+x_n^3 = x_1^4+x_2^4+\cdots+x_n^4.$$

Prove that $x_k \in \{0,1\}$.

750 Problem Let $n \geq 2$ an integer. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be such that

$$x_1^2+x_2^2+\cdots+x_n^2 = x_1x_2+x_2x_3+\cdots+x_{n-1}x_n+x_nx_1.$$

Prove that $x_1 = x_2 = \cdots = x_n$.

751 Problem If $b > 0$ and $B > 0$ prove that

$$\frac{a}{b} < \frac{A}{B} \implies \frac{a}{b} < \frac{a+A}{b+B} < \frac{A}{B}.$$

Further, if p and q are positive integers such that

$$\frac{7}{10} < \frac{p}{q} < \frac{11}{15},$$

what is the least value of q?

752 Problem Let $a < b$. Demonstrate that

$$|x-a| < |x-b| \iff x < \frac{a+b}{2}.$$

753 Problem Prove that if $r \geq s \geq t$ then

$$r^2 - s^2 + t^2 \geq (r-s+t)^2.$$

754 Problem Assume that $a_k, b_k, c_k, k = 1, \ldots, n$, are positive real numbers. Shew that

$$\left(\sum_{k=1}^{n}a_kb_kc_k\right)^4 \leq \left(\sum_{k=1}^{n}a_k^4\right)\left(\sum_{k=1}^{n}b_k^4\right)\left(\sum_{k=1}^{n}c_k^2\right)^2.$$

755 Problem Prove that for integer $n > 1$,

$$n! < \left(\frac{n+1}{2}\right)^n.$$

756 Problem Prove that for integer $n > 2$,

$$n^{n/2} < n!.$$

757 Problem Prove that $\forall (a,b,c) \in \mathbb{R}^3$,

$$a^2+b^2+c^2 \geq ab+bc+ca.$$

758 Problem Prove that $\forall (a,b,c) \in \mathbb{R}^3$, with $a \geq 0$, $b \geq 0$, $c \geq 0$, the following inequalities hold:

$$a^3+b^3+c^3 \geq \max(a^2b+b^2c+c^2a, a^2c+b^2a+c^2b),$$

$$a^3+b^3+c^3 \geq 3abc,$$

$$a^3+b^3+c^3 \geq \frac{1}{2}\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right).$$

759 Problem (Chebyshev's Inequality) Given sets of real numbers $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ prove that

$$\frac{1}{n}\sum_{1\leq k\leq n}\breve{a}_k\hat{b}_k \leq \left(\frac{1}{n}\sum_{1\leq k\leq n}a_k\right)\left(\frac{1}{n}\sum_{1\leq k\leq n}b_k\right) \leq \frac{1}{n}\sum_{1\leq k\leq n}\hat{a}_k\hat{b}_k.$$

760 Problem If $x > 0$, from

$$\sqrt{x+1}-\sqrt{x}=\frac{1}{\sqrt{x+1}+\sqrt{x}},$$

prove that

$$\frac{1}{2\sqrt{x+1}}<\sqrt{x+1}-\sqrt{x}<\frac{1}{2\sqrt{x}}.$$

Use this to prove that if $n > 1$ is a positive integer, then

$$2\sqrt{n+1}-2<1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}-1$$

761 Problem If $0 < a \leq b$, shew that

$$\frac{1}{8}\cdot\frac{(b-a)^2}{b}\leq\frac{a+b}{2}-\sqrt{ab}\leq\frac{1}{8}\cdot\frac{(b-a)^2}{a}$$

762 Problem Shew that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000}<\frac{1}{100}.$$

763 Problem Prove that for all $x > 0$,

$$\sum_{k=1}^{n}\frac{1}{(x+k)^2}<\frac{1}{x}-\frac{1}{x+n}.$$

764 Problem Let $x_i \in \mathbb{R}$ such that $\sum_{i=1} |x_i| = 1$ and $\sum_{i=1} x_i = 0$. Prove that

$$\left|\sum_{i=1}^{n}\frac{x_i}{i}\right|\leq\frac{1}{2}\left(1-\frac{1}{n}\right).$$

765 Problem Let n be a strictly positive integer. Let $x_i \geq 0$. Prove that

$$\prod_{k=1}^{n}(1+x_k)\geq 1+\sum_{k=1}^{n}x_k.$$

When does equality hold?

766 Problem (Nesbitt's Inequality) Let a, b, c be strictly positive real numbers. Then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}.$$

767 Problem Let a, b, c be positive real numbers. Prove that

$$(a+b)(b+c)(c+a)\geq 8abc.$$

768 Problem (IMO, 1978) Let a_k be a sequence of pairwise distinct positive integers. Prove that

$$\sum_{k=1}^{n}\frac{a_k}{k^2}\geq\sum_{k=1}^{n}\frac{1}{k}.$$

769 Problem (Harmonic Mean-Geometric Mean Inequality) Let $x_i > 0$ for $1 \leq i \leq n$. Then

$$\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}\leq(x_1x_2\cdots x_n)^{1/n},$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

770 Problem (Arithmetic Mean-Quadratic Mean Inequality) Let $x_i \geq 0$ for $1 \leq i \leq n$. Then

$$\frac{x_1+x_2+\cdots+x_n}{n}\leq\left(\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}\right)^{1/2},$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

771 Problem Given a set of real numbers $\{a_1, a_2, \ldots, a_n\}$ prove that there is an index $m \in \{0, 1, \ldots, n\}$ such that

$$\left|\sum_{1\leq k\leq m}a_k-\sum_{m<k\leq n}a_k\right|\leq\max_{1\leq k\leq n}|a_k|.$$

If $m = 0$ the first sum is to be taken as 0 and if $m = n$ the second one will be taken as 0.

Appendix A

Answers, Hints, and Solutions

10 Since their product is 1 the integers must be ± 1 and there must be an even number of -1's, say k of them. To make the sum of the numbers 0 we must have the same number of 1's. Thus we must have $k(1)+k(-1)=0$, and $k+k=34$, which means that $k=17$, which is not even.

11 Clearing denominators, there are 2000 summands on the sinistral side of the form $a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{2000}$, and the dextral side we simply have $a_1a_2\cdots a_{i-1}a_{i+1}\cdots a_{2000}$. If all the a_k were odd, the right hand side would be odd, and the left hand side would be even, being the sum of 2000 odd numbers, a contradiction.

12 If $log_2 3=\dfrac{a}{b}$, with integral $a,b\neq 0$ then $2^a=3^b$. By uniqueness of factorisation this is impossible unless $a=b=0$, which is not an allowed alternative.

13 If the palindrome were divisible by 10 then it would end in 0, and hence, by definition of being a palindrome, it would start in 0, which is not allowed in the definition.

14 Assume $AC\geq BC$ and locate point D on the line segment AC such that $AD=BD$. Then $\triangle ADB$ is isosceles at D and we must have $\angle A=\angle B$, a contradiction.

15 If $\sqrt{a}\leq\alpha$ then $\alpha\leq\alpha^2$, which implies that $\alpha(1-\alpha)\leq 0$, an impossible inequality if $0<\alpha<1$.

