CHAPTER 9

Mathematical Modeling with Differential Equations

EXERCISE SET 9.1

1. $y' = 2x^2 e^{x^3/3} = x^2 y$ and $y(0) = 2$ by inspection.

2. $y' = x^3 - 2\sin x,\ y(0) = 3$ by inspection.

3. **(a)** first order; $\dfrac{dy}{dx} = c; (1+x)\dfrac{dy}{dx} = (1+x)c = y$

(b) second order; $y' = c_1 \cos t - c_2 \sin t,\ y'' + y = -c_1 \sin t - c_2 \cos t + (c_1 \sin t + c_2 \cos t) = 0$

4. **(a)** first order; $2\dfrac{dy}{dx} + y = 2\Big(-\dfrac{c}{2}e^{-x/2} + 1\Big) + ce^{-x/2} + x - 3 = x - 1$

(b) second order; $y' = c_1 e^t - c_2 e^{-t},\ y'' - y = c_1 e^t + c_2 e^{-t} - \big(c_1 e^t + c_2 e^{-t}\big) = 0$

5. $\dfrac{1}{y}\dfrac{dy}{dx} = x\dfrac{dy}{dx} + y,\ \dfrac{dy}{dx}(1 - xy) = y^2,\ \dfrac{dy}{dx} = \dfrac{y^2}{1 - xy}$

6. $2x + y^2 + 2xy\dfrac{dy}{dx} = 0$, by inspection.

7. **(a)** IF: $\mu = e^{3\int dx} = e^{3x}, \dfrac{d}{dx}\left[ye^{3x}\right] = 0, ye^{3x} = C, y = Ce^{-3x}$

separation of variables: $\dfrac{dy}{y} = -3dx, \ln|y| = -3x + C_1, y = \pm e^{-3x}e^{C_1} = Ce^{-3x}$
including $C = 0$ by inspection

(b) IF: $\mu = e^{-2\int dt} = e^{-2t}, \dfrac{d}{dt}[ye^{-2t}] = 0, ye^{-2t} = C, y = Ce^{2t}$

separation of variables: $\dfrac{dy}{y} = 2dt,\ \ln|y| = 2t + C_1,\ y = \pm e^{C_1}e^{2t} = Ce^{2t}$
including $C = 0$ by inspection

8. **(a)** IF: $\mu = e^{-4\int x\,dx} = e^{-2x^2},\ \dfrac{d}{dx}\left[ye^{-2x^2}\right] = 0,\ y = Ce^{2x^2}$

separation of variables: $\dfrac{dy}{y} = 4x\,dx,\ \ln|y| = 2x^2 + C_1,\ y = \pm e^{C_1}e^{2x^2} = Ce^{2x^2}$
including $C = 0$ by inspection

(b) IF: $\mu = e^{\int dt} = e^t,\ \dfrac{d}{dt}\left[ye^t\right] = 0,\ y = Ce^{-t}$

separation of variables: $\dfrac{dy}{y} = -dt,\ \ln|y| = -t + C_1, y = \pm e^{C_1}e^{-t} = Ce^{-t}$
including $C = 0$ by inspection

9. $\mu = e^{\int 3dx} = e^{3x},\ e^{3x}y = \displaystyle\int e^x\,dx = e^x + C,\ y = e^{-2x} + Ce^{-3x}$

10. $\mu = e^{2\int x\,dx} = e^{x^2}, \dfrac{d}{dx}\left[ye^{x^2}\right] = xe^{x^2}, ye^{x^2} = \dfrac{1}{2}e^{x^2} + C, y = \dfrac{1}{2} + Ce^{-x^2}$

11. $\mu = e^{\int dx} = e^x$, $e^x y = \displaystyle\int e^x \cos(e^x)dx = \sin(e^x) + C$, $y = e^{-x}\sin(e^x) + Ce^{-x}$

12. $\dfrac{dy}{dx} + 2y = \dfrac{1}{2}$, $\mu = e^{\int 2dx} = e^{2x}$, $e^{2x}y = \displaystyle\int \frac{1}{2}e^{2x}dx = \frac{1}{4}e^{2x} + C$, $y = \dfrac{1}{4} + Ce^{-2x}$

13. $\dfrac{dy}{dx} + \dfrac{x}{x^2+1}y = 0, \mu = e^{\int(x/(x^2+1))dx} = e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1}$,
$\dfrac{d}{dx}\left[y\sqrt{x^2+1}\right] = 0$, $y\sqrt{x^2+1} = C$, $y = \dfrac{C}{\sqrt{x^2+1}}$

14. $\dfrac{dy}{dx} + y = \dfrac{1}{1+e^x}$, $\mu = e^{\int dx} = e^x$, $e^x y = \displaystyle\int \frac{e^x}{1+e^x}dx = \ln(1+e^x) + C$, $y = e^{-x}\ln(1+e^x) + Ce^{-x}$

15. $\dfrac{1}{y}dy = \dfrac{1}{x}dx$, $\ln|y| = \ln|x| + C_1$, $\ln\left|\dfrac{y}{x}\right| = C_1$, $\dfrac{y}{x} = \pm e^{C_1} = C$, $y = Cx$
including $C = 0$ by inspection

16. $\dfrac{dy}{1+y^2} = x^2\,dx, \tan^{-1}y = \dfrac{1}{3}x^3 + C, y = \tan\left(\dfrac{1}{3}x^3 + C\right)$

17. $\dfrac{dy}{1+y} = -\dfrac{x}{\sqrt{1+x^2}}dx, \ln|1+y| = -\sqrt{1+x^2} + C_1, 1+y = \pm e^{-\sqrt{1+x^2}}e^{C_1} = Ce^{-\sqrt{1+x^2}}$,
$y = Ce^{-\sqrt{1+x^2}} - 1, C \neq 0$

18. $y\,dy = \dfrac{x^3\,dx}{1+x^4}, \dfrac{y^2}{2} = \dfrac{1}{4}\ln(1+x^4) + C_1, 2y^2 = \ln(1+x^4) + C$, $y = \pm\sqrt{[\ln(1+x^4)+C]/2}$

19. $\left(\dfrac{1}{y} + y\right)dy = e^x dx$, $\ln|y| + y^2/2 = e^x + C$; by inspection, $y = 0$ is also a solution

20. $\dfrac{dy}{y} = -x\,dx$, $\ln|y| = -x^2/2 + C_1$, $y = \pm e^{C_1}e^{-x^2/2} = Ce^{-x^2/2}$, including $C = 0$ by inspection

21. $e^y dy = \dfrac{\sin x}{\cos^2 x}dx = \sec x \tan x\,dx$, $e^y = \sec x + C$, $y = \ln(\sec x + C)$

22. $\dfrac{dy}{1+y^2} = (1+x)\,dx, \tan^{-1}y = x + \dfrac{x^2}{2} + C$, $y = \tan(x + x^2/2 + C)$

23. $\dfrac{dy}{y^2-y} = \dfrac{dx}{\sin x}, \displaystyle\int\left[-\frac{1}{y} + \frac{1}{y-1}\right]dy = \int \csc x\,dx$, $\ln\left|\dfrac{y-1}{y}\right| = \ln|\csc x - \cot x| + C_1$,
$\dfrac{y-1}{y} = \pm e^{C_1}(\csc x - \cot x) = C(\csc x - \cot x)$, $y = \dfrac{1}{1 - C(\csc x - \cot x)}, C \neq 0$;
by inspection, $y = 0$ is also a solution, as is $y = 1$.

24. $\dfrac{1}{\tan y}dy = \dfrac{3}{\sec x}dx$, $\dfrac{\cos y}{\sin y}dy = 3\cos x\,dx$, $\ln|\sin y| = 3\sin x + C_1$,
$\sin y = \pm e^{3\sin x + C_1} = \pm e^{C_1}e^{3\sin x} = Ce^{3\sin x}, C \neq 0$,
$y = \sin^{-1}\left(Ce^{3\sin x}\right)$, as is $y = 0$ by inspection

25. $\dfrac{dy}{dx}+\dfrac{1}{x}y=1,\ \mu=e^{\int(1/x)dx}=e^{\ln x}=x,\ \dfrac{d}{dx}[xy]=x,\ xy=\dfrac{1}{2}x^2+C,\ y=x/2+C/x$

(a) $2=y(1)=\dfrac{1}{2}+C, C=\dfrac{3}{2}, y=x/2+3/(2x)$

(b) $2=y(-1)=-1/2-C, C=-5/2, y=x/2-5/(2x)$

26. $\dfrac{dy}{y}=x\,dx,\ \ln|y|=\dfrac{x^2}{2}+C_1,\ y=\pm e^{C_1}e^{x^2/2}=Ce^{x^2/2}$

(a) $1=y(0)=C$ so $C=1,\ y=e^{x^2/2}$

(b) $\dfrac{1}{2}=y(0)=C$, so $y=\dfrac{1}{2}e^{x^2/2}$

27. $\mu=e^{-\int x\,dx}=e^{-x^2/2},\ e^{-x^2/2}y=\displaystyle\int xe^{-x^2/2}dx=-e^{-x^2/2}+C,$

$y=-1+Ce^{x^2/2},\ 3=-1+C,\ C=4,\ y=-1+4e^{x^2/2}$

28. $\mu=e^{\int dt}=e^t,\ e^ty=\displaystyle\int 2e^t\,dt=2e^t+C,\ y=2+Ce^{-t},\ 1=2+C,\ C=-1,\ y=2-e^{-t}$

29. $(y+\cos y)\,dy=4x^2\,dx, \dfrac{y^2}{2}+\sin y=\dfrac{4}{3}x^3+C, \dfrac{\pi^2}{2}+\sin\pi=\dfrac{4}{3}(1)^3+C, \dfrac{\pi^2}{2}=\dfrac{4}{3}+C,$

$C=\dfrac{\pi^2}{2}-\dfrac{4}{3}, 3y^2+6\sin y=8x^3+3\pi^2-8$

30. $\dfrac{dy}{dx}=(x+2)e^y,\ e^{-y}dy=(x+2)dx,\ -e^{-y}=\dfrac{1}{2}x^2+2x+C,\ -1=C,$

$-e^{-y}=\dfrac{1}{2}x^2+2x-1,\ e^{-y}=-\dfrac{1}{2}x^2-2x+1,\ y=-\ln\left(1-2x-\dfrac{1}{2}x^2\right)$

31. $2(y-1)\,dy=(2t+1)\,dt, y^2-2y=t^2+t+C, 1+2=C, C=3, y^2-2y=t^2+t+3$

32. $y'+\dfrac{\sinh x}{\cosh x}y=\cosh x,\ \mu=e^{\int(\sinh x/\cosh x)dx}=e^{\ln\cosh x}=\cosh x,$

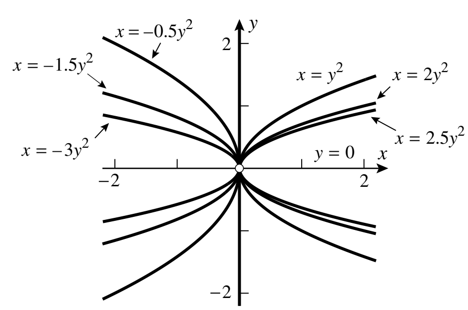
$(\cosh x)y=\displaystyle\int\cosh^2x\,dx=\int\frac{1}{2}(\cosh 2x+1)dx=\frac{1}{4}\sinh 2x+\frac{1}{2}x+C=\frac{1}{2}\sinh x\cosh x+\frac{1}{2}x+C,$

$y=\dfrac{1}{2}\sinh x+\dfrac{1}{2}x\operatorname{sech}x+C\operatorname{sech}x,\ \dfrac{1}{4}=C,\ y=\dfrac{1}{2}\sinh x+\dfrac{1}{2}x\operatorname{sech}x+\dfrac{1}{4}\operatorname{sech}x$

33. **(a)** $\dfrac{dy}{y}=\dfrac{dx}{2x},\ \ln|y|=\dfrac{1}{2}\ln|x|+C_1,$

$|y|=C|x|^{1/2},\ y^2=Cx;$

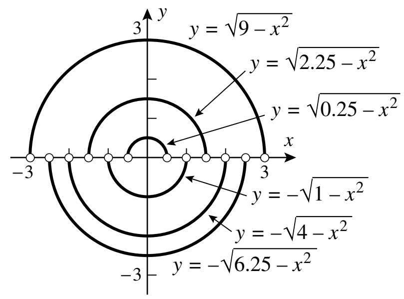
by inspection $y=0$ is also a solution.