16 We have $1-\dfrac{1}{10^{2000}}<\alpha<1$. Squaring,

$$1-\frac{2}{10^{2000}}+\frac{1}{10^{4000}}<\alpha^2.$$

Since $-\dfrac{1}{10^{2000}}+\dfrac{1}{10^{4000}<0}$, we have

$$1-\frac{1}{10^{2000}}<1-\frac{1}{10^{2000}}-\frac{1}{10^{2000}}+\frac{1}{10^{4000}}<\alpha^2.$$

26 There are n possible different remainders when an integer is divided by n, so among $n+1$ different integers there must be two integers in the group leaving the same remainder, and their difference is divisible by n.

27 20

66 (i) -3.5, (ii) 45

67 $x^2=(x+3-3)^2=(x+3)^2-6(x+3)+9$

89 Substitute t by $\sqrt{t^2-u^2+v^2}$, in 2.5.

98 Use the fact that $(b-a)^2=(\sqrt{b}-\sqrt{a})^2(\sqrt{b}+\sqrt{a})^2$.

104 Suppose that all these products are $> \frac{1}{4}$. Use the preceding problem to obtain a contradiction.

117 52

131 Write

$$\begin{aligned} 2222^{5555} + 5555^{2222} &= (2222^{5555} + 4^{5555}) \\ &\quad + (5555^{2222} - 4^{2222}) \\ &\quad - (4^{5555} - 4^{2222}). \end{aligned}$$

136 Consider $x = 2n - 1$.

139 we have

$$2^n - 1 = 2^{ab} - 1 = (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + \cdots + (2^a)^1 + 1).$$

Since $a > 1, 2^a - 1 > 1$. Since $b > 1$,

$$(2^a)^{b-1} + (2^a)^{b-2} + \cdots (2^a)^1 + 1) \geq 2^a + 1 > 1.$$

We have decomposed a prime number (the left hand side) into the product of two factors, each greater than 1, a contradiction. Thus n must be a prime.

140 We have

$$2^n + 1 = 2^{2^k m} + 1 = (2^{2^k} + 1)((2^{2^k})^{m-1} - (2^{2^k})^{m-2} + \cdots - (2^{2^k})^1 + 1).$$

Clearly, $2^{2^k} + 1 > 1$. Also if $m \geq 3$

$$(2^{2^k})^{m-1} - (2^{2^k})^{m-2} + \cdots - (2^{2^k})^1 + 1 \geq (2^{2^k})^2 - (2^{2^k})^1 + 1 > 1,$$

and so, we have produced two factors each greater than 1 for the prime $2^n + 1$, which is nonsense.

149 1

166 Observe that $(1+i)^{2004} = ((1+i)^2)^{1002} = (2i)^{1002}$, etc.

167 Group the summands in groups of four terms and observe that

$$\begin{aligned} & ki^{k+1} + (k+1)i^{k+2} + \\ & \quad (k+2)i^{k+3} + (k+4)i^{k+4} \\ &= i^{k+1}(k + (k+1)i - (k+2) - (k+3)i) \\ &= -2 - 2i. \end{aligned}$$

168 If k is an integer, $i^k + i^{k+1} + i^{k+2} + i^{k+3} = i^k(1 + i + i^2 + i^3) = 0$.

183 Argue by contradiction. Assume $a = 3k \pm 1$ or $b = 3m \pm 1$.

187 13

195 Think of $n - 6$ if n is even and $n - 9$ if n is odd.

197 Try $x = 36k + 14, y = (12k + 5)(18k + 7)$.

302 63

322 Consider, separately, the cases when n is and is not a perfect square.

347 $\frac{50}{99}$

348 9

356 3030

358 $\dfrac{5973}{1993}$

378 Shew first that $\csc 2x = \cot x - \cot 2x$.

379 Observe that
$$\frac{y}{1-y^2} = \frac{1}{1-y} - \frac{1}{1-y^2}.$$

392 $x_n = \dfrac{1}{3^n} + 2$.

393 $x_n = 5^n + 5n$.

394 $x_n = 6n^2 + 6n + 1$.

395 $x_n = 2^n + 3(5^n)$.

396 $a_{j+1} = 6^{2^j} - 1$.

399 Let $u_n = \cos v_n$.

413 $a_0 = 0,\ a_n = a_{n-1} + (n-1)3^n$.

423 Let $A_k \subseteq A$ be the set of those integers divisible by k.

❶ Notice that the elements are $2 = 2(1), 4 = 2(2), \ldots, 114 = 2(57)$. Thus $\text{card}(A) = 57$.

❷ There are $\lfloor\frac{57}{3}\rfloor = 19$ integers in A divisible by 3. They are
$$\{6, 12, 18, \ldots, 114\}.$$
Notice that $114 = 6(19)$. Thus $\text{card}(A_3) = 19$.

❸ There are $\lfloor\frac{57}{5}\rfloor = 11$ integers in A divisible by 5. They are
$$\{10, 20, 30, \ldots, 110\}.$$
Notice that $110 = 10(11)$. Thus $\text{card}(A_5) = 11$

❹ There are $\lfloor\frac{57}{15}\rfloor = 3$ integers in A divisible by 15. They are $\{30, 60, 90\}$. Notice that $90 = 30(3)$. Thus $\text{card}(A_{15}) = 3$, and observe that by Theorem **??** we have $\text{card}(A_{15}) = \text{card}(A_3 \cap A_5)$.

❺ We want $\text{card}(A_3 \cup A_5) = 19 + 11 = 30$.

❻ We want
$$\begin{aligned}\text{card}(A \setminus (A_3 \cup A_5)) &= \text{card}(A) - \text{card}(A_3 \cup A_5) \\ &= 57 - 30 \\ &= 27.\end{aligned}$$

❼ We want
$$\begin{aligned}\text{card}((A_3 \cup A_5) \setminus (A_3 \cap A_5)) &= \text{card}((A_3 \cup A_5)) \\ &\quad - \text{card}(A_3 \cap A_5) \\ &= 30 - 3 \\ &= 27.\end{aligned}$$

424 We have

❶ $\llbracket \frac{100}{2} \rrbracket = 50$

❷ $\llbracket \frac{100}{3} \rrbracket = 33$

❸ $\llbracket \frac{100}{7} \rrbracket = 14$

❹ $\llbracket \frac{100}{6} \rrbracket = 16$

❺ $\llbracket \frac{100}{14} \rrbracket = 7$

❻ $\llbracket \frac{100}{21} \rrbracket = 4$

❼ $\llbracket \frac{100}{42} \rrbracket = 2$

❽ $100-50-33-14+15+7+4-2=27$

❾ $16-2=14$

❿ 52

425 52%

426 22

427 Let A be the set of students liking Mathematics, B the set of students liking theology, and C be the set of students liking alchemy. We are given that

$$\text{card}(A)=14, \text{card}(B)=16,$$

$$\text{card}(C)=11, \text{card}(A\cap B)=7, \text{card}(B\cap C)=8, \text{card}(A\cap C)=5,$$

and

$$\text{card}(A\cap B\cap C)=4.$$

By the Principle of Inclusion-Exclusion,

$$\begin{aligned}\text{card}(A^c\cap B^c\cap C^c) &= 40-\text{card}(A)-\text{card}(B)-\text{card}(C)\\ &\quad +\text{card}(A\cap B)+\text{card}(A\cap C)+\text{card}(B\cap C)\\ &\quad -\text{card}(A\cap B\cap C).\end{aligned}$$