(b) $1=C(2)^2, C=1/4, y^2=x/4$

34. **(a)** $y\,dy = -x\,dx, \dfrac{y^2}{2} = -\dfrac{x^2}{2} + C_1, y = \pm\sqrt{C^2 - x^2}$

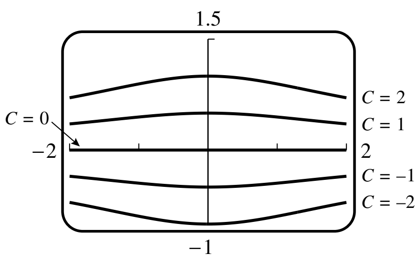
(b) $y = \sqrt{25 - x^2}$



35. $\dfrac{dy}{y} = -\dfrac{x\,dx}{x^2+4},$

$\ln|y| = -\dfrac{1}{2}\ln(x^2+4) + C_1,$

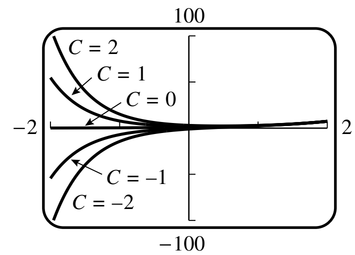
$y = \dfrac{C}{\sqrt{x^2+4}}$



36. $y' + 2y = 3e^t, \mu = e^{2\int dt} = e^{2t},$

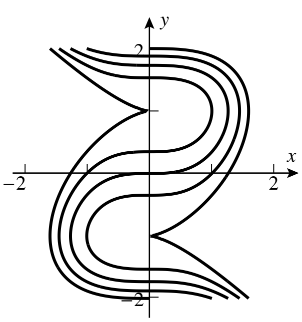
$\dfrac{d}{dt}\left[ye^{2t}\right] = 3e^{3t}, ye^{2t} = e^{3t} + C,$

$y = e^t + Ce^{-2t}$



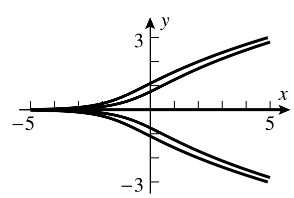
37. $(1 - y^2)\,dy = x^2\,dx,$

$y - \dfrac{y^3}{3} = \dfrac{x^3}{3} + C_1, x^3 + y^3 - 3y = C$



38. $\left(\dfrac{1}{y} + y\right)\,dy = dx,\ \ln|y| + \dfrac{y^2}{2} = x + C_1,$

$ye^{y^2/2} = \pm e^{C_1}e^x = Ce^x$ including $C = 0$



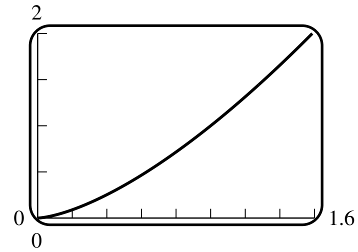
39. Of the solutions $y = \dfrac{1}{2x^2 - C}$, all pass through the point $\left(0, -\dfrac{1}{C}\right)$ and thus never through $(0,0)$. A solution of the initial value problem with $y(0) = 0$ is (by inspection) $y = 0$. The methods of Example 4 fail because the integrals there become divergent when the point $x = 0$ is included in the integral.

40. If $y_0 \neq 0$ then, proceeding as before, we get $C = 2x^2 - \dfrac{1}{y}, C = 2x_0^2 - \dfrac{1}{y_0}$, and

$y = \dfrac{1}{2x^2 - 2x_0^2 + 1/y_0}$, which is defined for all x provided $2x^2$ is never equal to $2x_0^2 - 1/y_0$; this last condition will be satisfied if and only if $2x_0^2 - 1/y_0 < 0$, or $0 < 2x_0^2 y_0 < 1$. If $y_0 = 0$ then $y = 0$ is, by inspection, also a solution for all real x.

41. $\dfrac{dy}{dx} = xe^y, e^{-y}\,dy = x\,dx, -e^{-y} = \dfrac{x^2}{2} + C, x = 2$ when $y = 0$ so $-1 = 2 + C, C = -3, x^2 + 2e^{-y} = 6$

42. $\dfrac{dy}{dx} = \dfrac{3x^2}{2y}, 2y\,dy = 3x^2\,dx, y^2 = x^3 + C, 1 = 1 + C, C = 0,$
$y^2 = x^3,\ y = x^{3/2}$ passes through $(1,1)$.



43. $\dfrac{dy}{dt} =$ rate in $-$ rate out, where y is the amount of salt at time t,

$\dfrac{dy}{dt} = (4)(2) - \left(\dfrac{y}{50}\right)(2) = 8 - \dfrac{1}{25}y$, so $\dfrac{dy}{dt} + \dfrac{1}{25}y = 8$ and $y(0) = 25$.

$\mu = e^{\int (1/25)dt} = e^{t/25},\ e^{t/25}y = \displaystyle\int 8e^{t/25}dt = 200e^{t/25} + C,$

$y = 200 + Ce^{-t/25},\ 25 = 200 + C,\ C = -175,$

(a) $y = 200 - 175e^{-t/25}$ oz **(b)** when $t = 25$, $y = 200 - 175e^{-1} \approx 136$ oz

44. $\dfrac{dy}{dt} = (5)(10) - \dfrac{y}{200}(10) = 50 - \dfrac{1}{20}y$, so $\dfrac{dy}{dt} + \dfrac{1}{20}y = 50$ and $y(0) = 0$.

$\mu = e^{\int \frac{1}{20}dt} = e^{t/20},\ e^{t/20}y = \displaystyle\int 50e^{t/20}dt = 1000e^{t/20} + C,$

$y = 1000 + Ce^{-t/20},\ 0 = 1000 + C,\ C = -1000;$

(a) $y = 1000 - 1000e^{-t/20}$ lb **(b)** when $t = 30$, $y = 1000 - 1000e^{-1.5} \approx 777$ lb

45. The volume V of the (polluted) water is $V(t) = 500 + (20 - 10)t = 500 + 10t$;
if $y(t)$ is the number of pounds of particulate matter in the water,
then $y(0) = 50$, and $\dfrac{dy}{dt} = 0 - 10\dfrac{y}{V} = -\dfrac{1}{50+t}y,\ \dfrac{dy}{dt} + \dfrac{1}{50+t}y = 0;\ \mu = e^{\int \frac{dt}{50+t}} = 50 + t;$
$\dfrac{d}{dt}[(50+t)y] = 0,\ (50+t)y = C,\ 2500 = 50y(0) = C,\ y(t) = 2500/(50+t).$
The tank reaches the point of overflowing when $V = 500 + 10t = 1000$, $t = 50$ min, so
$y = 2500/(50 + 50) = 25$ lb.

46. The volume of the lake (in gallons) is $V = 264\pi r^2 h = 264\pi(15)^2 3 = 178{,}200\pi$ gals. Let $y(t)$ denote the number of pounds of mercury salts at time t, then $\dfrac{dy}{dt} = 0 - 10^3\dfrac{y}{V} = -\dfrac{y}{178.2\pi}$ lb/h and $y_0 = 10^{-5}V = 1.782\pi$ lb; $\dfrac{dy}{y} = -\dfrac{dt}{178.2\pi}, \ln y = -\dfrac{t}{178.2\pi} + C_1,\ y = Ce^{-t/(178.2\pi)}$, and
$C = y(0) = y_0 10^{-5}$ V $= 1.782\pi,\ y = 1.782\pi e^{-t/(178.2\pi)}$ lb of mercury salts.

t	1	2	3	4	5	6	7	8	9	10	11	12
$y(t)$	5.588	5.578	5.568	5.558	5.549	5.539	5.529	5.519	5.509	5.499	5.489	5.480

47. (a) $\dfrac{dv}{dt} + \dfrac{c}{m}v = -g, \mu = e^{(c/m)\int dt} = e^{ct/m}, \dfrac{d}{dt}\left[ve^{ct/m}\right] = -ge^{ct/m}, ve^{ct/m} = -\dfrac{gm}{c}e^{ct/m} + C,$

$v = -\dfrac{gm}{c} + Ce^{-ct/m}$, but $v_0 = v(0) = -\dfrac{gm}{c} + C, C = v_0 + \dfrac{gm}{c}, v = -\dfrac{gm}{c} + \left(v_0 + \dfrac{gm}{c}\right)e^{-ct/m}$

(b) Replace $\dfrac{mg}{c}$ with v_τ and $-ct/m$ with $-gt/v_\tau$ in (23).

(c) From Part (b), $s(t) = C - v_\tau t - (v_0 + v_\tau)\dfrac{v_\tau}{g}e^{-gt/v_\tau}$;

$s_0 = s(0) = C - (v_0 + v_\tau)\dfrac{v_\tau}{g},\ C = s_0 + (v_0 + v_\tau)\dfrac{v_\tau}{g},\ s(t) = s_0 - v_\tau t + \dfrac{v_\tau}{g}(v_0 + v_\tau)\left(1 - e^{-gt/v_\tau}\right)$

48. Given $m = 240, g = 32, v_\tau = mg/c$: with a closed parachute $v_\tau = 120$ so $c = 64$, and with an open parachute $v_\tau = 24$, $c = 320$.

(a) Let t denote time elapsed in seconds after the moment of the drop. From Exercise 47(b), while the parachute is closed
$v(t) = e^{-gt/v_\tau}(v_0 + v_\tau) - v_\tau = e^{-32t/120}(0 + 120) - 120 = 120\left(e^{-4t/15} - 1\right)$ and thus $v(25) = 120\left(e^{-20/3} - 1\right) \approx -119.85$, so the parachutist is falling at a speed of 119.85 ft/s when the parachute opens. From Exercise 47(c), $s(t) = s_0 - 120t + \dfrac{120}{32}120\left(1 - e^{-4t/15}\right)$,
$s(25) = 10000 - 120 \cdot 25 + 450\left(1 - e^{-20/3}\right) \approx 7449.43$ ft.

(b) If t denotes time elapsed after the parachute opens, then, by Exercise 47(c),
$s(t) = 7449.43 - 24t + \dfrac{24}{32}(-119.85 + 24)\left(1 - e^{-32t/24}\right) = 0$, with the solution (Newton's Method) $t = 307.4$ s, so the sky diver is in the air for about $25 + 307.4 = 332.4$ s.

49. $\dfrac{dI}{dt} + \dfrac{R}{L}I = \dfrac{V(t)}{L}, \mu = e^{(R/L)\int dt} = e^{Rt/L}, \dfrac{d}{dt}(e^{Rt/L}I) = \dfrac{V(t)}{L}e^{Rt/L},$

$Ie^{Rt/L} = I(0) + \dfrac{1}{L}\displaystyle\int_0^t V(u)e^{Ru/L}du, I(t) = I(0)e^{-Rt/L} + \dfrac{1}{L}e^{-Rt/L}\int_0^t V(u)e^{Ru/L}du.$

(a) $I(t) = \dfrac{1}{4}e^{-5t/2}\displaystyle\int_0^t 12e^{5u/2}du = \dfrac{6}{5}e^{-5t/2}e^{5u/2}\Big]_0^t = \dfrac{6}{5}\left(1 - e^{-5t/2}\right)$ A.

(b) $\displaystyle\lim_{t\to+\infty} I(t) = \dfrac{6}{5}$ A

50. From Exercise 49 and Endpaper Table #42,

$$I(t) = 15e^{-2t} + \frac{1}{3}e^{-2t}\int_0^t 3e^{2u}\sin u\,du = 15e^{-2t} + e^{-2t}\frac{e^{2u}}{5}(2\sin u - \cos u)\Big]_0^t$$

$$= 15e^{-2t} + \frac{1}{5}(2\sin t - \cos t) + \frac{1}{5}e^{-2t}.$$

51. (a) $\dfrac{dv}{dt} = \dfrac{ck}{m_0 - kt} - g, v = -c\ln(m_0 - kt) - gt + C; v = 0$ when $t = 0$ so $0 = -c\ln m_0 + C,$

$C = c\ln m_0, v = c\ln m_0 - c\ln(m_0 - kt) - gt = c\ln\dfrac{m_0}{m_0 - kt} - gt.$

(b) $m_0 - kt = 0.2m_0$ when $t = 100$ so

$v = 2500\ln\dfrac{m_0}{0.2m_0} - 9.8(100) = 2500\ln 5 - 980 \approx 3044\,\text{m/s}.$

52. **(a)** By the chain rule, $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$ so $m\frac{dv}{dt} = mv\frac{dv}{dx}$.

(b) $\frac{mv}{kv^2 + mg}dv = -dx$, $\frac{m}{2k}\ln(kv^2 + mg) = -x + C$; $v = v_0$ when $x = 0$ so

$C = \frac{m}{2k}\ln(kv_0^2 + mg)$, $\frac{m}{2k}\ln(kv^2 + mg) = -x + \frac{m}{2k}\ln(kv_0^2 + mg)$, $x = \frac{m}{2k}\ln\frac{kv_0^2 + mg}{kv^2 + mg}$.

(c) $x = x_{max}$ when $v = 0$ so

$$x_{max} = \frac{m}{2k}\ln\frac{kv_0^2 + mg}{mg} = \frac{3.56 \times 10^{-3}}{2(7.3 \times 10^{-6})}\ln\frac{(7.3 \times 10^{-6})(988)^2 + (3.56 \times 10^{-3})(9.8)}{(3.56 \times 10^{-3})(9.8)} \approx 1298\,\text{m}$$

53. **(a)** $A(h) = \pi(1)^2 = \pi$, $\pi\frac{dh}{dt} = -0.025\sqrt{h}$, $\frac{\pi}{\sqrt{h}}dh = -0.025dt$, $2\pi\sqrt{h} = -0.025t + C$; $h = 4$ when $t = 0$, so $4\pi = C$, $2\pi\sqrt{h} = -0.025t + 4\pi$, $\sqrt{h} = 2 - \frac{0.025}{2\pi}t$, $h \approx (2 - 0.003979\,t)^2$.

(b) $h = 0$ when $t \approx 2/0.003979 \approx 502.6$ s ≈ 8.4 min.

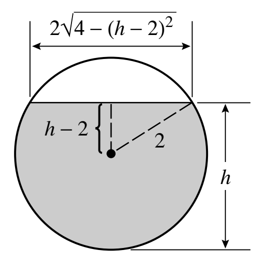
54. **(a)** $A(h) = 6\left[2\sqrt{4 - (h-2)^2}\right] = 12\sqrt{4h - h^2}$,

$12\sqrt{4h - h^2}\,\frac{dh}{dt} = -0.025\sqrt{h}$, $12\sqrt{4 - h}\,dh = -0.025dt$,

$-8(4 - h)^{3/2} = -0.025t + C$; $h = 4$ when $t = 0$ so $C = 0$,

$(4 - h)^{3/2} = (0.025/8)t$, $4 - h = (0.025/8)^{2/3}t^{2/3}$,

$h \approx 4 - 0.021375t^{2/3}$ ft



(b) $h = 0$ when $t = \frac{8}{0.025}(4 - 0)^{3/2} = 2560$ s ≈ 42.7 min

55. $\frac{dv}{dt} = -0.04v^2$, $\frac{1}{v^2}dv = -0.04dt$, $-\frac{1}{v} = -0.04t + C$; $v = 50$ when $t = 0$ so $-\frac{1}{50} = C$, $-\frac{1}{v} = -0.04t - \frac{1}{50}$, $v = \frac{50}{2t + 1}$ cm/s. But $v = \frac{dx}{dt}$ so $\frac{dx}{dt} = \frac{50}{2t + 1}$, $x = 25\ln(2t + 1) + C_1$; $x = 0$ when $t = 0$ so $C_1 = 0$, $x = 25\ln(2t + 1)$ cm.

56. $\frac{dv}{dt} = -0.02\sqrt{v}$, $\frac{1}{\sqrt{v}}dv = -0.02dt$, $2\sqrt{v} = -0.02t + C$; $v = 9$ when $t = 0$ so $6 = C$,

$2\sqrt{v} = -0.02t + 6$, $v = (3 - 0.01t)^2$ cm/s. But $v = \frac{dx}{dt}$ so $\frac{dx}{dt} = (3 - 0.01t)^2$,

$x = -\frac{100}{3}(3 - 0.01t)^3 + C_1$; $x = 0$ when $t = 0$ so $C_1 = 900$, $x = 900 - \frac{100}{3}(3 - 0.01t)^3$ cm.

57. Differentiate to get $\frac{dy}{dx} = -\sin x + e^{-x^2}$, $y(0) = 1$.

58. **(a)** Let $y = \frac{1}{\mu}[H(x) + C]$ where $\mu = e^{P(x)}$, $\frac{dP}{dx} = p(x)$, $\frac{d}{dx}H(x) = \mu q$, and C is an arbitrary constant. Then

$$\frac{dy}{dx} + p(x)y = \frac{1}{\mu}H'(x) - \frac{\mu'}{\mu^2}[H(x) + C] + p(x)y = q - \frac{p}{\mu}[H(x) + C] + p(x)y = q$$

(b) Given the initial value problem, let $C = \mu(x_0)y_0 - H(x_0)$. Then $y = \dfrac{1}{\mu}[H(x)+C]$ is a solution of the initial value problem with $y(x_0) = y_0$. This shows that the initial value problem has a solution.

To show uniqueness, suppose $u(x)$ also satisfies (5) together with $u(x_0) = y_0$. Following the arguments in the text we arrive at $u(x) = \dfrac{1}{\mu}[H(x) + C]$ for some constant C. The initial condition requires $C = \mu(x_0)y_0 - H(x_0)$, and thus $u(x)$ is identical with $y(x)$.

59. Suppose that $H(y) = G(x) + C$. Then $\dfrac{dH}{dy}\dfrac{dy}{dx} = G'(x)$. But $\dfrac{dH}{dy} = h(y)$ and $\dfrac{dG}{dx} = g(x)$, hence $y(x)$ is a solution of (10).

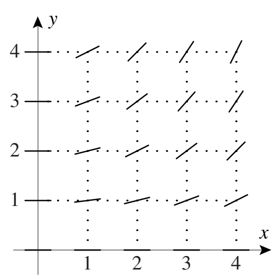
60. **(a)** $y = x$ and $y = -x$ are both solutions of the given initial value problem.

(b) $\displaystyle\int y\,dy = -\int x\,dx, y^2 = -x^2 + C$; but $y(0) = 0$, so $C = 0$. Thus $y^2 = -x^2$, which is impossible.

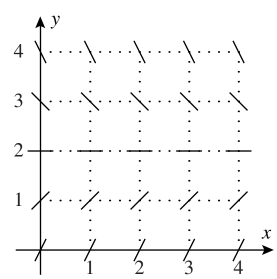
61. Suppose $I_1 \subset I$ is an interval with $I_1 \neq I$,and suppose $Y(x)$ is defined on I_1 and is a solution of (5) there. Let x_0 be a point of I_1. Solve the initial value problem on I with initial value $y(x_0) = Y(x_0)$. Then $y(x)$ is an extension of $Y(x)$ to the interval I, and by Exercise 58(b) applied to the interval I_1, it follows that $y(x) = Y(x)$ for x in I_1.

EXERCISE SET 9.2

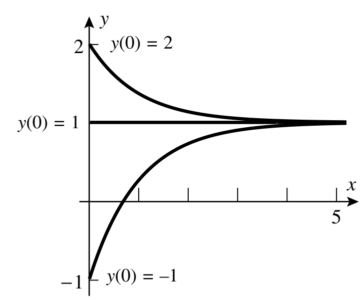
1.



2.



3.



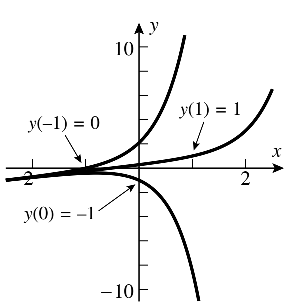
4. $\dfrac{dy}{dx} + y = 1, \mu = e^{\int dx} = e^x$, $\dfrac{d}{dx}[ye^x] = e^x$, $ye^x = e^x + C, y = 1 + Ce^{-x}$

(a) $-1 = 1 + C, C = -2, y = 1 - 2e^{-x}$

(b) $1 = 1 + C, C = 0, y = 1$

(c) $2 = 1 + C, C = 1, y = 1 + e^{-x}$

5.



6. $\dfrac{dy}{dx} - 2y = -x, \quad \mu = e^{-2\int dx} = e^{-2x}, \quad \dfrac{d}{dx}\left[ye^{-2x}\right] = -xe^{-2x},$

$ye^{-2x} = \dfrac{1}{4}(2x+1)e^{-2x} + C, \quad y = \dfrac{1}{4}(2x+1) + Ce^{2x}$

(a) $1 = 3/4 + Ce^2,\ C = 1/(4e^2),\ y = \dfrac{1}{4}(2x+1) + \dfrac{1}{4}e^{2x-2}$

(b) $-1 = 1/4 + C, C = -5/4, y = \dfrac{1}{4}(2x+1) - \dfrac{5}{4}e^{2x}$

(c) $0 = -1/4 + Ce^{-2},\ C = e^2/4,\ y = \dfrac{1}{4}(2x+1) + \dfrac{1}{4}e^{2x+2}$

7. $\lim_{x\to+\infty} y = 1$

8. $\lim_{x\to+\infty} y = \begin{cases} +\infty & \text{if } y_0 \ge 1/4 \\ -\infty, & \text{if } y_0 < 1/4 \end{cases}$

9. **(a)** IV, since the slope is positive for $x > 0$ and negative for $x < 0$.
(b) VI, since the slope is positive for $y > 0$ and negative for $y < 0$.
(c) V, since the slope is always positive.
(d) II, since the slope changes sign when crossing the lines $y = \pm 1$.
(e) I, since the slope can be positive or negative in each quadrant but is not periodic.
(f) III, since the slope is periodic in both x and y.

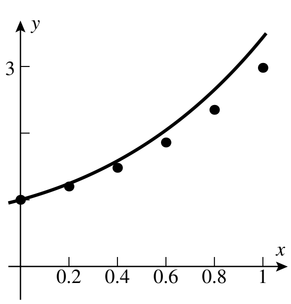
11. **(a)** $y_0 = 1,$
$y_{n+1} = y_n + (x_n + y_n)(0.2) = (x_n + 6y_n)/5$

n	0	1	2	3	4	5
x_n	0	0.2	0.4	0.6	0.8	1.0
y_n	1	1.20	1.48	1.86	2.35	2.98

(b) $y' - y = x,\ \mu = e^{-x},\ \dfrac{d}{dx}\left[ye^{-x}\right] = xe^{-x},$
$ye^{-x} = -(x+1)e^{-x} + C,\ 1 = -1 + C,$
$C = 2,\ y = -(x+1) + 2e^x$

x_n	0	0.2	0.4	0.6	0.8	1.0
$y(x_n)$	1	1.24	1.58	2.04	2.65	3.44
abs. error	0	0.04	0.10	0.19	0.30	0.46
perc. error	0	3	6	9	11	13

(c)



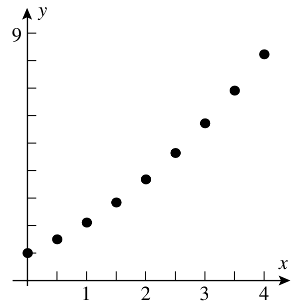
12. $h = 0.1, y_{n+1} = (x_n + 11y_n)/10$

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1.00	1.10	1.22	1.36	1.53	1.72	1.94	2.20	2.49	2.82	3.19

In Exercise 11, $y(1) \approx 2.98$; in Exercise 12, $y(1) \approx 3.19$; the true solution is $y(1) \approx 3.44$; so the absolute errors are approximately 0.46 and 0.25 respectively.