Substituting the numerical values of these cardinalities

$$40-14-16-11+7+5+8-4=15.$$

428 We have

❶ 31

❷ 10

❸ 3

❹ 3

❺ 1

❻ 1

❼ 1

❽ 960

429 Let Y, F, S, M stand for young, female, single, male, respectively, and let H stand for married.[1] We have

$$\begin{aligned} \text{card}(Y \cap F \cap S) &= \text{card}(Y \cap F) - \text{card}(Y \cap F \cap H) \\ &= \text{card}(Y) - \text{card}(Y \cap M) \\ &\qquad -(\text{card}(Y \cap H) - \text{card}(Y \cap H \cap M)) \\ &= 3000 - 1320 - (1400 - 600) \\ &= 880. \end{aligned}$$

430 34

431 $30; 7; 5; 18$

432 4

433 Let C denote the set of people who like candy, I the set of people who like ice cream, and K denote the set of people who like cake. We are given that $\text{card}(C) = 816$, $\text{card}(I) = 723$, $\text{card}(K) = 645$, $\text{card}(C \cap I) = 562$, $\text{card}(C \cap K) = 463$, $\text{card}(I \cap K) = 470$, and $\text{card}(C \cap I \cap K) = 310$. By Inclusion-Exclusion we have

$$\begin{aligned} \text{card}(C \cup I \cup K) &= \text{card}(C) + \text{card}(I) + \text{card}(K) \\ &\quad -\text{card}(C \cap I) - \text{card}(C \cap K) - \text{card}(I \cap C) \\ &\quad +\text{card}(C \cap I \cap K) \\ &= 816 + 723 + 645 - 562 - 463 - 470 + 310 \\ &= 999. \end{aligned}$$

The investigator miscounted, or probably did not report one person who may not have liked any of the three things.

434 A set with k elements has 2^k different subsets. We are given

$$2^{100} + 2^{100} + 2^{\text{card}(C)} = 2^{\text{card}(A \cup B \cup C)}.$$

This forces $\text{card}(C) = 101$, as $1 + 2^{\text{card}(C) - 101}$ is larger than 1 and a power of 2. Hence $\text{card}(A \cup B \cup C) = 102$. Using the Principle Inclusion-Exclusion, since $\text{card}(A) + \text{card}(B) + \text{card}(C) - \text{card}(A \cup B \cup C) = 199$,

$$\begin{aligned} \text{card}(A \cap B \cap C) &= \text{card}(A \cap B) + \text{card}(A \cap C) + \text{card}(B \cap C) - 199 \\ &= (\text{card}(A) + \text{card}(B) - \text{card}(A \cup B)) \\ &\qquad +(\text{card}(A) + \text{card}(C) \\ &\qquad -\text{card}(A \cup C)) + \text{card}(B) + \text{card}(C) \\ &\qquad -\text{card}(B \cup C) - 199 \\ &= 403 - \text{card}(A \cup B) - \text{card}(A \cup C) - \text{card}(B \cup C). \end{aligned}$$

As $A \cup B, A \cup C, B \cup C \subseteq A \cup B \cup C$, the cardinalities of all these sets are ≤ 102. Thus

$$\begin{aligned} \text{card}(A \cap B \cap C) &= 403 - \text{card}(A \cup B) - \text{card}(A \cup C) \\ &\qquad -\text{card}(B \cup C) \geq 403 - 3 \cdot 102 \\ &= 97. \end{aligned}$$

By letting

$$A = \{1, 2, \ldots, 100\}, B = \{3, 4, \ldots, 102\},$$

and

$$C = \{1, 2, 3, 4, 5, 6, \ldots, 101, 102\}$$

we see that the bound $\text{card}(A \cap B \cap C) = \text{card}(\{4, 5, 6, \ldots, 100\}) = 97$ is achievable.

[1] Or H for *hanged*, if you prefer.

435 Let A denote the set of those who lost an eye, B denote those who lost an ear, C denote those who lost an arm and D denote those losing a leg. Suppose there are n combatants. Then

$$\begin{aligned} n &\geq \operatorname{card}(A \cup B) \\ &= \operatorname{card}(A) + \operatorname{card}(B) - \operatorname{card}(A \cap B) \\ &= .7n + .75n - \operatorname{card}(A \cap B), \end{aligned}$$

$$\begin{aligned} n &\geq \operatorname{card}(C \cup D) \\ &= \operatorname{card}(C) + \operatorname{card}(D) - \operatorname{card}(C \cap D) \\ &= .8n + .85n - \operatorname{card}(C \cap D). \end{aligned}$$

This gives

$$\operatorname{card}(A \cap B) \geq .45n,$$

$$\operatorname{card}(C \cap D) \geq .65n.$$

This means that

$$\begin{aligned} n &\geq \operatorname{card}((A \cap B) \cup (C \cap D)) \\ &= \operatorname{card}(A \cap B) + \operatorname{card}(C \cap D) - \operatorname{card}(A \cap B \cap C \cap D) \\ &\geq .45n + .65n - \operatorname{card}(A \cap B \cap C \cap D), \end{aligned}$$

whence

$$\operatorname{card}(A \cap B \cap C \cap D) \geq .45 + .65n - n = .1n.$$

This means that at least 10% of the combatants lost all four members.

451 $2^{10} = 1024$

452 I can choose a right shoe in any of nine ways, once this has been done, I can choose a non-matching left shoe in eight ways, and thus I have 72 choices.
Aliter: I can choose any pair in $9 \times 9 = 81$ ways. Of these, 9 are matching pairs, so the number of non-matching pairs is $81 - 9 = 72$.

453 $= (20)(19)(20)(19)(20)(20) = 57760000$

454 $10^3 5^3 - 10^2 5^2 = 122500$

455 The number of different license plates is the number of different four-tuples (Letter $_1$, Letter $_2$, Digit $_1$, Digit $_2$). The first letter can be chosen in 26 ways, and so we have

26			

The second letter can be chosen in any of 26 ways:

26	26		

The first digit can be chosen in 10 ways:

26	26	10	

Finally, the last digit can be chosen in 10 ways:

26	26	10	10

By the multiplication principle, the number of different four-tuples is $26 \cdot 26 \cdot 10 \cdot 10 = 67600$.

456 (i) In this case we have a grid like

26	26	10	9

,

since after a digit has been used for the third position, it cannot be used again. Thus this can be done in $26 \cdot 26 \cdot 10 \cdot 9 = 60840$ ways.
(ii) In this case we have a grid like

26	25	10	10

,

since after a letter has been used for the first position, it cannot be used again. Thus this can be done in $26 \cdot 25 \cdot 10 \cdot 10 = 65000$ ways.
(iii) After a similar reasoning, we obtain a grid like

26	25	10	9

.

Thus this can be done in $26 \cdot 25 \cdot 10 \cdot 9 = 58500$ ways.

457 [1] 8, [2] $5^2 3^2 = 225$, [3] $5^2 \cdot 3 \cdot 2 = 150$, [4] $5 \cdot 4 \cdot 3^2 = 180$, [5] $8 \cdot 7 \cdot 6 \cdot 5 = 1680$.