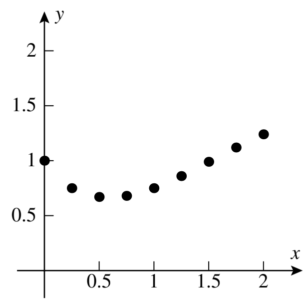
13. $y_0 = 1, y_{n+1} = y_n + \sqrt{y_n}/2$

n	0	1	2	3	4	5	6	7	8
x_n	0	0.5	1	1.5	2	2.5	3	3.5	4
y_n	1	1.50	2.11	2.84	3.68	4.64	5.72	6.91	8.23



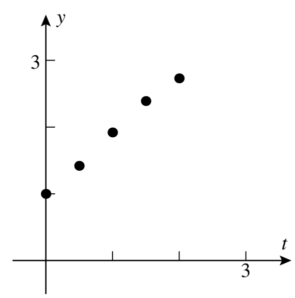
14. $y_0 = 1, y_{n+1} = y_n + (x_n - y_n^2)/4$

n	0	1	2	3	4	5	6	7	8
x_n	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
y_n	1	0.75	0.67	0.68	0.75	0.86	0.99	1.12	1.24



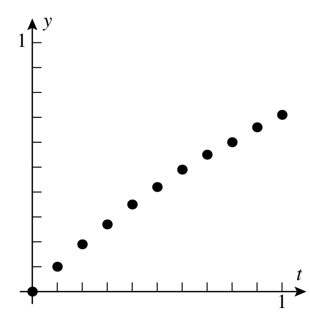
15. $y_0 = 1, y_{n+1} = y_n + \dfrac{1}{2}\sin y_n$

n	0	1	2	3	4
t_n	0	0.5	1	1.5	2
y_n	1	1.42	1.92	2.39	2.73



16. $y_0 = 0,\ y_{n+1} = y_n + e^{-y_n}/10$

n	0	1	2	3	4	5	6	7	8	9	10
t_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	0	0.10	0.19	0.27	0.35	0.42	0.49	0.55	0.60	0.66	0.71



17. $h = 1/5,\ y_0 = 1,\ y_{n+1} = y_n + \dfrac{1}{5}\cos(2\pi n/5)$

n	0	1	2	3	4	5
t_n	0	0.2	0.4	0.6	0.8	1.0
y_n	1.00	1.06	0.90	0.74	0.80	1.00

18. **(a)** By inspection, $\dfrac{dy}{dx} = e^{-x^2}$ and $y(0) = 0$.

(b) $y_{n+1} = y_n + e^{-x_n^2}/20 = y_n + e^{-(n/20)^2}/20$ and $y_{20} = 0.7625$. From a CAS, $y(1) = 0.7468$.

19. **(b)** $y\,dy = -x\,dx,\ y^2/2 = -x^2/2 + C_1,\ x^2 + y^2 = C$; if $y(0) = 1$ then $C = 1$ so $y(1/2) = \sqrt{3}/2$.

20. **(a)** $y_0 = 1, y_{n+1} = y_n + (\sqrt{y_n}/2)\Delta x$

$\Delta x = 0.2:\ y_{n+1} = y_n + \sqrt{y_n}/10;\ y_5 \approx 1.5489$

$\Delta x = 0.1: y_{n+1} = y_n + \sqrt{y_n}/20;\ y_{10} \approx 1.5556$

$\Delta x = 0.05: y_{n+1} = y_n + \sqrt{y_n}/40;\ y_{20} \approx 1.5590$

(c) $\dfrac{dy}{\sqrt{y}} = \dfrac{1}{2}dx,\ 2\sqrt{y} = x/2 + C,\ 2 = C,$

$\sqrt{y} = x/4 + 1, y = (x/4 + 1)^2,$

$y(1) = 25/16 = 1.5625$

EXERCISE SET 9.3

1. **(a)** $\dfrac{dy}{dt} = ky^2,\ y(0) = y_0, k > 0$ **(b)** $\dfrac{dy}{dt} = -ky^2,\ y(0) = y_0, k > 0$

3. **(a)** $\dfrac{ds}{dt} = \dfrac{1}{2}s$ **(b)** $\dfrac{d^2s}{dt^2} = 2\dfrac{ds}{dt}$

4. **(a)** $\dfrac{dv}{dt} = -2v^2$ **(b)** $\dfrac{d^2s}{dt^2} = -2\left(\dfrac{ds}{dt}\right)^2$

5. **(a)** $\dfrac{dy}{dt} = 0.01y,\ y_0 = 10{,}000$ **(b)** $y = 10{,}000e^{t/100}$

(c) $T = \dfrac{1}{k}\ln 2 = \dfrac{1}{0.01}\ln 2 \approx 69.31$ h **(d)** $45{,}000 = 10{,}000e^{t/100}$,

$t = 100\ln\dfrac{45{,}000}{10{,}000} \approx 150.41$ h

6. $k = \dfrac{1}{T}\ln 2 = \dfrac{1}{20}\ln 2$

(a) $\dfrac{dy}{dt} = ((\ln 2)/20)y,\ y(0) = 1$ **(b)** $y(t) = e^{t(\ln 2)/20} = 2^{t/20}$

(c) $y(120) = 2^6 = 64$ **(d)** $1{,}000{,}000 = 2^{t/20}$,

$t = 20\dfrac{\ln 10^6}{\ln 2} \approx 398.63$ min

7. **(a)** $\dfrac{dy}{dt} = -ky,\ y(0) = 5.0 \times 10^7$; $3.83 = T = \dfrac{1}{k}\ln 2$, so $k = \dfrac{\ln 2}{3.83} \approx 0.1810$

(b) $y = 5.0 \times 10^7 e^{-0.181t}$

(c) $y(30) = 5.0 \times 10^7 e^{-0.1810(30)} \approx 219{,}000$

(d) $y(t) = (0.1)y_0 = y_0e^{-kt},\ -kt = \ln 0.1,\ t = -\dfrac{\ln 0.1}{0.1810} = 12.72$ days

8. **(a)** $k = \dfrac{1}{T}\ln 2 = \dfrac{1}{140}\ln 2 \approx 0.0050$, so $\dfrac{dy}{dt} = -0.0050y,\ y_0 = 10.$

(b) $y = 10e^{-0.0050t}$

(c) 10 weeks = 70 days so $y = 10e^{-0.35} \approx 7$ mg.

(d) $0.3y_0 = y_0e^{-kt},\ t = -\dfrac{\ln 0.3}{0.0050} \approx 240.8$ days

9. $100e^{0.02t} = 5000,\ e^{0.02t} = 50,\ t = \dfrac{1}{0.02}\ln 50 \approx 196$ days

10. $y = 10{,}000e^{kt}$, but $y = 12{,}000$ when $t = 10$ so $10{,}000e^{10k} = 12{,}000,\ k = \dfrac{1}{10}\ln 1.2$. $y = 20{,}000$ when $2 = e^{kt},\ t = \dfrac{\ln 2}{k} = 10\dfrac{\ln 2}{\ln 1.2} \approx 38$, in the year 2025.

11. $y(t) = y_0e^{-kt} = 10.0e^{-kt},\ 3.5 = 10.0e^{-k(5)},\ k = -\dfrac{1}{5}\ln\dfrac{3.5}{10.0} \approx 0.2100,\ T = \dfrac{1}{k}\ln 2 \approx 3.30$ days

12. $y = y_0e^{-kt},\ 0.6y_0 = y_0e^{-5k},\ k = -\dfrac{1}{5}\ln 0.6 \approx 0.10$

(a) $T = \dfrac{\ln 2}{k} \approx 6.9$ yr

(b) $y(t) \approx y_0e^{-0.10t},\ \dfrac{y}{y_0} \approx e^{-0.10t}$, so $e^{-0.10t} \times 100$ percent will remain.

13. **(a)** $k = \dfrac{\ln 2}{5} \approx 0.1386; y \approx 2e^{0.1386t}$

(b) $y(t) = 5e^{0.015t}$

(c) $y = y_0e^{kt},\ 1 = y_0e^{k},\ 100 = y_0e^{10k}$. Divide: $100 = e^{9k},\ k = \dfrac{1}{9}\ln 100 \approx 0.5117$, $y \approx y_0e^{0.5117t}$; also $y(1) = 1$, so $y_0 = e^{-0.5117} \approx 0.5995, y \approx 0.5995e^{0.5117t}$.

(d) $k = \dfrac{\ln 2}{T} \approx 0.1386,\ 1 = y(1) \approx y_0e^{0.1386},\ y_0 \approx e^{-0.1386} \approx 0.8706,\ y \approx 0.8706e^{0.1386t}$

14. **(a)** $k = \dfrac{\ln 2}{T} \approx 0.1386,\ y \approx 10e^{-0.1386t}$

(b) $y = 10e^{-0.015t}$

(c) $100 = y_0e^{-k},\ 1 = y_0e^{-10k}$. Divide: $e^{9k} = 100,\ k = \dfrac{1}{9}\ln 100 \approx 0.5117$; $y_0 = e^{10k} \approx e^{5.117} \approx 166.83,\ y = 166.83e^{-0.5117t}$.

(d) $k = \dfrac{\ln 2}{T} \approx 0.1386,\ 10 = y(1) \approx y_0e^{-0.1386},\ y_0 \approx 10e^{0.1386} \approx 11.4866,\ y \approx 11.4866e^{-0.1386t}$

16. **(a)** None; the half-life is independent of the initial amount.
(b) $kT = \ln 2$, so T is inversely proportional to k.

17. **(a)** $T = \dfrac{\ln 2}{k}$; and $\ln 2 \approx 0.6931$. If k is measured in percent, $k' = 100k$, then $T = \dfrac{\ln 2}{k} \approx \dfrac{69.31}{k'} \approx \dfrac{70}{k'}$.

(b) 70 yr

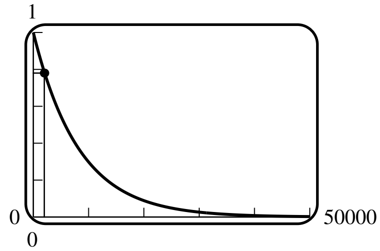
(c) 20 yr

(d) 7%

18. Let $y = y_0e^{kt}$ with $y = y_1$ when $t = t_1$ and $y = 3y_1$ when $t = t_1 + T$; then $y_0e^{kt_1} = y_1$ (i) and $y_0e^{k(t_1+T)} = 3y_1$ (ii). Divide (ii) by (i) to get $e^{kT} = 3,\ T = \dfrac{1}{k}\ln 3$.

19. From (12), $y(t) = y_0e^{-0.000121t}$. If $0.27 = \dfrac{y(t)}{y_0} = e^{-0.000121t}$ then $t = -\dfrac{\ln 0.27}{0.000121} \approx 10{,}820$ yr, and if $0.30 = \dfrac{y(t)}{y_0}$ then $t = -\dfrac{\ln 0.30}{0.000121} \approx 9950$, or roughly between 9000 B.C. and 8000 B.C.

20. **(a)**



(b) $t = 1988$ yields

$y/y_0 = e^{-0.000121(1988)} \approx 79\%$.

21. $y_0 \approx 2,\ L \approx 8$; since the curve $y = \dfrac{2 \cdot 8}{2 + 6e^{-kt}}$ passes through the point $(2, 4)$, $4 = \dfrac{16}{2 + 6e^{-2k}}$, $6e^{-2k} = 2,\ k = \dfrac{1}{2}\ln 3 \approx 0.5493$.

22. $y_0 \approx 400,\ L \approx 1000$; since the curve $y = \dfrac{400{,}000}{400 + 600e^{-kt}}$ passes through the point $(200, 600)$, $600 = \dfrac{400{,}000}{400 + 600e^{-200k}},\ 600e^{-200k} = \dfrac{800}{3},\ k = \dfrac{1}{200}\ln 2.25 \approx 0.00405$.

23. **(a)** $y_0 = 5$ **(b)** $L = 12$ **(c)** $k = 1$

(d) $L/2 = 6 = \dfrac{60}{5 + 7e^{-t}},\ 5 + 7e^{-t} = 10,\ t = -\ln(5/7) \approx 0.3365$

(e) $\dfrac{dy}{dt} = \dfrac{1}{12}y(12 - y),\ y(0) = 5$

24. **(a)** $y_0 = 1$ **(b)** $L = 1000$ **(c)** $k = 0.9$

(d) $750 = \dfrac{1000}{1 + 999e^{-0.9t}},\ 3(1 + 999e^{-0.9t}) = 4,\ t = \dfrac{1}{0.9}\ln(3 \cdot 999) \approx 8.8949$

(e) $\dfrac{dy}{dt} = \dfrac{0.9}{1000}y(1000 - y),\ y(0) = 1$

25. See (13):

(a) $L = 10$ **(b)** $k = 10$

(c) $\dfrac{dy}{dt} = 10(1 - 0.1y)y = 25 - (y - 5)^2$ is maximized when $y = 5$.