458 432

459 Solution:

❶ The first letter can be one of any 4. After choosing the first letter, we have 3 choices for the second letter, etc.. The total number of words is thus $4 \cdot 3 \cdot 2 \cdot 1 = 24$.

❷ The first letter can be one of any 4. Since we are allowed repetitions, the second letter can also be one of any 4, etc.. The total number of words so formed is thus $4^4 = 256$.

460 The last digit must perforce be 5. The other five digits can be filled with any of the six digits on the list: the total number is thus 6^5.

461 We have

❶ This is $5 \cdot 8^6 = 1310720$.

❷ This is $5 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 25200$.

❸ This is $5 \cdot 8^5 \cdot 4 = 655360$.

❹ This is $5 \cdot 8^5 \cdot 4 = 655360$.

❺ We condition on the last digit. If the last digit were 1 or 5 then we would have 5 choices for the first digit, and so we would have

$$5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 7200$$

phone numbers. If the last digit were either 3 or 7, then we would have 4 choices for the last digit and so we would have

$$4 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 5760$$

phone numbers. Thus the total number of phone numbers is

$$7200 + 5760 = 12960.$$

462 $26 \cdot 25^4 = 10156250$

463 For the leftmost digit cannot be 0 and so we have only the nine choices

$$\{1,2,3,4,5,6,7,8,9\}$$

for this digit. The other $n-1$ digits can be filled out in 10 ways, and so there are

$$9 \cdot \underbrace{10 \cdots 10}_{n-1\ 10'\text{s}} = 9 \cdot 10^{n-1}.$$

464 The leftmost digit cannot be 0 and so we have only the nine choices

$$\{1,2,3,4,5,6,7,8,9\}$$

for this digit. If the integer is going to be even, the last digit can be only one of the five $\{0,2,4,6,8\}$. The other $n-2$ digits can be filled out in 10 ways, and so there are

$$9\cdot\underbrace{10\cdots 10}_{n-2\ 10'\text{s}}\cdot 5 = 45\cdot 10^{n-2}.$$

465 9 1-digit numbers and $8\cdot 9^{n-1}$ n-digit numbers $n\geq 2$.

466 One can choose the last digit in 9 ways, one can choose the penultimate digit in 9 ways, etc. and one can choose the second digit in 9 ways, and finally one can choose the first digit in 9 ways. The total number of ways is thus 9^n.

467 m^2, $m(m-1)$

468 We will assume that the positive integers may be factorised in a unique manner as the product of primes. Expanding the product

$$(1+2+2^2+\cdots+2^8)(1+3+3^2+\cdots+3^9)(1+5+5^2)$$

each factor of $2^8 3^9 5^2$ appears and only the factors of this number appear. There are then, as many factors as terms in this product. This means that there are $(1+8)(1+9)(1+3)=320$ factors.

The sum of the divisors of this number may be obtained by adding up each geometric series in parentheses. The desired sum is then

$$\frac{2^9-1}{2-1}\cdot\frac{3^{10}-1}{3-1}\cdot\frac{5^3-1}{5-1}=467689684.$$

A similar argument gives the following. Let $p_1,p_2,\ldots,p_k$ be different primes. Then the integer

$$n=p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$$

has

$$\mathrm{d}(n)=(a_1+1)(a_2+1)\cdots(a_k+1)$$

positive divisors. Also, if $\sigma(n)$ denotes the sum of all positive divisors of n, then

$$\sigma(n)=\frac{p_1^{a_1+1}-1}{p_1-1}\cdot\frac{p_2^{a_2+1}-1}{p_2-1}\cdots\frac{p_k^{a_k+1}-1}{p_k-1}.$$

469 The 96 factors of 2^{95} are $1,2,2^2,\ldots,2^{95}$. Observe that $2^{10}=1024$ and so $2^{20}=1048576$. Hence

$$2^{19}=524288<1000000<1048576=2^{20}.$$

The factors greater than $1,000,000$ are thus $2^{20},2^{21},\ldots 2^{95}$. This makes for $96-20=76$ factors.

470 $(1+3)(1+2)(1+1)=24$; 18; 6; 4.

471 16

472 A. [1] 10000, [2] 5040, B. [1] 12 , [2] 10

473 $n=\underbrace{1+1+\cdots+1}_{n-1\ +'\text{s}}$. One either erases or keeps a plus sign.

474 There are 589 such values. The easiest way to see this is to observe that there is a bijection between the divisors of n^2 which are $>n$ and those $<n$. For if $n^2=ab$, with $a>n$, then $b<n$, because otherwise $n^2=ab>n\cdot n=n^2$, a contradiction. Also, there is exactly one decomposition $n^2=n\cdot n$. Thus the desired number is

$$\left\lfloor\!\left\lfloor\frac{d(n^2)}{2}\right\rfloor\!\right\rfloor+1-d(n)=\left\lfloor\!\left\lfloor\frac{(63)(39)}{2}\right\rfloor\!\right\rfloor+1-(32)(20)=589.$$

475 The total number of sequences is 3^n. There are 2^n sequences that contain no 0, 1 or 2. There is only one sequence that contains only 1's, one that contains only 2's, and one that contains only 0's. Obviously, there is no ternary sequence that contains no 0's or 1's or 2's. By the Principle of Inclusion-Exclusion, the number required is

$$3^n-(2^n+2^n+2^n)+(1+1+1)=3^n-3\cdot 2^n+3.$$

476 The conditions of the problem stipulate that both the region outside the circles in diagram 5.3 and R_3 will be empty. We are thus left with 6 regions to distribute 100 numbers. To each of the 100 numbers we may thus assign one of 6 labels. The number of sets thus required is 6^{100}.

484 21

485 56

486 $(26^2-25^2)+(26^3-25^3)=2002$

487

$$\begin{aligned}9&+9\cdot 9\\ &+9\cdot 9\cdot 8+9\cdot 9\cdot 8\cdot 7\\ &+9\cdot 9\cdot 8\cdot 7\cdot 6+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\\ &+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\\ &+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\\ &+9\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\\ &=8877690\end{aligned}$$

488 $2+4+8+16=30$.

489 8; $12(n-2)$; $6(n-2)^2$; $(n-2)^3$
Comment: This proves that $n^3=(n-2)^3+6(n-2)^2+12(n-2)+8$.

490 We condition on the first digit, which can be $4,5$, or 6. If the number starts with 4, in order to satisfy the conditions of the problem, we must choose the last digit from the set $\{0,2,6,8\}$. Thus we have four choices for the last digit. Once this last digit is chosen, we have 8 choices for the penultimate digit and 7 choices for the antepenultimate digit. There are thus $4\times 8\times 7=224$ even numbers which have their digits distinct and start with a 4. Similarly, there are 224 even numbers will all digits distinct and starting with a 6. When they start with a 5, we have 5 choices for the last digit, 8 for the penultimate and 7 for the antepenultimate. This gives $5\times 8\times 7=280$ ways. The total number is thus $224+224+280=728$.