26. $\dfrac{dy}{dt} = 50y\left(1 - \dfrac{1}{50{,}000}y\right)$; from (13), $k = 50,\ L = 50{,}000$.

(a) $L = 50{,}000$ **(b)** $k = 50$

(c) $\dfrac{dy}{dt}$ is maximized when $0 = \dfrac{d}{dy}\left(\dfrac{dy}{dt}\right) = 50 - y/500, y = 25{,}000$

27. Assume $y(t)$ students have had the flu t days after semester break. Then $y(0) = 20,\ y(5) = 35$.

(a) $\dfrac{dy}{dt} = ky(L - y) = ky(1000 - y),\ y_0 = 20$

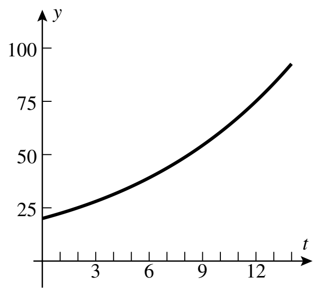
(b) Part (a) has solution $y = \dfrac{20000}{20 + 980e^{-kt}} = \dfrac{1000}{1 + 49e^{-kt}}$;

$35 = \dfrac{1000}{1 + 49e^{-5k}}$, $k = 0.115$, $y \approx \dfrac{1000}{1 + 49e^{-0.115t}}$.

(c)

t	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$y(t)$	20	22	25	28	31	35	39	44	49	54	61	67	75	83	93

(d)



28. **(a)** $\dfrac{dp}{dh} = -kp,\ p(0) = p_0$

(b) $p_0 = 1$, so $p = e^{-kh}$, but $p = 0.83$ when $h = 5000$ thus $e^{-5000k} = 0.83$,

$k = -\dfrac{\ln 0.83}{5000} \approx 0.0000373,\ p \approx e^{-0.0000373h}$ atm.

29. **(a)** $\dfrac{dT}{dt} = -k(T - 21),\ T(0) = 95,\ \dfrac{dT}{T - 21} = -k\,dt,\ \ln(T - 21) = -kt + C_1,$

$T = 21 + e^{C_1}e^{-kt} = 21 + Ce^{-kt},\ 95 = T(0) = 21 + C, C = 74,\ T = 21 + 74e^{-kt}$

(b) $85 = T(1) = 21 + 74e^{-k},\ k = -\ln\dfrac{64}{74} = -\ln\dfrac{32}{37},\ T = 21 + 74e^{t\ln(32/37)} = 21 + 74\left(\dfrac{32}{37}\right)^t,$

$T = 51$ when $\dfrac{30}{74} = \left(\dfrac{32}{37}\right)^t,\ t = \dfrac{\ln(30/74)}{\ln(32/37)} \approx 6.22$ min

30. $\dfrac{dT}{dt} = k(70 - T), T(0) = 40;\ -\ln(70 - T) = kt + C,\ 70 - T = e^{-kt}e^{-C},\ T = 40$ when $t = 0$, so

$30 = e^{-C},\ T = 70 - 30e^{-kt};\ 52 = T(1) = 70 - 30e^{-k},\ k = -\ln\dfrac{70 - 52}{30} = \ln\dfrac{5}{3} \approx 0.5,$

$T \approx 70 - 30e^{-0.5t}$

31. Let T denote the body temperature of McHam's body at time t, the number of hours elapsed after 10:06 P.M.; then $\dfrac{dT}{dt} = -k(T - 72),\ \dfrac{dT}{T - 72} = -kdt,\ \ln(T - 72) = -kt + C, T = 72 + e^Ce^{-kt},$

$77.9 = 72 + e^C,\ e^C = 5.9,\ T = 72 + 5.9e^{-kt},\ 75.6 = 72 + 5.9e^{-k},\ k = -\ln\dfrac{3.6}{5.9} \approx 0.4940,$

$T = 72 + 5.9e^{-0.4940t}$. McHam's body temperature was last $98.6°$ when $t = -\dfrac{\ln(26.6/5.9)}{0.4940} \approx -3.05$, so around 3 hours and 3 minutes before 10:06; the death took place at approximately 7:03 P.M., while Moore was on stage.

32. If $T_0 < T_a$ then $\dfrac{dT}{dt} = k(T_a - T)$ where $k > 0$. If $T_0 > T_a$ then $\dfrac{dT}{dt} = -k(T - T_a)$ where $k > 0$; both cases yield $T(t) = T_a + (T_0 - T_a)e^{-kt}$ with $k > 0$.

33. **(a)** $y = y_0 b^t = y_0 e^{t\ln b} = y_0 e^{kt}$ with $k = \ln b > 0$ since $b > 1$.

(b) $y = y_0 b^t = y_0 e^{t\ln b} = y_0 e^{-kt}$ with $k = -\ln b > 0$ since $0 < b < 1$.

(c) $y = 4(2^t) = 4e^{t\ln 2}$ **(d)** $y = 4(0.5^t) = 4e^{t\ln 0.5} = 4e^{-t\ln 2}$

34. If $y = y_0 e^{kt}$ and $y = y_1 = y_0 e^{kt_1}$ then $y_1/y_0 = e^{kt_1}$, $k = \dfrac{\ln(y_1/y_0)}{t_1}$; if $y = y_0 e^{-kt}$ and $y = y_1 = y_0 e^{-kt_1}$ then $y_1/y_0 = e^{-kt_1}$, $k = -\dfrac{\ln(y_1/y_0)}{t_1}$.

EXERCISE SET 9.4

1. **(a)** $y = e^{2x}$, $y' = 2e^{2x}$, $y'' = 4e^{2x}$; $y'' - y' - 2y = 0$
$y = e^{-x}$, $y' = -e^{-x}$, $y'' = e^{-x}$; $y'' - y' - 2y = 0$.

(b) $y = c_1 e^{2x} + c_2 e^{-x}$, $y' = 2c_1 e^{2x} - c_2 e^{-x}$, $y'' = 4c_1 e^{2x} + c_2 e^{-x}$; $y'' - y' - 2y = 0$

2. **(a)** $y = e^{-2x}$, $y' = -2e^{-2x}$, $y'' = 4e^{-2x}$; $y'' + 4y' + 4y = 0$
$y = xe^{-2x}$, $y' = (1-2x)e^{-2x}$, $y'' = (4x-4)e^{-2x}$; $y'' + 4y' + 4y = 0$.

(b) $y = c_1 e^{-2x} + c_2 x e^{-2x}$, $y' = -2c_1 e^{-2x} + c_2(1-2x)e^{-2x}$,
$y'' = 4c_1 e^{-2x} + c_2(4x-4)e^{-2x}$; $y'' + 4y' + 4y = 0$.

3. $m^2 + 3m - 4 = 0$, $(m-1)(m+4) = 0$; $m = 1, -4$ so $y = c_1 e^x + c_2 e^{-4x}$.

4. $m^2 + 6m + 5 = 0$, $(m+1)(m+5) = 0$; $m = -1, -5$ so $y = c_1 e^{-x} + c_2 e^{-5x}$.

5. $m^2 - 2m + 1 = 0$, $(m-1)^2 = 0$; $m = 1$, so $y = c_1 e^x + c_2 x e^x$.

6. $m^2 + 6m + 9 = 0$, $(m+3)^2 = 0$; $m = -3$ so $y = c_1 e^{-3x} + c_2 x e^{-3x}$.

7. $m^2 + 5 = 0$, $m = \pm\sqrt{5}\,i$ so $y = c_1 \cos\sqrt{5}\,x + c_2 \sin\sqrt{5}\,x$.

8. $m^2 + 1 = 0$, $m = \pm i$ so $y = c_1 \cos x + c_2 \sin x$.

9. $m^2 - m = 0$, $m(m-1) = 0$; $m = 0, 1$ so $y = c_1 + c_2 e^x$.

10. $m^2 + 3m = 0$, $m(m+3) = 0$; $m = 0, -3$ so $y = c_1 + c_2 e^{-3x}$.

11. $m^2 + 4m + 4 = 0$, $(m+2)^2 = 0$; $m = -2$ so $y = c_1 e^{-2t} + c_2 t e^{-2t}$.

12. $m^2 - 10m + 25 = 0$, $(m-5)^2 = 0$; $m = 5$ so $y = c_1 e^{5t} + c_2 t e^{5t}$.

13. $m^2 - 4m + 13 = 0$, $m = 2 \pm 3i$ so $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$.

14. $m^2 - 6m + 25 = 0$, $m = 3 \pm 4i$ so $y = e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$.

15. $8m^2 - 2m - 1 = 0$, $(4m+1)(2m-1) = 0$; $m = -1/4, 1/2$ so $y = c_1 e^{-x/4} + c_2 e^{x/2}$.

16. $9m^2 - 6m + 1 = 0$, $(3m-1)^2 = 0$; $m = 1/3$ so $y = c_1 e^{x/3} + c_2 x e^{x/3}$.

17. $m^2 + 2m - 3 = 0$, $(m+3)(m-1) = 0$; $m = -3, 1$ so $y = c_1 e^{-3x} + c_2 e^x$ and $y' = -3c_1 e^{-3x} + c_2 e^x$.
Solve the system $c_1 + c_2 = 1$, $-3c_1 + c_2 = 5$ to get $c_1 = -1$, $c_2 = 2$ so $y = -e^{-3x} + 2e^x$.

18. $m^2 - 6m - 7 = 0$, $(m+1)(m-7) = 0$; $m = -1, 7$ so $y = c_1e^{-x} + c_2e^{7x}$, $y' = -c_1e^{-x} + 7c_2e^{7x}$. Solve the system $c_1 + c_2 = 5$, $-c_1 + 7c_2 = 3$ to get $c_1 = 4$, $c_2 = 1$ so $y = 4e^{-x} + e^{7x}$.

19. $m^2 - 6m + 9 = 0$, $(m-3)^2 = 0$; $m = 3$ so $y = (c_1 + c_2x)e^{3x}$ and $y' = (3c_1 + c_2 + 3c_2x)e^{3x}$. Solve the system $c_1 = 2$, $3c_1 + c_2 = 1$ to get $c_1 = 2$, $c_2 = -5$ so $y = (2 - 5x)e^{3x}$.

20. $m^2 + 4m + 1 = 0$, $m = -2 \pm \sqrt{3}$ so $y = c_1e^{(-2+\sqrt{3})x} + c_2e^{(-2-\sqrt{3})x}$,
$y' = (-2+\sqrt{3})c_1e^{(-2+\sqrt{3})x} + (-2-\sqrt{3})c_2e^{(-2-\sqrt{3})x}$. Solve the system $c_1 + c_2 = 5$,
$(-2+\sqrt{3})c_1 + (-2-\sqrt{3})c_2 = 4$ to get $c_1 = \frac{5}{2} + \frac{7}{3}\sqrt{3}$, $c_2 = \frac{5}{2} - \frac{7}{3}\sqrt{3}$ so
$y = \left(\frac{5}{2} + \frac{7}{3}\sqrt{3}\right)e^{(-2+\sqrt{3})x} + \left(\frac{5}{2} - \frac{7}{3}\sqrt{3}\right)e^{(-2-\sqrt{3})x}$.

21. $m^2 + 4m + 5 = 0$, $m = -2 \pm i$ so $y = e^{-2x}(c_1\cos x + c_2\sin x)$,
$y' = e^{-2x}[(c_2 - 2c_1)\cos x - (c_1 + 2c_2)\sin x]$. Solve the system $c_1 = -3$, $c_2 - 2c_1 = 0$ to get $c_1 = -3$, $c_2 = -6$ so $y = -e^{-2x}(3\cos x + 6\sin x)$.

22. $m^2 - 6m + 13 = 0$, $m = 3 \pm 2i$ so $y = e^{3x}(c_1\cos 2x + c_2\sin 2x)$,
$y' = e^{3x}[(3c_1 + 2c_2)\cos 2x - (2c_1 - 3c_2)\sin 2x]$. Solve the system $c_1 = -1$, $3c_1 + 2c_2 = 1$ to get $c_1 = -1$, $c_2 = 2$ so $y = e^{3x}(-\cos 2x + 2\sin 2x)$.