491 When the number 99 is written down, we have used

$$1\cdot 9+2\cdot 90=189$$

digits. If we were able to write 999, we would have used

$$1\cdot 9+2\cdot 90+3\cdot 900=2889$$

digits, which is more than 1002 digits. The 1002nd digit must be among the three-digit positive integers. We have $1002-189=813$ digits at our disposal, from which we can make $\llbracket\frac{813}{3}\rrbracket=271$ three-digit integers, from 100 to 270. When the 0 in 270 is written, we have used $189+3\cdot 271=1002$ digits. The 1002nd digit is the 0 in 270.

492 4

493 There is 1 such number with 1 digit, 10 such numbers with 2 digits, 100 with three digits, 1000 with four digits, etc. Starting with 2 and finishing with 299 we have used $1\cdot 1+2\cdot 10+3\cdot 100=321$ digits. We need $1978-321=1657$ more digits from among the 4-digit integers starting with 2. Now $\llbracket\frac{1657}{4}\rrbracket=414$, so we look at the 414th 4-digit integer starting with 2, namely, at 2413. Since the 3 in 2413 constitutes the $321+4\cdot 414=1977$-th digit used, the 1978-th digit must be the 2 starting 2414.

494 19990

495 [1] 125, [2] 25, [3] 25, [4] $5+2\cdot 3+3\cdot 6=29$.

496 8

497 4095

498 144

499 First observe that $1+7=3+5=8$. The numbers formed have either one, two, three or four digits. The sum of the numbers of 1 digit is clearly $1+7+3+5=16$.

There are $4\times 3=12$ numbers formed using 2 digits, and hence 6 pairs adding to 8 in the units and the tens. The sum of the 2 digits formed is $6((8)(10)+8)=6\times 88=528$.

There are $4\times 3\times 2=24$ numbers formed using 3 digits, and hence 12 pairs adding to 8 in the units, the tens, and the hundreds. The sum of the 3 digits formed is $12(8(100)+(8)(10)+8)=12\times 888=10656$.

There are $4\times 3\times 2\cdot 1=24$ numbers formed using 4 digits, and hence 12 pairs adding to 8 in the units, the tens the hundreds, and the thousands. The sum of the 4 digits formed is $12(8(1000)+8(100)+(8)(10)+8)=12\times 8888=106656$.

The desired sum is finally

$$16+528+10656+106656=117856.$$

500 Observe that

❶ We find the pairs

$$\{1,6\},\{2,7\},\{3,8\},\ldots,\{45,50\},$$

so there are 45 in total. (Note: the pair $\{a,b\}$ is indistinguishable from the pair $\{b,a\}$.

❷ If $|a-b|=1$, then we have

$$\{1,2\},\{2,3\},\{3,4\},\ldots,\{49,50\},$$

or 49 pairs. If $|a-b|=2$, then we have

$$\{1,3\},\{2,4\},\{3,5\},\ldots,\{48,50\},$$

or 48 pairs. If $|a-b|=3$, then we have

$$\{1,4\},\{2,5\},\{3,6\},\ldots,\{47,50\},$$

or 47 pairs. If $|a-b|=4$, then we have

$$\{1,5\},\{2,6\},\{3,7\},\ldots,\{46,50\},$$

or 46 pairs. If $|a-b|=5$, then we have

$$\{1,6\},\{2,7\},\{3,8\},\ldots,\{45,50\},$$

or 45 pairs.

The total required is thus

$$49+48+47+46+45=235.$$

501 If $x=0$, put $m(x)=1$, otherwise put $m(x)=x$. We use three digits to label all the integers, from 000 to 999 If a,b,c are digits, then clearly $p(100a+10b+c)=m(a)m(b)m(c)$. Thus

$$p(000)+\cdots+p(999)=m(0)m(0)m(0)+\cdots+m(9)m(9)m(9),$$

which in turn

$$\begin{aligned} &= (m(0)+m(1)+\cdots+m(9))^3 \\ &= (1+1+2+\cdots+9)^3 \\ &= 46^3 \\ &= 97336. \end{aligned}$$

Hence

$$\begin{aligned} S &= p(001)+p(002)+\cdots+p(999) \\ &= 97336-p(000) \\ &= 97336-m(0)m(0)m(0) \\ &= 97335. \end{aligned}$$

510 120

511 479001600; 4838400; 33868800

512 720; 24; 120; 144

513 1440

514 128

515 81729648000

516 249

517 We have

❶ This is 8!.

❷ Permute XY in 2! and put them in any of the 7 spaces created by the remaining 6 people. Permute the remaining 6 people. This is $2!\cdot 7\cdot 6!$.

❸ In this case, we alternate between sexes. Either we start with a man or a woman (giving 2 ways), and then we permute the men and the women. This is $2\cdot 4!4!$.

❹ Glue the couples into 4 separate blocks. Permute the blocks in 4! ways. Then permute each of the 4 blocks in 2!. This is $4!(2!)^4$.

❺ Sit the women first, creating 5 spaces in between. Glue the men together and put them in any of the 5 spaces. Permute the men in 4! ways and the women in 4!. This is $5\cdot 4!4!$.

525 1816214400

526 548

527 18

528 We have

❶ This is

$$\frac{10!}{4!3!2!}$$

❷ This is

$$\frac{9!}{4!3!2!}$$

❸ This is

$$\frac{8!}{2!3!2!}$$

529 36

530 25

531 126126; 756756

545 $\binom{10}{2} = 45$

546 $\binom{7}{1}\binom{5}{3} = (7)(10) = 70$

547 $\Big(^{N}$

❶ $(E+F+S+I)!$

❷ $4!\cdot E!F!S!I!$

❸ $\binom{E+F+I+1}{1}S!(E+F+I)!$

❹ $\binom{E+F+I+1}{S}S!(E+F+I)!$

❺ $2!\binom{F+I+1}{2}S!E!(F+I)!$

557 We can choose the seven people in $\binom{20}{7}$ ways. Of the seven, the chairman can be chosen in seven ways. The answer is thus

$$7\binom{20}{7}=542640.$$

Aliter: Choose the chairman first. This can be done in twenty ways. Out of the nineteen remaining people, we just have to choose six, this can be done in $\binom{19}{6}$ ways. The total number of ways is hence $20\binom{19}{6}=542640$.

558 We can choose the seven people in $\binom{20}{7}$ ways. Of these seven people chosen, we can choose the chairman in seven ways and the secretary in six ways. The answer is thus $7\cdot 6\binom{20}{7}=3255840$.
Aliter: If one chooses the chairman first, then the secretary and finally the remaining five people of the committee, this can be done in $20\cdot 19\cdot \binom{18}{5}=3255840$ ways.

559 For a string of three-digit numbers to be decreasing, the digits must come from $\{0,1,\ldots,9\}$ and so there are $\binom{10}{3}=120$ three-digit numbers with all its digits in decreasing order. If the string of three-digit numbers is increasing, the digits have to come from $\{1,2,\ldots,9\}$, thus there are $\binom{9}{3}=84$ three-digit numbers with all the digits increasing. The total asked is hence $120+84=204$.