23. **(a)** $m = 5, -2$ so $(m-5)(m+2) = 0$, $m^2 - 3m - 10 = 0$; $y'' - 3y' - 10y = 0$.

(b) $m = 4, 4$ so $(m-4)^2 = 0$, $m^2 - 8m + 16 = 0$; $y'' - 8y' + 16y = 0$.

(c) $m = -1 \pm 4i$ so $(m + 1 - 4i)(m + 1 + 4i) = 0$, $m^2 + 2m + 17 = 0$; $y'' + 2y' + 17y = 0$.

24. $c_1e^x + c_2e^{-x}$ is the general solution, but $\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$ and $\sinh x = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$ so $\cosh x$ and $\sinh x$ are also solutions.

25. $m^2 + km + k = 0$, $m = \left(-k \pm \sqrt{k^2 - 4k}\right)/2$

(a) $k^2 - 4k > 0$, $k(k-4) > 0$; $k < 0$ or $k > 4$

(b) $k^2 - 4k = 0$; $k = 0, 4$

(c) $k^2 - 4k < 0$, $k(k-4) < 0$; $0 < k < 4$

26. $z = \ln x$; $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx} = \dfrac{1}{x}\dfrac{dy}{dz}$ and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz},$$

substitute into the original equation to get $\dfrac{d^2y}{dz^2} + (p-1)\dfrac{dy}{dz} + qy = 0$.

27. **(a)** $\dfrac{d^2y}{dz^2} + 2\dfrac{dy}{dz} + 2y = 0$, $m^2 + 2m + 2 = 0$; $m = -1 \pm i$ so
$y = e^{-z}(c_1\cos z + c_2\sin z) = \dfrac{1}{x}[c_1\cos(\ln x) + c_2\sin(\ln x)]$.

(b) $\dfrac{d^2y}{dz^2} - 2\dfrac{dy}{dz} - 2y = 0$, $m^2 - 2m - 2 = 0$; $m = 1 \pm \sqrt{3}$ so
$y = c_1e^{(1+\sqrt{3})z} + c_2e^{(1-\sqrt{3})z} = c_1x^{1+\sqrt{3}} + c_2x^{1-\sqrt{3}}$

28. $m^2 + pm + q = 0$, $m = \frac{1}{2}(-p \pm \sqrt{p^2 - 4q})$. If $0 < q < p^2/4$ then $y = c_1e^{m_1x} + c_2e^{m_2x}$ where $m_1 < 0$ and $m_2 < 0$, if $q = p^2/4$ then $y = c_1e^{-px/2} + c_2xe^{-px/2}$, if $q > p^2/4$ then $y = e^{-px/2}(c_1\cos kx + c_2\sin kx)$ where $k = \frac{1}{2}\sqrt{4q - p^2}$. In all cases $\lim_{x\to+\infty} y(x) = 0$.

29. **(a)** Neither is a constant multiple of the other, since, e.g. if $y_1 = ky_2$ then $e^{m_1x} = ke^{m_2x}$, $e^{(m_1-m_2)x} = k$. But the right hand side is constant, and the left hand side is constant only if $m_1 = m_2$, which is false.

(b) If $y_1 = ky_2$ then $e^{mx} = kxe^{mx}, kx = 1$ which is impossible. If $y_2 = y_1$ then $xe^{mx} = ke^{mx}$, $x = k$ which is impossible.

30. $y_1 = e^{ax}\cos bx$, $y_1' = e^{ax}(a\cos bx - b\sin bx)$, $y_1'' = e^{ax}[(a^2 - b^2)\cos bx - 2ab\sin bx]$ so $y_1'' + py_1' + qy_1 = e^{ax}[(a^2 - b^2 + ap + q)\cos bx - (2ab + bp)\sin bx]$. But $a = -\frac{1}{2}p$ and $b = \frac{1}{2}\sqrt{4q - p^2}$ so $a^2 - b^2 + ap + q = 0$ and $2ab + bp = 0$ thus $y_1'' + py_1' + qy_1 = 0$. Similarly, $y_2 = e^{ax}\sin bx$ is also a solution.
Since $y_1/y_2 = \cot bx$ and $y_2/y_1 = \tan bx$ it is clear that the two solutions are linearly independent.

31. **(a)** The general solution is $c_1e^{\mu x} + c_2e^{mx}$; let $c_1 = 1/(\mu - m)$, $c_2 = -1/(\mu - m)$.

(b) $\displaystyle\lim_{\mu\to m}\frac{e^{\mu x} - e^{mx}}{\mu - m} = \lim_{\mu\to m} xe^{\mu x} = xe^{mx}$.

32. **(a)** If $\lambda = 0$, then $y'' = 0$, $y = c_1 + c_2x$. Use $y(0) = 0$ and $y(\pi) = 0$ to get $c_1 = c_2 = 0$. If $\lambda < 0$, then let $\lambda = -a^2$ where $a > 0$ so $y'' - a^2y = 0$, $y = c_1e^{ax} + c_2e^{-ax}$. Use $y(0) = 0$ and $y(\pi) = 0$ to get $c_1 = c_2 = 0$.

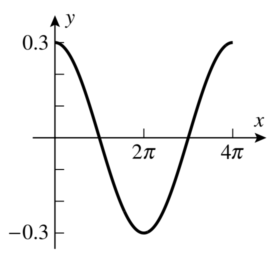
(b) If $\lambda > 0$, then $m^2 + \lambda = 0$, $m^2 = -\lambda = \lambda i^2$, $m = \pm\sqrt{\lambda}i$, $y = c_1\cos\sqrt{\lambda}\,x + c_2\sin\sqrt{\lambda}\,x$. If $y(0) = 0$ and $y(\pi) = 0$, then $c_1 = 0$ and $c_1\cos\pi\sqrt{\lambda} + c_2\sin\pi\sqrt{\lambda} = 0$ so $c_2\sin\pi\sqrt{\lambda} = 0$. But $c_2\sin\pi\sqrt{\lambda} = 0$ for arbitrary values of c_2 if $\sin\pi\sqrt{\lambda} = 0$, $\pi\sqrt{\lambda} = n\pi$, $\lambda = n^2$ for $n = 1, 2, 3, \ldots$, otherwise $c_2 = 0$.

33. $k/M = 0.25/1 = 0.25$

(a) From (20), $y = 0.3\cos(t/2)$

(b) $T = 2\pi\cdot 2 = 4\pi$ s, $f = 1/T = 1/(4\pi)$ Hz

(c)



(d) $y = 0$ at the equilibrium position, so $t/2 = \pi/2, t = \pi$ s.

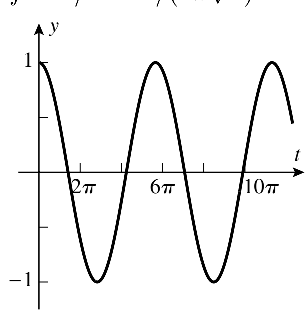
(e) $t/2 = \pi$ at the maximum position below the equlibrium position, so $t = 2\pi$ s.

34. $64 = w = -Mg$, $M = 2$, $k/M = 0.25/2 = 1/8$, $\sqrt{k/M} = 1/(2\sqrt{2})$

(a) From (20), $y = \cos(t/(2\sqrt{2}))$

(b) $T = 2\pi\sqrt{\dfrac{M}{k}} = 2\pi(2\sqrt{2}) = 4\pi\sqrt{2}$ s,
$f = 1/T = 1/(4\pi\sqrt{2})$ Hz

(c)



(d) $y = 0$ at the equilibrium position, so $t/(2\sqrt{2}) = \pi/2, t = \pi\sqrt{2}$ s

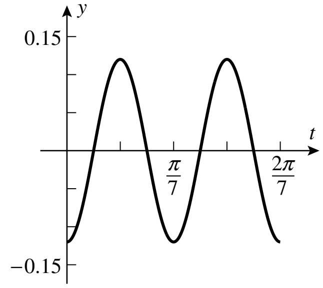
(e) $t/(2\sqrt{2}) = \pi, t = 2\pi\sqrt{2}$ s

35. $l = 0.05,\ k/M = g/l = 9.8/0.05 = 196\ \text{s}^{-2}$

(a) From (20), $y = -0.12\cos 14t$.

(b) $T = 2\pi\sqrt{M/k} = 2\pi/14 = \pi/7$ s, $f = 7/\pi$ Hz

(c)



(d) $14t = \pi/2,\ t = \pi/28$ s

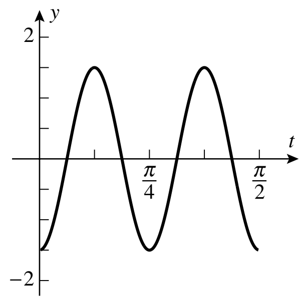
(e) $14t = \pi,\ t = \pi/14$ s

36. $l = 0.5,\ k/M = g/l = 32/0.5 = 64,\ \sqrt{k/M} = 8$

(a) From (20), $y = -1.5\cos 8t$.

(b) $T = 2\pi\sqrt{M/k} = 2\pi/8 = \pi/4$ s; $f = 1/T = 4/\pi$ Hz

(c)



(d) $8t = \pi/2,\ t = \pi/16$ s

(e) $8t = \pi,\ t = \pi/8$ s

37. Assume $y = y_0\cos\sqrt{\dfrac{k}{M}}\,t$, so $v = \dfrac{dy}{dt} = -y_0\sqrt{\dfrac{k}{M}}\sin\sqrt{\dfrac{k}{M}}\,t$

(a) The maximum speed occurs when $\sin\sqrt{\dfrac{k}{M}}\,t = \pm 1,\ \sqrt{\dfrac{k}{M}}\,t = n\pi + \pi/2$, so $\cos\sqrt{\dfrac{k}{M}}\,t = 0,\ y = 0$.

(b) The minimum speed occurs when $\sin\sqrt{\dfrac{k}{M}}\,t = 0,\ \sqrt{\dfrac{k}{M}}\,t = n\pi$, so $\cos\sqrt{\dfrac{k}{M}}\,t = \pm 1,\ y = \pm y_0$.

38. **(a)** $T = 2\pi\sqrt{\dfrac{M}{k}},\ k = \dfrac{4\pi^2}{T^2}M = \dfrac{4\pi^2}{T^2}\dfrac{w}{g}$, so $k = \dfrac{4\pi^2}{g}\dfrac{w}{9} = \dfrac{4\pi^2}{g}\dfrac{w+4}{25},\ 25w = 9(w+4)$,

$25w = 9w + 36, w = \dfrac{9}{4}, k = \dfrac{4\pi^2}{g}\dfrac{w}{9} = \dfrac{4\pi^2}{32}\dfrac{1}{4} = \dfrac{\pi^2}{32}$

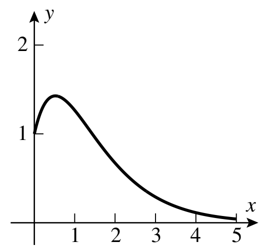
(b) From Part (a), $w = \dfrac{9}{4}$

39. By Hooke's Law, $F(t) = -kx(t)$, since the only force is the restoring force of the spring. Newton's Second Law gives $F(t) = Mx''(t)$, so $Mx''(t) + kx(t) = 0,\ x(0) = x_0, x'(0) = 0$.

40. $0 = v(0) = y'(0) = c_2\sqrt{\dfrac{k}{M}}$, so $c_2 = 0$; $y_0 = y(0) = c_1$, so $y = y_0\cos\sqrt{\dfrac{k}{M}}\,t$.

41. **(a)** $m^2 + 2.4m + 1.44 = 0, (m + 1.2)^2 = 0, m = -1.2,\ y = C_1 e^{-6t/5} + C_2 t e^{-6t/5}$,

$C_1 = 1,\ 2 = y'(0) = -\frac{6}{5}C_1 + C_2, C_2 = \frac{16}{5},\ y = e^{-6t/5} + \frac{16}{5}te^{-6t/5}$



(b) $y'(t) = 0$ when $t = t_1 = 25/48 \approx 0.520833,\ y(t_1) = 1.427364$ cm

(c) $y = \frac{16}{5}\ e^{-6t/5}(t + 5/16) = 0$ only if $t = -5/16$, so $y \neq 0$ for $t \geq 0$.

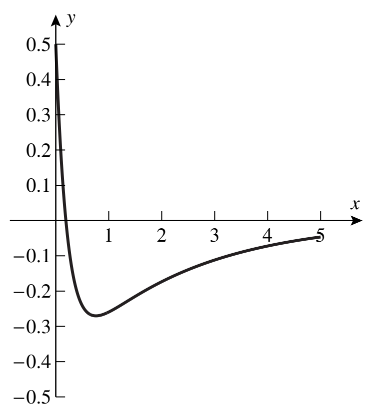
42. **(a)** $m^2 + 5m + 2 = (m + 5/2)^2 - 17/4 = 0, m = -5/2 \pm \sqrt{17}/2$,

$y = C_1 e^{(-5+\sqrt{17})t/2} + C_2 e^{(-5-\sqrt{17})t/2}$,

$C_1 + C_2 = 1/2,\ -4 = y'(0) = \frac{-5+\sqrt{17}}{2}C_1 + \frac{-5-\sqrt{17}}{2}C_2$

$C_1 = \frac{17 - 11\sqrt{17}}{68}, C_2 = \frac{17 + 11\sqrt{17}}{68}$

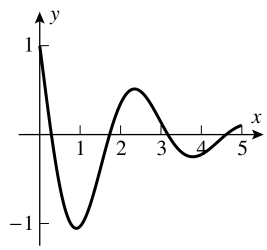
$y = \frac{17 - 11\sqrt{17}}{68}e^{(-5+\sqrt{17})t/2} + \frac{17 + 11\sqrt{17}}{68}e^{-(5+\sqrt{17})t/2}$



(b) $y'(t) = 0$ when $t = t_1 = 0.759194, y(t_1) = -0.270183$ cm so the maximum distance below the equilibrium position is 0.270183 cm.

(c) $y(t) = 0$ when $t = t_2 = 0.191132,\ y'(t_2) = -1.581022$ cm/sec so the speed is $|y'(t_2)| = 1.581022$ cm/s.

43. **(a)** $m^2+m+5=0, m=-1/2\pm(\sqrt{19}/2)i,\ y=e^{-t/2}\left[C_1\cos(\sqrt{19}t/2)+C_2\sin(\sqrt{19}t/2)\right]$,
$1=y(0)=C_1, -3.5=y'(0)=-(1/2)C_1+(\sqrt{19}/2)C_2,\ C_2=-6/\sqrt{19}$,
$y=e^{-t/2}\cos(\sqrt{19}\,t/2)-(6/\sqrt{19}\,)e^{-t/2}\sin(\sqrt{19}\,t/2)$



(b) $y'(t)=0$ for the first time when $t=t_1=0.905533$, $y(t_1)=-1.054466$ cm so the maximum distance below the equilibrium position is 1.054466 cm.

(c) $y(t)=0$ for the first time when $t=t_2=0.288274$, $y'(t_2)=-3.210357$ cm/s.

(d) The acceleration is $y''(t)$ so from the differential equation $y''=-y'-5y$. But $y=0$ when the object passes through the equilibrium position, thus $y''=-y'=3.210357$ cm/s^2.

44. **(a)** $m^2+m+3m=0, m=-1/2\pm\sqrt{11}i/2,\ y=e^{-t/2}\left[C_1\cos(\sqrt{11}t/2)+C_2\sin(\sqrt{11}t/2)\right]$,
$-2=y(0)=C_1, v_0=y'(0)=-(1/2)C_1+(\sqrt{11}/2)C_2,\ C_2=(v_0-1)(2/\sqrt{11})$,
$y(t)=e^{-t/2}\left[-2\cos(\sqrt{11}\,t/2)+(2/\sqrt{11}\,)(v_0-1)\sin(\sqrt{11}\,t/2)\right]$
$y'(t)=e^{-t/2}\left[v_0\cos(\sqrt{11}t/2)+\left[(12-v_0)/\sqrt{11}\right]\sin(\sqrt{11}t/2)\right]$

(b) We wish to find v_0 such that $y(t)=1$ but no greater. This implies that $y'(t)=0$ at that point. So find the largest value of v_0 such that there is a solution of $y'(t)=0, y(t)=1$. Note that $y'(t)=0$ when $\tan\frac{\sqrt{11}}{2}t=\frac{v_0\sqrt{11}}{v_0-12}$. Choose the smallest positive solution t_0 of this equation. Then

$$\sec^2\frac{\sqrt{11}}{2}t_0=1+\tan^2\frac{\sqrt{11}}{2}t_0=\frac{12[(v_0-1)^2+11]}{(v_0-12)^2}.$$

Assume for now that $v_0<12$; if not, we will deal with it later. Then $\tan\frac{\sqrt{11}}{2}t_0<0$, so

$\frac{\pi}{2}<\frac{\sqrt{11}}{2}t_0<\pi$, and $\sec\frac{\sqrt{11}}{2}t_0=\frac{2\sqrt{3}\sqrt{(v_0-1)^2+11}}{v_0-12}$

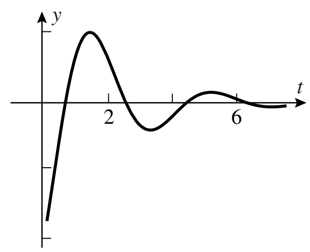
and $\cos\frac{\sqrt{11}}{2}t_0=\frac{v_0-12}{\sqrt{3}\,\sqrt{(v_0-1)^2+11}}$,

$\sin\frac{\sqrt{11}}{2}t_0=\tan\frac{\sqrt{11}}{2}t_0\cos\frac{\sqrt{11}}{2}t_0=\frac{v_0\sqrt{11}}{2\sqrt{3}\sqrt{(v_0-1)^2+11}}$, and

$$y(t_0)=e^{-t_0/2}\left[-2\cos\frac{\sqrt{11}}{2}t_0+\frac{2(v_0-1)}{\sqrt{11}}\sin\frac{\sqrt{11}}{2}t_0\right]=e^{-t_0/2}\frac{\sqrt{(v_0-1)^2+11}}{\sqrt{3}}.$$

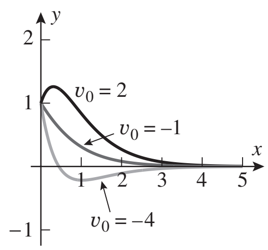
Use various values of $v_0, 0<v_0<12$ to determine the transition point from $y<1$ to $y>1$ and then refine the partition on the values of v to arrive at $v\approx 2.44$ cm/s.

(c)



45. **(a)** $m^2 + 3.5m + 3 = (m + 1.5)(m + 2), y = C_1 e^{-3t/2} + C_2 e^{-2t}$,
$1 = y(0) = C_1 + C_2, v_0 = y'(0) = -(3/2)C_1 - 2C_2,\ C_1 = 4 + 2v_0, C_2 = -3 - 2v_0$,
$y(t) = (4 + 2v_0)e^{-3t/2} - (3 + 2v_0)e^{-2t}$

(b) $v_0 = 2, y(t) = 8e^{-3t/2} - 7e^{-2t},\ v_0 = -1, y(t) = 2e^{-3t/2} - e^{-2t}$,
$v_0 = -4, y(t) = -4e^{-3t/2} + 5e^{-2t}$



46. $\dfrac{dy}{dt} + p(x)y = c\dfrac{dy_1}{dt} + p(x)(cy_1) = c\left[\dfrac{dy_1}{dt} + p(x)y_1\right] = c \cdot 0 = 0$

CHAPTER 9 SUPPLEMENTARY EXERCISES

4. The differential equation in Part (c) is not separable; the others are.

5. **(a)** linear **(b)** linear and separable **(c)** separable **(d)** neither

6. IF: $\mu = e^{-2x^2},\ \dfrac{d}{dx}\left[ye^{-2x^2}\right] = xe^{-2x^2},\ ye^{-2x^2} = -\dfrac{1}{4}e^{-2x^2} + C,\ y = -\dfrac{1}{4} + Ce^{2x^2}$

Sep of var: $\dfrac{dy}{4y+1} = x\,dx,\ \dfrac{1}{4}\ln|4y+1| = \dfrac{x^2}{2} + C_1,\ 4y+1 = \pm e^{4C_1}e^{2x^2} = C_2 e^{2x^2};\ y = -\dfrac{1}{4} + Ce^{2x^2}$, including $C = 0$

7. The parabola $ky(L - y)$ opens down and has its maximum midway between the y-intercepts, that is, at the point $y = \dfrac{1}{2}(0 + L) = L/2$, where $\dfrac{dy}{dt} = k(L/2)^2 = kL^2/4$.

8. **(a)** If $y = y_0e^{kt}$, then $y_1 = y_0e^{kt_1}$, $y_2 = y_0e^{kt_2}$, divide: $y_2/y_1 = e^{k(t_2-t_1)}$, $k = \dfrac{1}{t_2 - t_1}\ln(y_2/y_1)$, $T = \dfrac{\ln 2}{k} = \dfrac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}$. If $y = y_0e^{-kt}$, then $y_1 = y_0e^{-kt_1}$, $y_2 = y_0e^{-kt_2}$, $y_2/y_1 = e^{-k(t_2-t_1)}, k = -\dfrac{1}{t_2 - t_1}\ln(y_2/y_1)$, $T = \dfrac{\ln 2}{k} = -\dfrac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}$.

In either case, T is positive, so $T = \left|\dfrac{(t_2 - t_1)\ln 2}{\ln(y_2/y_1)}\right|$.

(b) In Part (a) assume $t_2 = t_1 + 1$ and $y_2 = 1.25y_1$. Then $T = \dfrac{\ln 2}{\ln 1.25} \approx 3.1$ h.

9. $\dfrac{dV}{dt} = -kS$; but $V = \dfrac{4\pi}{3}r^3, \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$, $S = 4\pi r^2$, so $dr/dt = -k, r = -kt + C, 4 = C$, $r = -kt + 4, 3 = -k + 4, k = 1, r = 4 - t$ m.

10. Assume the tank contains $y(t)$ oz of salt at time t. Then $y_0 = 0$ and for $0 < t < 15$, $\dfrac{dy}{dt} = 5 \cdot 10 - \dfrac{y}{1000}10 = (50 - y/100)$ oz/min, with solution $y = 5000 + Ce^{-t/100}$. But $y(0) = 0$ so $C = -5000$, $y = 5000(1 - e^{-t/100})$ for $0 \le t \le 15$, and $y(15) = 5000(1 - e^{-0.15})$. For $15 < t < 30$, $\dfrac{dy}{dt} = 0 - \dfrac{y}{1000}5$, $y = C_1e^{-t/200}$, $C_1e^{-0.075} = y(15) = 5000(1 - e^{-0.15})$, $C_1 = 5000(e^{0.075} - e^{-0.075})$, $y = 5000(e^{0.075} - e^{-0.075})e^{-t/100}, y(30) = 5000(e^{0.075} - e^{-0.075})e^{-0.3} \approx 556.13$ oz.

11. **(a)** Assume the air contains $y(t)$ ft^3 of carbon monoxide at time t. Then $y_0 = 0$ and for $t > 0$, $\dfrac{dy}{dt} = 0.04(0.1) - \dfrac{y}{1200}(0.1) = 1/250 - y/12000$, $\dfrac{d}{dt}\left[ye^{t/12000}\right] = \dfrac{1}{250}e^{t/12000}$, $ye^{t/12000} = 48e^{t/12000} + C$, $y(0) = 0$, $C = -48$; $y = 48(1 - e^{-t/12000})$. Thus the percentage of carbon monoxide is $P = \dfrac{y}{1200}100 = 4(1 - e^{-t/12000})$ percent.