560 We can choose the four students who are going to take the first test in $\binom{20}{4}$ ways. From the remaining ones, we can choose students in $\binom{16}{4}$ ways to take the second test. The third test can be taken in $\binom{12}{4}$ ways. The fourth in $\binom{8}{4}$ ways and the fifth in $\binom{4}{4}$ ways. The total number is thus

$$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}.$$

561 We align the thirty-nine cards which are not hearts first. There are thirty-eight spaces between them and one at the beginning and one at the end making a total of forty spaces where the hearts can go. Thus there are $\binom{40}{13}$ ways of choosing the *places* where the hearts can go. Now, since we are interested in arrangements, there are 39! different configurations of the non-hearts and 13! different configurations of the hearts. The total number of arrangements is thus $\binom{40}{13}39!13!$.

562 The equality signs cause us trouble, since allowing them would entail allowing repetitions in our choices. To overcome that we establish a one-to-one correspondence between the vectors $(a,b,c,d), 0\leq a\leq b\leq c\leq d\leq n$ and the vectors $(a',b',c',d'), 0\leq a'<b'<c'<d'\leq n+3$. Let $(a',b',c',d')=(a,b+1,c+2,d+3)$. Now we just have to pick four different numbers from the set $\{0,1,2,3,\ldots,n,n+1,n+2,n+3\}$. This can be done in $\binom{n+4}{4}$ ways.

563 We have

❶ $(T+L+W)!$

❷ $3!T!L!W! = 6T!L!W!$

❸ $\binom{T+L+1}{W}(T+L)!W!$

❹ $\binom{T+L+1}{1}(T+L)!W!$

564 The required number is

$$\binom{20}{1}+\binom{20}{2}+\cdots+\binom{20}{20} = 2^{20}-\binom{20}{0} = 1048576-1 = 1048575.$$

565 The required number is

$$\binom{20}{4}+\binom{20}{6}+\cdots+\binom{20}{20} = 2^{19}-\binom{20}{0}-\binom{20}{2} = 524288-1-190 = 524097.$$

566 We have

❶ $\frac{13!}{2!3!3!} = 86486400$

❷ $\frac{11!}{2!3!} = 3326400$

❸ $\frac{11!}{2!2!2!} = 4989600$

❹ $\binom{12}{1}\frac{11!}{3!3!} = 13305600$

❺ $\binom{12}{2}\frac{11!}{3!3!} = 73180800$

❻ $\binom{10}{1}\frac{9!}{3!3!2!} = 50400$

567 We have

❶ $\binom{M+W}{C}$

❷ $\binom{M}{C-T}\binom{W}{T}$

❸ $\binom{M+W-2}{C-2}$

❹ $\binom{M+W-2}{C}$

568

$$\binom{M+W}{C}-\binom{M+W-2}{C-2} = 2\binom{M+W-2}{C-1}+\binom{M+W-2}{C}.$$

569 2030

570 $2\binom{50}{2}$

571 $\binom{n+k-1}{k}$

572 [1] For the first column one can put any of 4 checkers, for the second one, any of 3, etc. hence there are $4\cdot 3\cdot 2\cdot 1 = 24$.
[2] If there is a column without a checker then there must be a column with 2 checkers. There are 3 choices for this column. In this column we can put the two checkers in $\binom{4}{2} = 6$ ways. Thus there are $4\cdot 3\binom{4}{2}4\cdot 4 = 1152$ ways of putting the checkers.
[3] The number of ways of filling the board with no restrictions is $\binom{16}{4}$. The number of ways of of of filling the board so that there is one checker per column is 4^4. Hence the total is $\binom{16}{4} - 4^4 = 1564$.

573 7560.

574 $\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}$.

575 $\binom{15}{7}\binom{8}{4}$.

575 There are 6513215600 of former and 3486784400 of the latter.

576 $\binom{17}{5}\binom{12}{5}\binom{7}{4}\binom{3}{3}$; $\binom{17}{3}\binom{14}{4}2^{10}$.

577 $\sum_{k=3}^{7}\binom{7}{k} = 99$

577 $2^{10}-1-1-\binom{10}{5} = 1024-2-252 = 770$

577 $\binom{n}{2}$; $n-1$; $\binom{n}{3}$; $\binom{n-1}{2}$

578 $\binom{12}{1}\binom{11}{5}\binom{6}{2}\binom{4}{4}$

579 $\binom{6}{3}^{20} = 104857600000000000000000000$

580 $\binom{9}{3}\binom{5}{3} = 840$

581 $\binom{b}{c}\binom{g}{c}c!$

582 $(2^3-1)(2^4-1)(2^2-1) = 315$

585 $\binom{10}{2}2^8$

586 We have

$$\begin{aligned}\binom{n-1}{k-1}+\binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!}+\frac{(n-1)!}{k!(n-k-1)!}\\ &= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{n-k}+\frac{1}{k}\right)\\ &= \frac{(n-1)!}{(n-k-1)!(k-1)!}\frac{n}{(n-k)k}\\ &= \frac{n!}{(n-k)!k!}.\\ &= \binom{n}{k}.\end{aligned}$$

A combinatorial interpretation can be given as follows. Suppose we have a bag with n red balls. The number of ways of choosing k balls is n. If we now paint one of these balls blue, the number of ways of choosing k balls is the number of ways of choosing balls if we always *include* the blue ball (and this can be done in $\binom{n-1}{k-1}$) ways, plus the number of ways of choosing k balls if we always *exclude* the blue ball (and this can be done in $\binom{n-1}{k}$ ways).

587 The sinistral side counts the number of ways of selecting r elements from a set of n, then selecting k elements from those r. The dextral side counts how many ways to select the k elements first, then select the remaining $r-k$ elements to be chosen from the remaining $n-k$ elements.

588 The dextral side sums

$$\binom{n}{0}\binom{n}{0}+\binom{n}{1}\binom{n}{1}+\binom{n}{2}\binom{n}{2}+\cdots+\binom{n}{n}\binom{n}{n}.$$

By the symmetry identity, this is equivalent to summing

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}.$$

Now consider a bag with $2n$ balls, n of them red and n of them blue. The above sum is counting the number of ways of choosing 0 red balls and n blue balls, 1 red ball and $n-1$ blue balls, 2 red balls and $n-2$ blue balls, etc.. This is clearly the number of ways of choosing n balls of either colour from the bag, which is $\binom{2n}{n}$.

589 11754

590 2844

591 432

592 $\binom{15}{9}$;15!/6!

593 29.

595 2^4

596 $\binom{8}{5}5!$

597 175308642

598 Hint: There are k occupied boxes and $n-k$ empty boxes. Align the balls first! $\binom{k+1}{n-k}$.

599 There are $n-k$ empty seats. Sit the people in between those seats. $\binom{n-k+1}{k}$.

608 36

609 $36-9=25$

610 $\binom{14}{4}$

612 $\binom{50}{3}=19600$

633 $\binom{17}{5}\binom{12}{5}\binom{7}{4}\binom{3}{3}$; $\binom{17}{3}\binom{14}{4}2^{10}$

641 Write $k=m-(m-k)$. Use the absorption identity to evaluate

$$\sum_{k=0}^{n}(m-k)\binom{m-k-1}{n-k-1}.)$$

643 11

644 $a=1990$

645 True.

646 True.

647 $\binom{20}{8}(2^8)(3^{12})$

648 $\binom{15}{8}$

649 840

651 The 166-th

655 $\binom{1000}{6}$

656 0, as $\binom{15}{1}=\binom{15}{14}$, $\binom{15}{2}=\binom{15}{13}$, etc.