(b) $0.012 = 4(1 - e^{-t/12000})$, $t = 36.05$ min

12. $\dfrac{dy}{y^2 + 1} = dx$, $\tan^{-1} y = x + C$, $\pi/4 = C$; $y = \tan(x + \pi/4)$

13. $\left(\dfrac{1}{y^5} + \dfrac{1}{y}\right) dy = \dfrac{dx}{x}$, $-\dfrac{1}{4}y^{-4} + \ln|y| = \ln|x| + C$; $-\dfrac{1}{4} = C$, $y^{-4} + 4\ln(x/y) = 1$

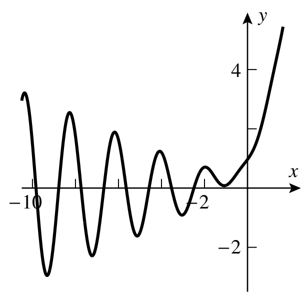
14. $\dfrac{dy}{dx} + \dfrac{2}{x}y = 4x$, $\mu = e^{\int(2/x)dx} = x^2$, $\dfrac{d}{dx}\left[yx^2\right] = 4x^3$, $yx^2 = x^4 + C$, $y = x^2 + Cx^{-2}$, $2 = y(1) = 1 + C$, $C = 1$, $y = x^2 + 1/x^2$

15. $\dfrac{dy}{y^2} = 4\sec^2 2x\,dx,\ -\dfrac{1}{y} = 2\tan 2x + C,\ -1 = 2\tan\left(2\dfrac{\pi}{8}\right) + C = 2\tan\dfrac{\pi}{4} + C = 2 + C,\ C = -3,$
$y = \dfrac{1}{3 - 2\tan 2x}$

16. $\dfrac{dy}{y^2 - 5y + 6} = dx,\ \dfrac{dy}{(y-3)(y-2)} = dx,\ \left[\dfrac{1}{y-3} - \dfrac{1}{y-2}\right] dy = dx,\ \ln\left|\dfrac{y-3}{y-2}\right| = x + C_1,$
$\dfrac{y-3}{y-2} = Ce^x;\ y = \ln 2 \text{ if } x = 0, \text{ so } C = \dfrac{\ln 2 - 3}{\ln 2 - 2};\ y = \dfrac{3 - 2Ce^x}{1 - Ce^x} = \dfrac{3\ln 2 - 6 - (2\ln 2 - 6)e^x}{\ln 2 - 2 - (\ln 2 - 3)e^x}$

17. **(a)** $\mu = e^{-\int dx} = e^{-x},\ \dfrac{d}{dx}\left[ye^{-x}\right] = xe^{-x}\sin 3x,$
$ye^{-x} = \displaystyle\int xe^{-x}\sin 3x\,dx = \left(-\frac{3}{10}x - \frac{3}{50}\right)e^{-x}\cos 3x + \left(-\frac{1}{10}x + \frac{2}{25}\right)e^{-x}\sin 3x + C;$
$1 = y(0) = -\dfrac{3}{50} + C,\ C = \dfrac{53}{50},\ y = \left(-\dfrac{3}{10}x - \dfrac{3}{50}\right)\cos 3x + \left(-\dfrac{1}{10}x + \dfrac{2}{25}\right)\sin 3x + \dfrac{53}{50}e^x$

(c)



19. **(a)** Let $T_1 = 5730 - 40 = 5690, k_1 = \dfrac{\ln 2}{T_1} \approx 0.00012182; T_2 = 5730 + 40 = 5770, k_2 \approx 0.00012013.$ With $y/y_0 = 0.92, 0.93,\ t_1 = -\dfrac{1}{k_1}\ln\dfrac{y}{y_0} = 684.5, 595.7; t_2 = -\dfrac{1}{k_2}\ln(y/y_0) = 694.1, 604.1;$ in 1988 the shroud was at most 695 years old, which places its creation in or after the year 1293.

(b) Suppose T is the true half-life of carbon-14 and $T_1 = T(1 + r/100)$ is the false half-life. Then with $k = \dfrac{\ln 2}{T},\ k_1 = \dfrac{\ln 2}{T_1}$ we have the formulae $y(t) = y_0e^{-kt},\ y_1(t) = y_0e^{-k_1t}$. At a certain point in time a reading of the carbon-14 is taken resulting in a certain value y, which in the case of the true formula is given by $y = y(t)$ for some t, and in the case of the false formula is given by $y = y_1(t_1)$ for some t_1.

If the true formula is used then the time t since the beginning is given by $t = -\dfrac{1}{k}\ln\dfrac{y}{y_0}$. If the false formula is used we get a false value $t_1 = -\dfrac{1}{k_1}\ln\dfrac{y}{y_0}$; note that in both cases the value y/y_0 is the same. Thus $t_1/t = k/k_1 = T_1/T = 1 + r/100$, so the percentage error in the time to be measured is the same as the percentage error in the half-life.

20. **(a)** $y_{n+1} = y_n + 0.1(1 + 5t_n - y_n),\ y_0 = 5$

n	0	1	2	3	4	5	6	7	8	9	10
t_n	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
y_n	5.00	5.10	5.24	5.42	5.62	5.86	6.13	6.41	6.72	7.05	7.39

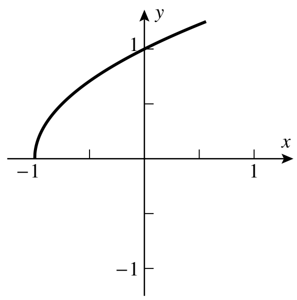
(b) The true solution is $y(t) = 5t - 4 + 4e^{1-t}$, so the percentage errors are given by

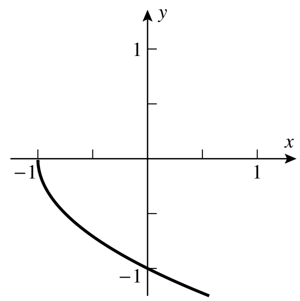
t_n	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
y_n	5.00	5.10	5.24	5.42	5.62	5.86	6.13	6.41	6.72	7.05	7.39
$y(t_n)$	5.00	5.12	5.27	5.46	5.68	5.93	6.20	6.49	6.80	7.13	7.47
abs. error	0.00	0.02	0.03	0.05	0.06	0.06	0.07	0.07	0.08	0.08	0.08
rel. error (%)	0.00	0.38	0.66	0.87	1.00	1.08	1.12	1.13	1.11	1.07	1.03

21. (a) $y = C_1 e^x + C_2 e^{2x}$ **(b)** $y = C_1 e^{x/2} + C_2 x e^{x/2}$

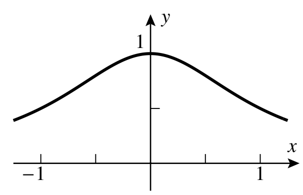
(c) $y = e^{-x/2}\left[C_1 \cos \frac{\sqrt{7}}{2} x + C_2 \sin \frac{\sqrt{7}}{2} x\right]$

22. (a) $2y dy = dx, y^2 = x + C$; if $y(0) = 1$ then $C = 1$, $y^2 = x + 1, y = \sqrt{x+1}$; if $y(0) = -1$ then $C = 1, y^2 = x + 1, y = -\sqrt{x+1}$.





(b) $\frac{dy}{y^2} = -2x\,dx,\ -\frac{1}{y} = -x^2 + C,\ -1 = C,\ y = 1/(x^2+1)$



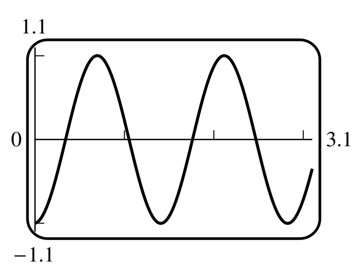
23. (a) Use (15) in Section 9.3 with $y_0 = 19, L = 95$: $y(t) = \frac{1805}{19 + 76e^{-kt}}$, $25 = y(1) = \frac{1805}{19 + 76e^{-k}}$, $k \approx 0.3567$; when $0.8L = y(t) = \frac{y_0 L}{19 + 76e^{-kt}}$, $19 + 76e^{-kt} = \frac{5}{4}y_0 = \frac{95}{4}$, $t \approx 7.77$ yr.

(b) From (13), $\frac{dy}{dt} = k\left(1 - \frac{y}{95}\right)y,\ y(0) = 19.$

24. (a) $y_0 = y(0) = c_1,\ v_0 = y'(0) = c_2\sqrt{\frac{k}{M}},\ c_2 = \sqrt{\frac{M}{k}}v_0,\ y = y_0 \cos\sqrt{\frac{k}{M}}\,t + v_0\sqrt{\frac{M}{k}}\sin\sqrt{\frac{k}{M}}\,t$

(b) $l = 0.5, k/M = g/l = 9.8/0.5 = 19.6,$

$y = -\cos(\sqrt{19.6}\,t) + 0.25\frac{1}{\sqrt{19.6}}\sin(\sqrt{19.6}\,t)$



(c) $y = -\cos(\sqrt{19.6}\,t) + 0.25\dfrac{1}{\sqrt{19.6}}\sin(\sqrt{19.6}\,t)$, so

$$|y\text{max}| = \sqrt{(-1)^2 + \left(\frac{0.25}{\sqrt{19.6}}\right)^2} \approx 1.10016 \text{ m is the maximum displacement.}$$

25. $y = y_0 \cos\sqrt{\dfrac{k}{M}}\,t,\ T = 2\pi\sqrt{\dfrac{M}{k}},\ y = y_0\cos\dfrac{2\pi t}{T}$

(a) $v = y'(t) = -\dfrac{2\pi}{T}y_0\sin\dfrac{2\pi t}{T}$ has maximum magnitude $2\pi|y_0|/T$ and occurs when $2\pi t/T = n\pi + \pi/2,\ y = y_0\cos(n\pi + \pi/2) = 0$.

(b) $a = y''(t) = -\dfrac{4\pi^2}{T^2}y_0\cos\dfrac{2\pi t}{T}$ has maximum magnitude $4\pi^2|y_0|/T^2$ and occurs when $2\pi t/T = j\pi,\ y = y_0\cos j\pi = \pm y_0$.

26. **(a)** In t years the interest will be compounded nt times at an interest rate of r/n each time. The value at the end of 1 interval is $P + (r/n)P = P(1 + r/n)$, at the end of 2 intervals it is $P(1+r/n) + (r/n)P(1+r/n) = P(1+r/n)^2$, and continuing in this fashion the value at the end of nt intervals is $P(1+r/n)^{nt}$.

(b) Let $x = r/n$, then $n = r/x$ and $\lim_{n\to+\infty} P(1+r/n)^{nt} = \lim_{x\to 0^+} P(1+x)^{rt/x} = \lim_{x\to 0^+} P[(1+x)^{1/x}]^{rt} = Pe^{rt}$.

(c) The rate of increase is $dA/dt = rPe^{rt} = rA$.

27. **(a)** $A = 1000e^{(0.08)(5)} = 1000e^{0.4} \approx \$1,491.82$

(b) $Pe^{(0.08)(10)} = 10,000,\ Pe^{0.8} = 10,000,\ P = 10,000e^{-0.8} \approx \$4,493.29$

(c) From (11) of Section 9.3 with $k = r = 0.08$, $T = (\ln 2)/0.08 \approx 8.7$ years.

28. The case $p(x) = 0$ has solutions $y = C_1y_1 + C_2y_2 = C_1x + C_2$. So assume now that $p(x) \neq 0$. The differential equation becomes $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} = 0$. Let $Y = \dfrac{dy}{dx}$ so that the equation becomes $\dfrac{dY}{dx} + p(x)Y = 0$, which is a first order separable equation in the unknown Y. We get $\dfrac{dY}{Y} = -p(x)\,dx, \ln|Y| = -\displaystyle\int p(x)\,dx,\ Y = \pm e^{-\int p(x)dx}$.
Let $P(x)$ be a specific antiderivative of $p(x)$; then any solution Y is given by $Y = \pm e^{-P(x)+C_1}$ for some C_1. Thus all solutions are given by $Y(t) = C_2e^{-P(x)}$ including $C_2 = 0$. Consequently $\dfrac{dy}{dx} = C_2e^{-P(x)},\ y = C_2\displaystyle\int e^{-P(x)}\,dx + C_3$. If we let $y_1(x) = \displaystyle\int e^{-P(x)}\,dx$ and $y_2(x) = 1$ then y_1 and y_2 are both solutions, and they are linearly independent (recall $P(x) \neq 0$) and hence $y(x) = c_1y_1(x) + c_2y_2(x)$.

29. $\dfrac{d}{dt}\left[\dfrac{1}{2}k[y(t)]^2 + \dfrac{1}{2}M(y'(t))^2\right] = ky(t)y'(t) + My'(t)y''(t) = My'(t)[\dfrac{k}{M}y(t) + y''(t)] = 0$, as required.