657 0

658 False. Sinistral side = $\binom{200}{5}$, dextral side = $\binom{200}{17}$

659 The 62-nd.

663 $\binom{12}{2}$

664 $6\binom{10}{1,1,8}+3^4\binom{10}{0,4,6}$.

665 $\binom{10}{2,3,5}$

681 $\cos^2 x=1-\sin^2 x$

683 $\log_e x > 1$ if $x > e$

697 Let $y = mx$ and divide the equations obtained and solve for m.

6.2 Put $u = x+2, v = y+3$. Divide one equation by the other.

699 Let $u = x+y, v = x-y$.

749 The given equalities entail t

$$\sum_{k=1}^{n}(x_k^2 - x_k)^2 = 0.$$

A sum of squares is 0 if and only if every term is 0. This gives the result.

750 The given equality entails that

$$\frac{1}{2}\left((x_1-x_2)^2+(x_2-x_3)^2+\cdots+(x_{n-1}-x_n)^2+(x_n-x_1)^2\right)=0.$$

A sum of squares is 0 if and only if every term is 0. This gives the result.

751 Since $aB < Ab$ one has $a(b+B) = ab+aB < ab+Ab = (a+A)b$ so $\frac{a}{b} < \frac{a+A}{b+B}$. Similarly $B(a+A) = aB+AB < Ab+AB = A(b+B)$ and so $\frac{a+A}{b+B} < \frac{A}{B}$.

We have

$$\frac{7}{10} < \frac{11}{15} \implies \frac{7}{10} < \frac{18}{25} < \frac{11}{15} \implies \frac{7}{10} < \frac{25}{35} < \frac{18}{25} < \frac{11}{15}.$$

Since $\frac{25}{35} = \frac{5}{7}$, we have $q \leq 7$. Could it be smaller? Observe that $\frac{5}{6} > \frac{11}{15}$ and that $\frac{4}{6} < \frac{7}{10}$. Thus by considering the cases with denominators $q = 1,2,3,4,5,6$, we see that no such fraction lies in the desired interval. The smallest denominator is thus 7.

753 We have

$$(r-s+t)^2 - t^2 = (r-s+t-t)(r-s+t+t) = (r-s)(r-s+2t).$$

Since $t-s \leq 0$, $r-s+2t = r+s+2(t-s) \leq r+s$ and so

$$(r-s+t)^2 - t^2 \leq (r-s)(r+s) = r^2 - s^2$$

which gives

$$(r-s+t)^2 \leq r^2 - s^2 + t^2.$$

754 Using the CBS Inequality (Theorem 742) on $\sum_{k=1}^{n}(a_k b_k)c_k$ once we obtain

$$\sum_{k=1}^{n} a_k b_k c_k \leq \left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2}\left(\sum_{k=1}^{n} c_k^2\right)^{1/2}.$$

Using CBS again on $\left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2}$ we obtain

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k c_k &\leq \left(\sum_{k=1}^{n} a_k^2 b_k^2\right)^{1/2}\left(\sum_{k=1}^{n} c_k^2\right)^{1/2}\\ &\leq \left(\sum_{k=1}^{n} a_k^4\right)^{1/4}\left(\sum_{k=1}^{n} b_k^4\right)^{1/4}\left(\sum_{k=1}^{n} c_k^2\right)^{1/2},\end{aligned}$$

which gives the required inequality.

755 This follows directly from the AM-GM Inequality applied to $1, 2, \ldots, n$:

$$n!^{1/n}(1\cdot 2\cdots n)^{1/n} < \frac{1+2+\cdots+n}{n} = \frac{n+1}{2},$$

where strict inequality follows since the factors are unequal for $n > 1$.

756 First observe that for integer k, $1 < k < n$, $\quad k(n-k+1) = k(n-k)+k > 1(n-k)+k = n$. Thus

$$n!^2 = (1\cdot n)(2\cdot(n-1))(3\cdot(n-2))\cdots((n-1)\cdot 2)(n\cdot 1) > n\cdot n\cdot n\cdots n = n^n.$$

757 Assume without loss of generality that $a \geq b \geq c$. Then $a \geq b \geq c$ is similarly sorted as itself, so by the Rearrangement Inequality

$$a^2+b^2+c^2 = aa+bb+cc \geq ab+bc+ca.$$

This also follows directly from the identity

$$a^2+b^2+c^2-ab-bc-ca = \left(a-\frac{b+c}{2}\right)^2+\frac{3}{4}(b-c)^2.$$

One can also use the AM-GM Inequality thrice:

$$a^2+b^2 \geq 2ab; \quad b^2+c^2 \geq 2bc; \quad c^2+a^2 \geq 2ca,$$

and add.

758 Assume without loss of generality that $a \geq b \geq c$. Then $a \geq b \geq c$ is similarly sorted as $a^2 \geq b^2 \geq c^2$, so by the Rearrangement Inequality

$$a^3+b^3+c^3 = aa^2+bb^2+cc^2 \geq a^2b+b^2c+c^2a,$$

and

$$a^3+b^3+c^3 = aa^2+bb^2+cc^2 \geq a^2c+b^2a+c^2b.$$

Upon adding

$$a^3+b^3+c^3 = aa^2+bb^2+cc^2 \geq \frac{1}{2}\left(a^2(b+c)+b^2(c+a)+c^2(a+b)\right).$$

Again, if $a \geq b \geq c$ then

$$ab \geq ac \geq bc,$$

thus

$$a^3+b^3+c^3 = \geq a^2b+b^2c+c^2a = (ab)a+(bc)b+(ac)c \geq (ab)c+(bc)a+(ac)b = 3abc.$$

This last inequality also follows directly from the AM-GM Inequality, as

$$(a^3b^3c^3)^{1/3} \leq \frac{a^3+b^3+c^3}{3},$$

or from the identity

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca),$$

and the inequality of problem 757.

759 We apply n times the Rearrangement Inequality

$$\begin{array}{ccccc}
\breve{a}_1\hat{b}_1+\breve{a}_2\hat{b}_2+\cdots+\breve{a}_n\hat{b}_n & \leq & a_1b_1+a_2b_2+\cdots+a_nb_n & \leq & \hat{a}_1\hat{b}_1+\hat{a}_2\hat{b}_2+\cdots+\hat{a}_n\hat{b}_n \\
\breve{a}_1\hat{b}_1+\breve{a}_2\hat{b}_2+\cdots+\breve{a}_n\hat{b}_n & \leq & a_1b_2+a_2b_3+\cdots+a_nb_1 & \leq & \hat{a}_1\hat{b}_1+\hat{a}_2\hat{b}_2+\cdots+\hat{a}_n\hat{b}_n \\
\breve{a}_1\hat{b}_1+\breve{a}_2\hat{b}_2+\cdots+\breve{a}_n\hat{b}_n & \leq & a_1b_3+a_2b_4+\cdots+a_nb_2 & \leq & \hat{a}_1\hat{b}_1+\hat{a}_2\hat{b}_2+\cdots+\hat{a}_n\hat{b}_n \\
 & & \vdots & & \\
\breve{a}_1\hat{b}_1+\breve{a}_2\hat{b}_2+\cdots+\breve{a}_n\hat{b}_n & \leq & a_1b_n+a_2b_1+\cdots+a_nb_{n-1} & \leq & \hat{a}_1\hat{b}_1+\hat{a}_2\hat{b}_2+\cdots+\hat{a}_n\hat{b}_n
\end{array}$$

Adding we obtain the desired inequalities.

761 Use the fact that $(b-a)^2 = (\sqrt{b}-\sqrt{a})^2(\sqrt{b}+\sqrt{a})^2$.

762 Let

$$A = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000}$$

and

$$B = \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{10000}{10001}.$$

Clearly, $x^2-1 < x^2$ for all real numbers x. This implies that

$$\frac{x-1}{x} < \frac{x}{x+1}$$

whenever these four quantities are positive. Hence

$$\begin{array}{ccc} 1/2 & < & 2/3 \\ 3/4 & < & 4/5 \\ 5/6 & < & 6/7 \\ \vdots & \vdots & \vdots \\ 9999/10000 & < & 10000/10001 \end{array}$$

As all the numbers involved are positive, we multiply both columns to obtain

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{9999}{10000} < \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{10000}{10001},$$

or $A < B$. This yields $A^2 = A\cdot A < A\cdot B$. Now

$$A\cdot B = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdot\frac{7}{8}\cdots\frac{9999}{10000}\cdot\frac{10000}{10001} = \frac{1}{10001},$$

and consequently, $A^2 < A\cdot B = 1/10001$. We deduce that $A < 1/\sqrt{10001} < 1/100$.

763 Observe that for $k \geq 1$, $(x+k)^2 > (x+k)(x+k-1)$ and so

$$\frac{1}{(x+k)^2} < \frac{1}{(x+k)(x+k-1)} = \frac{1}{x+k-1} - \frac{1}{x+k}.$$

Hence

$$\begin{aligned} \frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}+\frac{1}{(x+3)^2}+\cdots+\frac{1}{(x+n-1)^2}+\frac{1}{(x+n)^2} &< \frac{1}{x(x+1)}+\frac{1}{(x+1)(x+2)}+\frac{1}{(x+2)((x+3))} \\ &\quad +\cdots+\frac{1}{(x+n-2)(x+n-1)}+\frac{1}{(x+n-1)(x+n)} \\ &= \frac{1}{x}-\frac{1}{x+1}+\frac{1}{x+1}-\frac{1}{x+2}+\frac{1}{x+2}-\frac{1}{x+3} \\ &\quad +\cdots+\frac{1}{x+n-2}-\frac{1}{x+n-1}+\frac{1}{x+n-1}-\frac{1}{x+n} \\ &= \frac{1}{x}-\frac{1}{x+n}. \end{aligned}$$

764 For $1 \leq i \leq n$, we have

$$\left|\frac{2}{i}-1-\frac{1}{n}\right| \leq 1-\frac{1}{n} \iff \left(\frac{2}{i}-\left(1+\frac{1}{n}\right)\right)^2 \leq \left(1-\frac{1}{n}\right)^2 \iff \frac{4}{i^2}-\frac{4}{i}\left(1+\frac{1}{n}\right)+\frac{4}{n} \leq 0 \iff \frac{(i-n)(i-1)}{i^2 n} \leq 0.$$

Thus

$$\left|\sum_{i=1}^{n}\frac{x_i}{i}\right| = \frac{1}{2}\left|\sum_{i=1}^{n}\left(\frac{2}{i}-\left(1+\frac{1}{n}\right)\right)x_i\right|,$$

as $\sum_{i=1} x_i = 0$. Now

$$\left|\sum_{i=1}^{n}\left(\frac{2}{i}-\left(1+\frac{1}{n}\right)\right)x_i\right| \leq \sum_{i=1}^{n}\left|\frac{2}{i}-1-\frac{1}{n}\right||x_i| \leq \left(1-\frac{1}{n}\right)\sum_{i=1}^{n}|x_i| = \left(1-\frac{1}{n}\right).$$

765 Expanding the product

$$\prod_{k=1}^{n}(1+x_k) = 1+\sum_{k=1}^{n}x_k+\sum_{1\leq i<j\leq n}x_ix_j+\cdots \geq 1+\sum_{k=1}^{n}x_k,$$

since the $x_k \geq 0$. When $n=1$ equality is obvious. When $n>1$ equality is achieved when $\sum_{1\leq i<j\leq n}^{n} x_ix_j = 0$.

766 Assume $a \geq b \geq c$. Put $s = a+b+c$. Then

$$-a \leq -b \leq -c \implies s-a \leq s-b \leq s-c \implies \frac{1}{s-a} \geq \frac{1}{s-b} \geq \frac{1}{s-c}$$

and so the sequences a,b,c and $\frac{1}{s-a},\frac{1}{s-b},\frac{1}{s-c}$ are similarly sorted. Using the Rearrangement Inequality twice:

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c} \geq \frac{a}{s-c}+\frac{b}{s-a}+\frac{c}{s-b}; \qquad \frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c} \geq \frac{a}{s-b}+\frac{b}{s-c}+\frac{c}{s-a}.$$

Adding these two inequalities

$$2\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right) \geq \frac{b+c}{s-a}+\frac{c+a}{s-b}+\frac{c+a}{s-c},$$

whence

$$2\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 3,$$

from where the result follows.

767 From the AM-GM Inequality,

$$a+b \geq 2\sqrt{ab}; \quad b+c \geq 2\sqrt{bc}; c+a \geq 2\sqrt{ca},$$

and the desired inequality follows upon multiplication of these three inequalities.

768 By the Rearrangement inequality

$$\sum_{k=1}^{n}\frac{a_k}{k^2} \geq \sum_{k=1}^{n}\frac{\breve{a}_k}{k^2} \geq \sum_{k=1}^{n}\frac{1}{k},$$

as $\breve{a}_k \geq k$, the a's being pairwise distinct positive integers.

769 By the AM-GM Inequality,

$$\left(\frac{1}{x_1}\frac{1}{x_2}\cdots\frac{1}{x_n}\right)^{1/n} \leq \frac{\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}}{n},$$

whence the inequality.

770 By the CBS Inequality,

$$(1\cdot x_1+1\cdot x_2+\cdots+1\cdot x_n)^2 \leq \left(1^2+1^2+\cdots+1^2\right)\left(x_1^2+x_2^2+\cdots+x_n^2\right),$$

which gives the desired inequality.

771 Put

$$T_m = \sum_{1 \le k \le m} a_k - \sum_{m < k \le n} a_k.$$

Clearly $T_0 = -T_n$. Since the sequence $T_0, T_1, \dots, T_n$ changes signs, choose an index p such that T_{p-1} and T_p have different signs. Thus either $T_{p-1} - T_p = 2|a_p|$ or $T_p - T_{p-1} = 2|a_p|$. We claim that

$$\min\left(\left|T_{p-1}\right|, \left|T_p\right|\right) =\le \max_{1 \le k \le n} |a_k|.$$

For, if contrariwise both $\left|T_{p-1}\right| > \max_{1 \le k \le n} |a_k|$ and $\left|T_p\right| > \max_{1 \le k \le n} |a_k|$, then $2|a_p| = |T_{p-1} - T_p| > 2 \max_{1 \le k \le n} |a_k|$, a contradiction.